May 5, 2020

Numerical Analysis

Re-cap: Last time

Initial Value Problem: $y'(t) = f(t, y(t))$

$y(t_0) = y_0$

Euler's Method: $\frac{y_{k+1} - y_k}{h} = f(t_k, y_k)$

$\Rightarrow y_{k+1} = y_k + h \cdot f(t_k, y_k)$

Recall: $t_k = t_0 + k \cdot h$

$y_k \approx y(t_k)$.

Other topics:

- Richardson extrapolation
- Local vs. Global error for Euler's Method

Additional Schemes for solving $y' = f(t, y(t))$:

- Midpoint method: Half step of Euler, then a full step:

$$y_{k+1/2} = y_k + \frac{h}{2} f(t_k, y_k)$$

$$y_{k+1} = y_k + h \cdot f(t_{k+\frac{1}{2}}, y_{k+\frac{1}{2}})$$

Quadrature Method: Trapezoidal Method

$$(*) \quad y_{k+1} = y_k + \underbrace{\frac{h}{2}\left(f(t_k, y_k) + f(t_{k+1}, y_{k+1})\right)}_{\approx \int_{t_k}^{t_{k+1}} f(t, y(t))\, dt}$$

This equation must be solved for y_{k+1}

IMPLICIT METHOD

Depending on how complicated f is, it may be expensive to solve (*) for y_{k+1}.

We can avoid solving for y_{k+1} by using instead an approximation to the Trapezoidal rule.

Heun's Method ① Take a Euler step to obtain $\tilde{y}_{k+1}$

② Use $\tilde{y}_{k+1}$ in the trapezoidal rule.

$$\tilde{y}_{k+1} = y_k + h \cdot f(t_k, y_k) \qquad \boxed{\text{Euler step}}$$

$$y_{k+1} = y_k + \frac{h}{2}\left(f(t_k, y_k) + f(t_{k+1}, \tilde{y}_{k+1}) \right) \qquad \text{EXPLICIT METHOD.}$$

Analysis of Explicit One-Step Methods

All explicit one-step methods are of the form:

$$y_{k+1} = y_k + h \cdot \underbrace{\psi(t_k, y_k, h)}_{\substack{\text{varies by how we approximate } \int_{t_k}^{t_{k+1}} f \\ \text{and how we approximate } y'.}}$$

Example:

$$y_{k+1/2} = y_k + \frac{h}{2} f(t_k, y_k)$$

$$y_{k+1} = y_k + h \cdot f(t_{k+\frac{1}{2}}, y_{k+1/2})$$

$$= y_k + h \cdot f\left(t_k + \frac{h}{2},\ y_k + h/2 \cdot f(t_k, y_k) \right)$$

$\Rightarrow$ Therefore: $\psi(t, y, h) = f\left(t + \frac{h}{2},\ y + \frac{h}{2} \cdot f(t, y)\right)$.

Definition A one-step method is consistent if

$$\lim_{h \to 0} \psi(t, y, h) = f(t, y)$$

$$\Leftrightarrow \lim_{h \to 0} \left(\underbrace{\frac{y(t_{k+1}) - y(t_k)}{h}}_{\approx y'(t_k)} - \underbrace{\psi(t_k, y(t_k), h)}_{\approx f(t_k, y(t_k))} \right) = 0$$

$\boxed{2}$

Definition A one-step method is stable if there is some $K>0$ and $h_0>0$ such that two solutions $y_n, \tilde{y}_n$ have $|y_n - \tilde{y}_n| \leq K |y_0 - \tilde{y}_0|$ whenever $h \leq h_0$ and $nh \leq T - t_0$

$\uparrow$ number of steps.

Consider the two initial value problems:

$$\begin{aligned} y'(t) &= f(t,y) \\ y(t_0) &= y_0 \end{aligned} \qquad \Bigg| \qquad \begin{aligned} \tilde{y}'(t) &= f(t,y) \\ \tilde{y}(t_0) &= \tilde{y}_0. \end{aligned}$$

$$\Rightarrow \quad \begin{aligned} y_n &\simeq y(t_n) \\ \tilde{y}_n &\approx \tilde{y}(t_n) \end{aligned}$$

Theorem If an explicit one-step method is stable and consistent and it has a local truncation error of $\mathcal{O}(h^p)$, then the global error is $\mathcal{O}(h^p)$.

