Numerical Analysis

March 26, 2020.

New Grading Scheme

HW 1-3	10%
Exam 1	25%
HW 4-6	65%

Last Class: Least Squares Problems

$$\min_{\vec{x}} \| A\vec{x} - \vec{b} \|_2$$

Instead of solving the normal equations, solve the system $A\vec{x} = \text{proj}_{\text{col} A} \vec{b}$

(1) QR factorization

(2) SVD

(3) Example application of linear regression.

Next topic: Eigenvalue Problems

Recall: $\lambda, \vec{v}$ are an eigenvalue pair if $A\vec{v} = \lambda\vec{v}$.

Direct computation: form characteristic equation:

$$\underbrace{\det(A - \lambda I)}_{\substack{\text{polynomial in } \lambda \text{ of degree } n \text{ if } A \in \mathbb{R}^{n \times n}. \\ p(\lambda)}} = 0$$

The solutions to $p(\lambda) = 0$ are the eigenvalues.

This is expensive for various reasons: forming $p(\lambda)$ cost $n!$ flops.

Then, a nonlinear root finding algorithm must be used to solve $p(\lambda) = 0$. (Bisection, Newton, etc.)

Application: Systems of linear Initial Value problems:

$$\vec{y}\,' = A\vec{y} \qquad \vec{y}(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_n(t) \end{pmatrix} \qquad \vec{y}\,'(t) = \begin{pmatrix} y_1'(t) \\ y_2'(t) \\ \vdots \\ y_n'(t) \end{pmatrix}$$

One solution method is to diagonalize A. (Investigate diagonalization of matrices.)

$$A = P D P^{-1}$$

$$P = (\vec{v}_1\, \vec{v}_2 \cdots \vec{v}_n) \qquad D = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$

$$\vec{y}\,' = PDP^{-1}\vec{y}$$

$$\underbrace{P^{-1}\vec{y}\,'}_{\vec{u}\,'} = D \underbrace{P^{-1}\vec{y}}_{\vec{u}}$$

$$\Rightarrow \quad \vec{u}\,' = D\vec{u}$$

$$\Rightarrow \quad \begin{aligned} u_1' &= \lambda_1 u_1 \\ u_2' &= \lambda_2 u_2 \\ &\vdots \\ u_n' &= \lambda_n u_n \end{aligned} \quad \Rightarrow \quad \begin{aligned} u_i &= c_i e^{\lambda_i t} \\ u_i' &= \lambda_i c_i e^{\lambda_i t} \\ &= \lambda_i u_i \end{aligned}$$

$\Rightarrow$ Change variables back.

$$\vec{u} = \begin{pmatrix} c_1 u_1 \\ \vdots \\ c_n u_n \end{pmatrix} = P^{-1}\vec{y}$$

$$\Rightarrow \quad \vec{y} = P\vec{u}.$$

Solving $\vec{y}\,' = A\vec{y}$ was reduced to finding eigenvalues and eigenvectors of A.

Big Important Theorem:

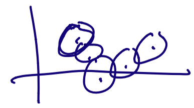
Gerschgorin's Theorem Let A be an $n \times n$ matrix (real or complex). Then, all the eigenvalues of A lie in

$$\bigcup_{i=1}^{n} D_i \,, \quad \text{where}$$

i^{th} diagonal of A

$$D_i = \left\{ z \in \mathbb{C} \text{ such that } |z - a_{ii}| \le \underbrace{\sum_{j \ne i} |a_{ij}|}_{\text{sum of off-diagonal entries in row } i.} \right\}$$

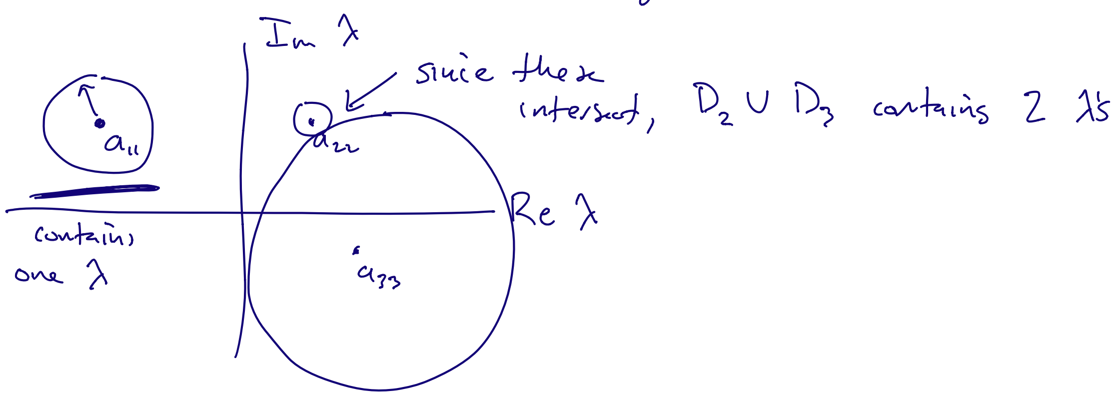
referred to as a "Gerschgorin Disk".



← 5 disks connected,

If m disks are connected, then m eigenvalues are located in this region.

Graphically



Simplest case:

A is diagonal: $A = \begin{pmatrix} a_{11} & & 0 \\ & a_{22} & \\ 0 & & \ddots \\ & & & a_{nn} \end{pmatrix} \Rightarrow \lambda_i = a_{ii}$

Proof: Let $\lambda, \vec{v}$ be an eigenvalue/vector pair.

