Numerical analysis
March 3, 2020

Pivoting

This algorithm for $A = LU$ may <u>fail</u> if a pivot is equal to 0, <u>or</u> if it is very small.

Thm For any $A \in \mathbb{R}^{n\times n}$, $\exists$ P, L, U s.t. $PA = LU$.

Solution: Interchange rows.

Ex: $A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 4 & 4 \\ 4 & 8 & 3 \end{pmatrix}$ is obviously <u>singular</u>

Ex: ~~A~~ What goes wrong with

$$A = \begin{pmatrix} 10^{-20} & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Exact solution: $\vec{x} \approx \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$x_1 = 1 + \frac{1}{10^{20}-1}$$

$$x_2 = 1 - \frac{1}{10^{20}-1}$$

With our Gaussian elimination procedure: (in double prec. arith.)

$$\left(\begin{array}{cc|c} 10^{-20} & 1 & 1 \\ 1 & 1 & 2 \end{array}\right) \sim \left(\begin{array}{cc|c} 10^{-20} & 1 & 1 \\ 0 & -10^{20} & -10^{20} \end{array}\right)$$

$$\Rightarrow \quad x_2 = 1$$

$$x_1 = \frac{1 - 1}{10^{-20}} = 0 \qquad \left(\text{or multiple } \frac{\epsilon}{10^{-20}} \sim 10^4\right)$$

The problem is completely avoided by interchanging:

$$\left(\begin{array}{cc|c} 1 & 1 & 2 \\ 10^{-20} & 1 & 1 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & 1 \end{array}\right)$$

Interchanging with <u>largest</u> element in column: <u>Partial Pivoting</u>

Op. Count. - no cost for pivoting
- $O(n^2)$ cost for searching

<u>Full pivoting</u> search column <u>and</u> row (requires re-ordering of RHS as well)

In this case (part. piv.): $A = PLU$ ← permutation matrix swapping rows.

<u>Special Case</u> <u>Symmetric Positive Definite matrices</u>

<u>No pivoting</u> is necessary → one of the earliest results in the field of numerical ~~analy~~ analysis.

At the time: Solve $A^*A\vec{x} = A^*\vec{b}$ instead (not a good idea).

If A is . sym. pos. def. ~~then~~ then

$$A = LU$$

$$= A^t = U^t L^t$$

Can we make $L = U^t$, $L^t = U$? Yes. Scale things differently.

&

<u>Cholesky Decomposition</u>: $A = L^t L$ for A s.p.d.

$$L = \begin{pmatrix} l_{11} & & & \\ l_{21} & l_{22} & & \\ \vdots & \vdots & \ddots & \\ & & & l_{nn} \end{pmatrix}$$

$$LL^t = \begin{pmatrix} l_{11} & & \\ l_{21} & l_{22} & \\ \vdots & & \ddots \end{pmatrix} \begin{pmatrix} l_{11} & l_{21} & \cdots \\ 0 & l_{22} & \\ & & \ddots \end{pmatrix}$$

$$= \begin{pmatrix} l_{11}^2 & & \\ l_{21} l_{11} & \ddots & \\ l_{31} l_{11} & & \end{pmatrix}$$

$$\Rightarrow l_{11}^2 = a_{11}$$

$$l_{21} = a_{21} / l_{11}$$

$$\vdots$$

We will now discuss the analogue for linear systems: When solving $A\vec{x} = \vec{b}$, what is the sensitivity of $\vec{x} = A^{-1}\vec{b}$ on $\vec{b}$?

Norms for vectors:

Def: $\| \cdot \|$ is a norm if

(1) $\| \vec{u} \| \geq 0$, $\| \vec{u} \| = 0$ iff $\vec{u} = \vec{0}$

(2) $\| \alpha \underline{u} \| = |\alpha| \, \| \underline{u} \|$

(3) $\| \underline{u} + \underline{v} \| \leq \| \underline{u} \| + \| \underline{v} \|$ (triangle inequality)

Think about norms as lengths $\Rightarrow$ $\| \underline{u} \| = \sqrt{\underline{u} \cdot \underline{u}}$

$= \sqrt{\sum u_i^2}$ (ℓ^2 norm)

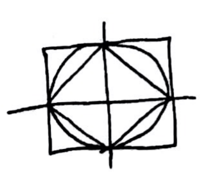
Alternative norms

ℓ^∞-norm: $\| \underline{u} \|_\infty = \max |u_i|$

ℓ^1-norm: $\| \underline{u} \|_1 = \sum |u_i|$

ℓ^p-norm: $\| \underline{u} \|_p = \left(\sum |u_i|^p \right)^{1/p}$

Ex: "unit circles in $1, \infty, 2$ norms"



Only the 2-norm comes from an inner product (dot product)