Definition An explicit one-step method is convergent if

$$\underbrace{\max_{t_n \in [t_0, T]} |y(t_n) - y_n|}_{\text{Global error.}} \to 0 \quad \text{as} \quad y_0 \to y(t_0) \quad \text{and} \quad h \to 0.$$

Overall take home message:

$$\text{Stable} + \text{consistent} \Rightarrow \text{convergent}$$

Stiff Initial Value Problems

A prime example of when things go wrong is stiff IVPs. Consider the system of initial value problems:

$$\left.\begin{aligned} y_1'(t) &= -100\, y_1(t) + y_2(t) \\ y_2'(t) &= -\tfrac{1}{10}\, y_2(t) \end{aligned}\right\}$$

Can be written in matrix form as

$$\vec{y}\,' = A\vec{y},$$

$$\vec{y}(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$$

$$A(t) = \begin{pmatrix} -100 & 1 \\ 0 & -1/10 \end{pmatrix}$$

The solution to this system can be obtained analytically:

$$\begin{cases} y_2(t) = y_2(0)\, e^{-t/10} \\ y_2'(t) = -\frac{1}{10}\, y_2(0)\, e^{-t/10} = -\frac{1}{10}\, y_2(t) \quad \checkmark \end{cases}$$

$$y_1(t) = \underbrace{c_1 e^{-100t}}_{\hookrightarrow \text{ decays very fast.}} + c_2 e^{-t/10}$$

$$t = 1 \implies e^{-100} \approx 10^{-44}.$$

$$e^{-1/10} \approx 0.9$$

Try Euler's Method:

2nd Equation: $$y_{2,k+1} = y_{2,k} - \frac{h}{10}\, y_{2,k}.$$

$$= \left(1 - \frac{h}{10}\right) y_{2,k}.$$

$$= \left(1 - \frac{h}{10}\right)^{k+1} y_2(0). \quad \to 0 \quad \text{if} \quad \frac{h}{10} < 1$$

$$= \underline{h < 10}.$$

1st Equation: $$y_{1,k+1} = y_{1,k} + h\left(-100\, y_{1,k} + y_{2,k}\right)$$

$$= (1 - 100h)\, y_{1,k} + \underbrace{h\, y_{2,k}}_{= h\left(1 - \frac{h}{10}\right)^{k} y_{2}(0)}$$

... continue back substituting for $y_{1,k}$ ...

We finally obtain that

$$y_{1,k+1} = (1-100h)^{k+1} y_1(0) + h\left(1-\frac{h}{10}\right)^k \underbrace{\left(\sum_{\ell=0}^{k}\left(\frac{1-100h}{1-h/10}\right)^{\ell}\right)}_{\text{can be computed, some number.}} y_2(0)$$

$$= d_1(1-100h)^{k+1} + d_2\left(1-\frac{h}{10}\right)^{k+1}$$

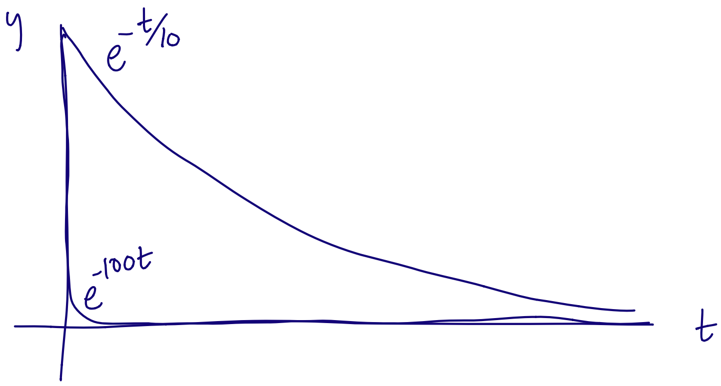
In order for $y_{1,k} \to 0$ as $k\to\infty$, we need both $(1-100h)^{k+1}$ and $\left(1-\frac{h}{10}\right)^{k+1}$ to go to zero.

So $|1-100h| < 1 \implies h < \frac{1}{50}$.

If $h > \frac{1}{50}$, then $(1-100h)^{k+1}$ will grow even after the analytic solution e^{-100t} has decayed to near zero.

Stiff equations: ① Generally only arise in systems of equations

The defining characteristic of stiff equations. ② The solutions have components that behave on different time scales.



If $y \approx a e^{-t/10} + b e^{-100t}$

$\approx a e^{-t/10}$ for $t > 0$,

but the numerical method still requires time-steps dictated by e^{-100t}.

More stability analysis

Consider the model problem:

$y' = \lambda y$. The solution is $y(t) = c e^{\lambda t}$,

$y(t) \to 0$ iff $\mathrm{Re}(\lambda) < 0$.

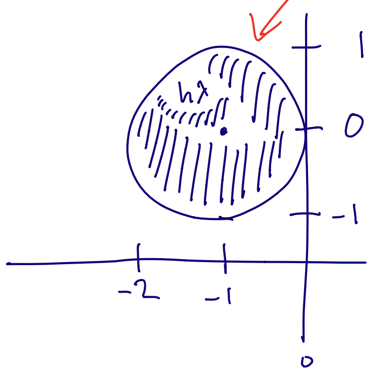
Definition Region of absolute stability: All numbers $z = h\lambda \in \mathbb{C}$ such that $y_k \to 0$ as $k \to \infty$ when the ODE method is applied to $y' = \lambda y$.

Example Euler's Method

$$
\begin{aligned}
y_{k+1} &= y_k + h\lambda y_k \\
&= (1 + h\lambda)\, y_k \\
&= (1 + h\lambda)^{k+1}\, y_0
\end{aligned}
$$

$y_k \to 0$ iff $\quad |1 + h\lambda| < 1$

$\iff$



The interior of this disk is the region of absolute stability.