Then $A\vec{v} = \lambda \vec{v} \Rightarrow$ i^{th} component

$$\sum a_{ij} v_j = \lambda v_i \quad \text{for all } i$$

$$\Rightarrow (\lambda - a_{ii}) v_i = \sum_{j \ne i} a_{ij} v_j$$

Pick largest element of $\vec{v}$, call it v_k. (in absolute value)

$$\Rightarrow (\lambda - a_{kk}) v_k = \sum_{j \neq k} a_{kj} v_j$$

$$|\lambda - a_{kk}| \cancel{|v_k|} \leq \sum_{j \neq k} |a_{kj}| \underbrace{\frac{|v_j|}{|v_k|}}_{\leq 1}$$

$$\leq \sum_{j \neq k} |a_{kj}|$$

We will revisit this theorem in detail when discussing Jacobi's method.

The Power Method

Goal Calculate the eigenvalue with largest magnitude and associated eigenvector. (Assume that A is diagonalizable.)

Start with a random vector $\vec{w}$.

If $\vec{w}$ is truly random, then it is a linear combination of every eigenvector of A.

$$\Rightarrow \vec{w} = \sum_{j=1}^{n} c_j \vec{v}_j$$

eigenvectors.

Apply A to $\vec{w}$: $A\vec{w} = A\left(\sum_j c_j \vec{v}_j\right)$

$$= \sum_j c_j A\vec{v}_j$$

$$= \boxed{\sum_j c_j \lambda_j \vec{v}_j}$$

$$A^2\vec{w} = A(A\vec{w})$$

$$= \sum_j c_j \lambda_j^2 \vec{v}_j$$

$$A^k \vec{w} = \sum_j c_j \lambda_j^k \vec{v}_j$$

For sufficiently large k, this is dominated by the largest λ, $\approx c_1 \lambda_1^k \vec{v}_1$.

(Assume that $|\lambda_1| > |\lambda_2| > |\lambda_3| \cdots$)

If $|\lambda_1| > |\lambda_2|$, sufficiently larger, then λ_1 dominates.

So eventually, if $\vec{y}^{(k)} = A^k \vec{w}$, then

$$\vec{y}^{(k+1)} = A^{k+1}\vec{w}$$
$$= A\,\vec{y}^{(k)} \approx \underline{\lambda_1}\,\vec{y}^{(k)}$$

Normalize these iterates on every step:

$$\vec{w}_0 = \vec{w}/\|\vec{w}\|$$

$$\vec{w}_1 = \frac{A\vec{w}_0}{\|A\vec{w}_0\|} \quad \cdots \quad \vec{w}_k = \frac{A\vec{w}_{k-1}}{\|A\vec{w}_{k-1}\|}.$$

Under this normalization, the eigenvector λ_1 is approximately equal to

① $w_{ki}/w_{(k-1)i} \approx \lambda_1$

i^{th} component of $\vec{w}_k$

② Better option is to estimate λ_1 as

$$\lambda_1 \approx (A\vec{w}_k, \vec{w}_k)$$

Since $A\vec{w}_k \approx \lambda_1 \vec{w}_k$

$$\vec{w}_k^T A \vec{w}_k \approx \lambda_1 \underbrace{\vec{w}_k^T \vec{w}_k}_{=1}$$

$=1$ since $\vec{w}_k$ is a unit vector.

The $\vec{w}_k$'s approach $\vec{v}_1$ as $k \to \infty$.

How fast does the power method converge?

Examine the quantity $A^k\vec{w} - \vec{v}_1$:

If k is sufficiently large, then $A^k\vec{w} \approx c_1 \lambda_1^k \vec{v}_1$ (assume $c_1 > 0$)

$$A^k\vec{w} \simeq c_1 \lambda_1^k \vec{v}_1$$

$$\Rightarrow \vec{v}_1 \approx \frac{1}{c_1\lambda_1^k} A^k \vec{w}$$

$$= \frac{1}{c_1\lambda_1^k}\left(c_1\lambda_1^k\vec{v}_1 + c_2\lambda_2^k\vec{v}_2 + \dots + c_n\lambda_n^k\vec{v}_n\right)$$

$$= \vec{v}_1 + \frac{c_2}{c_1}\left(\frac{\lambda_2}{\lambda_1}\right)^k\vec{v}_2 + \frac{c_3}{c_1}\left(\frac{\lambda_3}{\lambda_1}\right)^k\vec{v}_3 + \dots + \frac{c_n}{c_1}\left(\frac{\lambda_n}{\lambda_1}\right)^k\vec{v}_n .$$

If $|\lambda_j| < |\lambda_1|$ for $j > 1$, then $\left(\frac{\lambda_j}{\lambda_1}\right)^k \to 0$ as $k \to \infty$

$$\vec{w}_k = \frac{A^k\vec{w}}{\|A^k\vec{w}\|}$$

$$\|\vec{w}_k - \vec{v}_1\| \approx \left|\frac{c_2}{c_1}\right| \left|\frac{\lambda_2}{\lambda_1}\right|^k$$

$$\sim O\left(\left|\frac{\lambda_2}{\lambda_1}\right|^k\right)$$

The convergence of the power method depends on the gap in the eigenvalues.

I.e. the relative size of λ_2 to λ_1.

This means that if $\left|\frac{\lambda_2}{\lambda_1}\right| \approx 1$, then convergence is very slow.