Jan 30, 2020 — Numerical Analysis

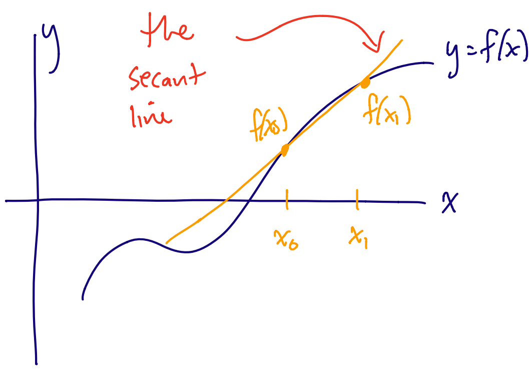
- Secant
- Deriv Newtons method

The Bisection Method used two pieces of information to approximate the solution to $f(x)=0$ — the sign of the function.

What if we use values? The result is the Secant Method.

Start with two guesses for the root, x_0, x_1:

Graphically:



Find the root of the secant line:

$$s(x) = \frac{f(x_1)-f(x_0)}{x_1 - x_0}(x-x_0) + f(x_0)$$

The root of the secant line satisfies $s(x)=0$:

$$s(x)=0 \implies x = x_1 - f(x_1)\left(\frac{x_1-x_0}{f(x_1)-f(x_0)}\right)$$

Call this root x_2, the next approximation to the root of f. Also, define $f(x_k) = f_k$.

The secant method generates a sequence of approximations to the root of f by:

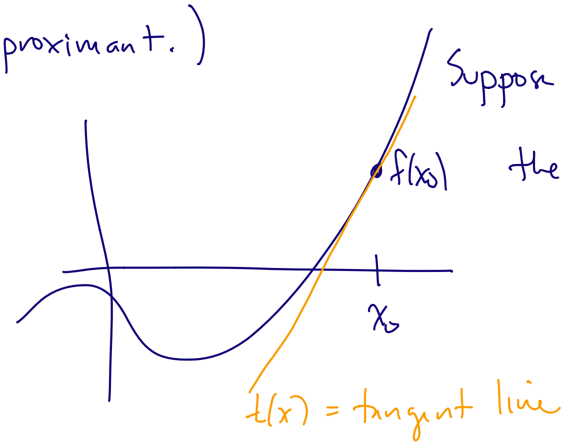
$$x_{k+1} = x_k - f_k\left(\frac{x_k - x_{k-1}}{f_k - f_{k-1}}\right).$$

We will revisit the convergence properties of this method later.

Summary: Approximate f by a secant line, find root of secant line, repeat using new approximate root.

Newton's Method

What if we are now allowed to use derivative information to approximate f? (And then find the root of this approximant.)



Suppose we know $f(x_0)$ and $f'(x_0)$, then we can draw the tangent line.

The equation of the tangent line is given as:

$$t(x) = f(x_0) + f'(x_0)(x - x_0)$$

The root of the tangent line satisfies $t(x) = 0$

$$\Rightarrow \quad x = x_0 - \frac{f(x_0)}{f'(x_0)} \equiv x_1$$

x_1 is the next approximation to the root of f.

Repeating this procedure at x_1 we obtain Newton's Method:

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}.$$

Alternative interpretation: the tangent line is a first-order Taylor approximation to f.

Recall: The Taylor series of f about x_0 (assuming f has infinite derivatives):

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

$$= f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f^{(3)}(x_0)}{3!}(x - x_0)^3 + \ldots$$

Truncating this series after the first two terms gives:

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$

If x_0 is close to the root of f, ξ, $f(\xi) = 0$, then we get that

$$f(\xi) = 0 \approx f(x_0) + f'(x_0)(\xi - x_0)$$

Solve for $\xi \Rightarrow$ $\quad \xi \approx x_0 - \dfrac{f(x_0)}{f'(x_0)}$. This is Newton's Method.

approximately

Summary The Secant Method and Newton's Method both work by the same mechanism: approximate f by a linear function, find the root of that linear function. Update and repeat.

Convergence Behavior

Let ξ be the true root of f, ie $f(\xi) = 0$.

We are interested in how the absolute error of x_k changes from iteration to iteration. $e_k = |\xi - x_k|$.

For the bisection method:

$$e_{k+1} \approx \frac{1}{2} e_k$$

It turns out that Newton converges quadratically:

$$e_{k+1} \approx A\, e_k^2$$ ← this is fast!

If $e_0 = 10^{-1}$ $\quad e_3 \sim 10^{-8}$

$e_1 \sim 10^{-2}$ $\quad e_4 \sim 10^{-16}$

$e_2 \sim 10^{-4}$

We will prove this rate next class.