Jan 28, 2020 Numerical Analysis

Logistics:

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Old definition of Numerical analysis: "the study of rounding errors"

This is boring, and not very meaningful.

Better definition - Trefethen '92: "the study of algorithms for the problems of continuous mathematics"

Much of the field of numerical analysis came out of trying to efficiently, and stably, solve $A\vec{x} = \vec{b}$ in floating-point arithmetic.

NA touches all fields now: ODEs, PDEs, physics, etc.

General overview of topics to be covered:

- solving nonlinear systems of equations
- numerical linear algebra
- polynomial interpolation
- numerical integration
- ODEs: initial value problems
- Monte Carlo methods
- Fast Fourier Transform

• There will be computing! Familiarize yourself with MATLAB.

• Textbook: Suli & Mayers, Intro to Numerical Analysis (free from NYU)

Many numerical analysis / Math failures can be found at

ima.umn.edu/~arnold/disasters

This is an important field!

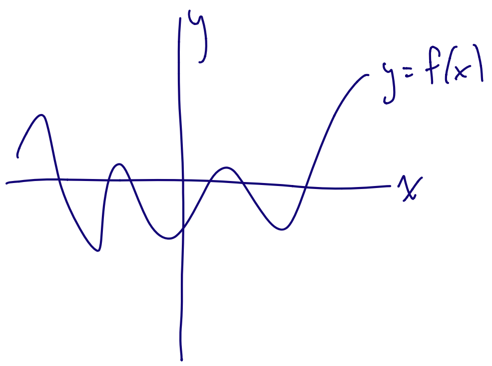
First topic Solving a nonlinear equation

Linear: $3x + 7 = 2$ Can solve by hand, explicit form of solution

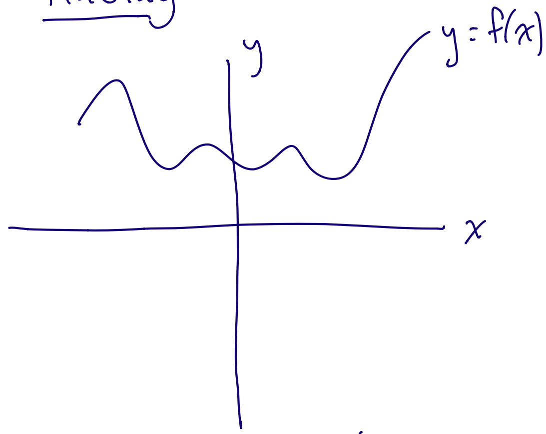
Nonlinear: $\cos x + x^2 - 7 = 5$ No closed form solution, must use a numerical method

General form of the problem:

Solve $f(x) = 0 \Rightarrow$ Root finding




vs.




Many solutions

No solution (at least if $x$ is required to be real-valued)

Ie. $x^2 + 1 = 0 \Rightarrow x = \pm i$

A sufficient condition for a solution to exist on the interval $[a,b]$: $f(a) < 0$ & $f(b) > 0$

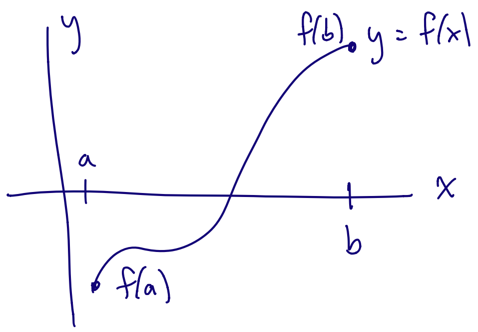
OR $f(a) > 0$ & $f(b) < 0$

**Thm:** If $f$ is continuous and real-valued on $[a,b]$, and if $f(a)\cdot f(b) < 0$, then there exists an $x \in (a,b)$ s.t. $f(x)=0$.

**Proof:** Merely apply the Intermediate Value Thm. (Calc I).

Can we use this Thm to design a numerical method for solving $f(x)=0$ ?

**Bisection**




$f(a) < 0$, $f(b) > 0$ $\Rightarrow$ $f(x)=0$ has a solution on $[a,b]$.

**Idea:** Split the interval in half, apply the same Thm:

**If** $f\left(\frac{a+b}{2}\right) < 0$, then $f(x)=0$ has a solution on $\left[\frac{a+b}{2}, b\right]$

**If** $f\left(\frac{a+b}{2}\right) > 0$, then $f(x)=0$ has a solution on $\left[a, \frac{a+b}{2}\right]$.

Split interval in half, and repeat.

Let $a_0 = a$, $b_0 = b$, the original interval.

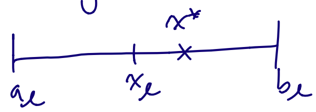
$[a_\ell, b_\ell]$ be the interval obtained after $\ell$ splittings.

Then $b_\ell - a_\ell = \dfrac{b_0 - a_0}{2^\ell} = \dfrac{L}{2^\ell}$ with $L = b_0 - a_0$.

Let $x_\ell = \dfrac{a_\ell + b_\ell}{2}$ be our approximation of the solution to $f(x)=0$ on step $\ell$.

When do we stop the splittings? How many steps of bisection do we take?

If we want to guarantee that $|x_\ell - x^*| < \epsilon$, (some small precision)




($x^*$: true solution, $f(x^*) = 0$.)

then we need to choose $\ell$ such that

$$|x_\ell - x^*| \leq \frac{b_\ell - a_\ell}{2} = \frac{1}{2}\,\frac{b_0 - a_0}{2^\ell} = \frac{1}{2^{\ell+1}} L < \epsilon$$

$$\Rightarrow \quad 2^{\ell+1} > \frac{L}{\epsilon} \quad \Rightarrow \quad \ell > 1 + \log_2 L/\epsilon .$$

If $e_\ell$ = error on $\ell^{th}$ step

$= |x_\ell - x^*|$ = <u>absolute error</u> in $x_\ell$.

then $e_{\ell+1} \leq \frac{1}{2} e_\ell$.

$\Rightarrow$ The error goes down by a factor of 2.

This is <u>not</u> very fast.

Bisection only used the <u>sign</u> of the function $f$ at $a$ and $b$.
Can we derive a better (faster) method by using the actual values $f(a)$ and $f(b)$?

Next time: Secant method & Newton's Method.