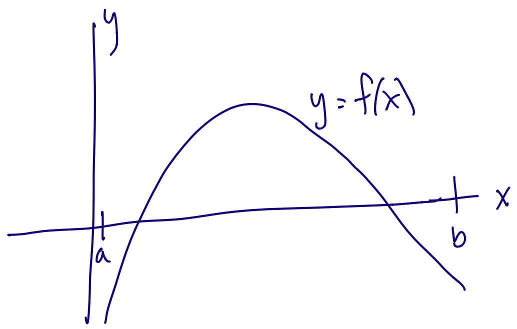
February 11, 2020 Numerical Analysis

One more application of root finding: Optimization

One may wish to find x such that $f(x)$ is as large (or small) as possible on some interval $[a,b]$:



If f is convex and smooth then the max occurs at a, b or one location in (a,b) when $f'(x)=0$

$\Rightarrow$ find the root of f', check values at a, b.

If $f'(\xi)=0$, then apply Newton's Method to the function $g = f'$:

$$x_{k+1} = x_k - g(x_k)/g'(x_k)$$

$$= x_k - f'(x_k)/f''(x_k)$$

Same idea...

Stopping Criteria:

Since we never know the true value of the root, how do we determine when to stop the root finding iterations?

$\Rightarrow$ examine successive iterations

If $x_k \to \xi$, then for k sufficiently large,

$$|\xi - x_{k+1}| \leq |\xi - x_k|$$

Therefore, $|x_{k+1} - x_k| \to 0$, use x_{k+1} as best guess for ξ, and stop when $|x_{k+1} - x_k| \leq \epsilon$.

Generalization: Fixed point iterations

Recall: Newton's Method

$$x_{k+1} = x_k - f(x_k)/f'(x_k) = g(x_k)$$

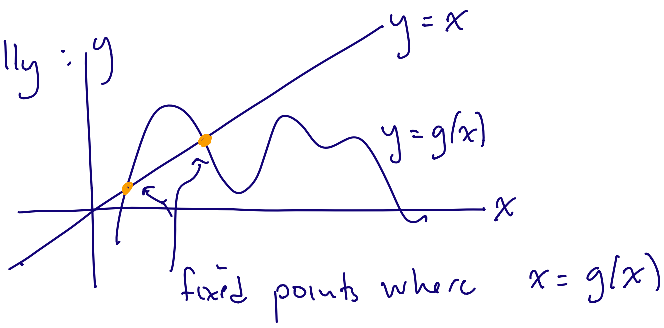
If the true root is ξ, then $\xi = g(\xi)$.

$\Rightarrow$ If $x_k = \xi$, then $x_{k+1} = \xi$. $\leftarrow$ We say ξ is a fixed point of the iteration $x_{k+1} = g(x_k)$.

For arbitrary functions g, one could ask: (1) are there fixed-points?

(2) Does the iteration converge?

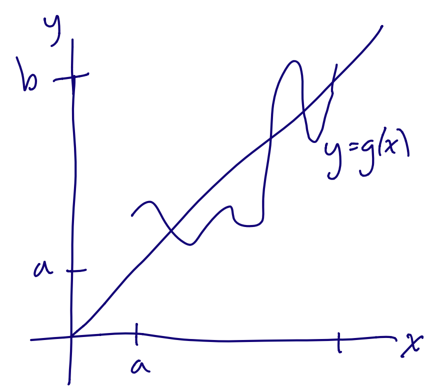
Graphically:



Note Solving $f(x) = 0$ is equivalent to finding a fixed point of $g(x) = f(x) + x$

since $g(x) = x \Rightarrow f(x) + x = x$

$f(x) = 0$



Definition A simple iteration is given by

$$x_{k+1} = g(x_k),$$

when we assume g is continuous on $[a,b]$, and that $g(x) \in [a,b]$ for all $x \in [a,b]$ (this condition guarantees the existence of a fixed point by the following theorem).

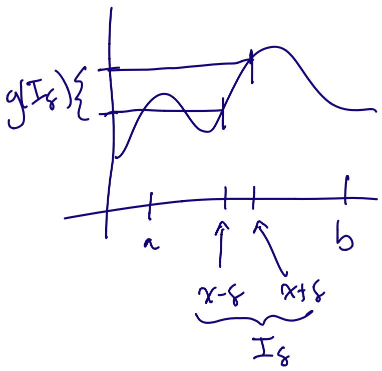
Brouwer's Fixed Point Theorem: If g is as above, then there exists $\xi \in [a,b]$ such that $\xi = g(\xi)$.

Proof: Since $g(x) \in [a,b]$ for all $x \in [a,b]$, it is easy to show that $g(x) - x = h(x)$ has at least one root in $[a,b]$:

Hint: show that $h(a)\,h(b) < 0$.

Next we are most interested in showing that some type of iteration converges, and therefore let's examine how g maps small neighborhoods of points:

I.e.: if $y \in [x-\delta, x+\delta]$, then where is $g(y)$?



Definition [Contraction]: Let g be continuous on $[a,b]$. The function g is a contraction on $[a,b]$ if there exists a number L with $0 < L < 1$ such that:

$$\underbrace{|g(x) - g(y)| \leq L\,|x-y|}_{} \quad \text{for all } x,y \in [a,b].$$

This means that g maps points to values which are closer together.

If L is allowed to be any positive number, then this is known as a Lipschitz condition. (I.e. if L is also allowed to be ≥ 1.) (Compare with Lipschitz continuous from Analysis.)

Theorem [Contraction Mapping Theorem]

Let g be continuous on $[a,b]$, $g(x) \in [a,b]$ for all $x \in [a,b]$, and assume that g is a contraction on $[a,b]$.

Then, g has a unique fixed point $\xi = g(\xi)$.

Furthermore, $x_{n+1} = g(x_n)$ converges to ξ for any initial starting value $x_0 \in [a,b]$.

Proof next time...