Lecture notes

Thursday Feb 13, 2020

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1 Fixed point iterations

1.1 Contractions

Definition 1.1 (Contraction)**.** Let g be continuous on $[a, b]$. The function g is *a contraction* on $[a, b]$ if there exists a number L with $0 < L < 1$ such that $|g(x) - g(y)| < L|x - y$ for all $x, y \in [a, b]$.

The above definition means that g maps points to values which are closer together.

The above definition is also related to the concept of *Lipschitz continuity*, in which the restriction of $L < 1$ is lifted.

Theorem 1 (Contraction Mapping Theorem (CMT))**.** *If g is a contraction on $[a, b]$ and $g(x) \in [a, b]$ for all $x \in [a, b]$, then g has a unique fixed point $\xi = g(\xi) \in [a, b]$. Furthermore, the sequence defined by $x_{k+1} = g(x_k)$ converges to ξ for any $x_0 \in [a, b]$.*

Proof. A fixed point ξ exists due to Brouwer's Fixed Point Theorem. Now, we show that it is unique by contradiction. Assume that ξ is not unique; this means that there exists another $\xi' \neq \xi$ such that $\xi' = g(\xi')$. Furthermore, we have that

$$\begin{aligned} |\xi - \xi'| &= |g(\xi) - g(\xi')| \\ &\leq L|\xi - \xi'|. \end{aligned} \tag{1}$$

Dividing each side by $|\xi - \xi'|$ implies that $L \geq 1$, and therefore g cannot be a contraction. This is a contradiction, and therefore $\xi = \xi'$, proving that ξ is the unique fixed point.

Next, we will show that the x_k's converge to ξ for any initial $x_0 \in [a, b]$. Since $x_k = g(x_{k-1})$ and g is a contraction,

$$\begin{aligned} |x_k - \xi| &= |g(x_{k-1}) - g(\xi)| \\ &\leq L|x_{k-1} - \xi|. \end{aligned}$$

Repeating this argument k times, we have that

$$|x_k - \xi| \leq L^k |x_0 - \xi|,$$

and since $0 < L < 1$, $L^k \to 0$ and therefore $|x_k - \xi| \to 0$ as $k \to \infty$. This proves convergence of the x_k's. □

Example 1.1. Take the function $f(x) = e^x - 2x - 1$ on the interval $[1, 2]$. It is easy to show that $f(x) = 0$ has a unique solution on this interval, call it ξ, i.e. $f(\xi) = 0$. This root finding problem can be re-written as a fixed point problem:

$$\begin{aligned} e^x &= 2x+1 \\ \log e^x &= \log(2x+1) \\ x &= \log(2x+1) \\ &= g(x). \end{aligned}$$

Clearly the function g is continuous on the interval $[1,2]$, as well as differentiable. By the Mean Value Theorem (MVT), there exists some $\eta = \eta(x,y) \in [1,2]$ such that for any $x, y \in [1,2]$:

$$\begin{aligned} |g(x) - g(y)| &= |g'(\eta)(x-y)| \\ &= |g'(\eta)||x-y|. \end{aligned}$$

Since $g'(x) = 2/(2x+1)$, clearly $|g'(x)| \leq 2/3$ for $x \in [1,2]$. Therefore we have that

$$|g(x) - g(y)| \leq \frac{2}{3}|x-y|,$$

and therefore $g(x) = \log(2x+1)$ is a contraction.

By the CMT, we know that the sequence defined by $x_{k+1} = g(x_k)$ converges to ξ (the root of f) for any $x_0 \in [1,2]$.

How many iterations do we need to guarantee that $e_k = |x_k - \xi| \leq \epsilon$, where $\epsilon > 0$ is some desired precision?

Using the proof of the CMT, we have that:

$$|x_k - \xi| \leq \left(\frac{2}{3}\right)^k |2-1|.$$

Setting the expression above to be less than or equal to ϵ we have that:

$$\left(\frac{2}{3}\right)^k \leq \epsilon \qquad \Longrightarrow \qquad k \geq \frac{\log \epsilon}{\log 2/3} \approx 2.5|\log \epsilon|.$$

1.2 Stability of fixed points

Some fixed points are attracting and some are repelling, the following definitions classify these fixed points.

Definition 1.2 (Stable fixed point)**.** A fixed point $\xi = g(\xi)$ is stable if $x_k \to \xi$ for **every** x_0 in some sufficiently small neighborhood of ξ.

Definition 1.3 (Unstable fixed point)**.** A fixed point $\xi = g(\xi)$ is unstable if the **only** initial condition that yields a convergent sequence is $x_0 = \xi$, i.e., the sequence x_k diverges for every x_0 in a neighborhood of ξ.

1.3 Rates of convergence

In the case that ξ is a stable fixed point, at what rate do we expect the sequence to converge? Examine the limit of the ratio of successive errors:

$$\begin{aligned}\lim_{k\to\infty} \frac{|x_{k+1}-\xi|}{x_k-\xi|} &= \lim_{k\to\infty} \frac{|g(x_k)-\xi|}{x_k-\xi|} \\ &= |g'(\xi)|.\end{aligned}$$

So the derivative of g dictates how fast the sequence will converge.

Remark. Of course it has to be the case that $|g'(\xi)| < 1$ otherwise we have that for some sufficiently large k the errors obey $e_{k+1} > e_k$, and therefore the sequence diverges. Go back and contrast this situation with the discussion of *order of convergence* from a few lectures ago.

Definition 1.4 (Rate of convergence)**.** As before, denote by e_k the error $e_k = |x_k - \xi|$, and furthermore set

$$\lim_{k\to\infty} \frac{e_{k+1}}{e_k} = \mu.$$

If $0 < \mu < 1$ then the x_k's converge **linearly**. This is a **first order convergent sequence**.

Now, set $\rho = -\log_{10}\mu$. The number ρ is known as the **asymptotic rate of convergence**.

The above definition can be applied to *any* sequence, not just those obtained from fixed point iterations.

Example 1.2. Define a sequence by:

$$x_k = 1 + \frac{1}{10^k}.$$

Clearly the limit of this sequence as $k \to \infty$ is 1, and we therefore have that

$$\lim_{k\to\infty} \frac{e_{k+1}}{e_k} = \lim_{k\to\infty} \frac{1/10^{k+1}}{1/10^k} = \frac{1}{10},$$

and the rate of convergence is given by

$$\rho = -\log_{10}\frac{1}{10} = 1.$$

What exactly does ρ measure? The value of ρ measures the number of correct decimal digits gained on every successive iteration. E.g., if $\rho = 2$, then every iteration has two more digits of agreement with its limit.

For fixed point iterations, since

$$\lim_{k\to\infty} \frac{e_{k+1}}{e_k} = |g'(\xi)|,$$

we have that $\mu = |g'(\xi)|$ and $\rho = -\log_{10}|g'(\xi)|$. This means that the *flatter* the function g is near the fixed point, the *faster* $x_{k+1} = g(x_k)$ converges.