The Polyfolds of Gromov-Witten Theory

Helmut Hofer

Sunday, April 27th, 9:00am–∞pm

In this Sunday lecture series I shall describe the polyfold structure on the space of stable curves Z (not necessarily pseudoholomorphic) and the strong bundle structure for a natural bundle $p : W \to Z$, so that the nonlinear Cauchy-Riemann operator can be viewed as (generalized) Fredholm section whose solution set is the Gromov-compactified moduli space of pseudoholomorphic stable curves. The same ideas work for symplectic field theory essentially without any additional technical difficulties.

There are some prerequisites. If you understand the words in the glossary you are prepared. If not try to read

$$http://www.cims.nyu.edu/\sim hofer/stanford/lecture-1-5.pdf$$

In particular the subsection about Deligne-Mumford theory via Lie groupoids is useful for the polyfold constructions.

1 Glossary

We recall some of the basic notions from [1].

- **sc-structure**. An sc-structure on the Banach space E is a nested sequence
$$E = E_0 \supset E_1 \supset E_2 \cdots \supset E_\infty = \bigcap_{k\geq 0} E_k$$
of Banach spaces E_m, $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, having the following properties.
  (1) If $m < n$, the inclusion $E_n \to E_m$ is a compact operator.
  (2) The vector space E_∞ is dense in E_m for every $m \geq 0$.

A Banach space E equipped with an sc-structure (E_m) is called an **sc-Banach space**. Each of the spaces E_m is an sc-Banach space and denoted by E^m. The sc-structure on E^m is given by $(E_{m+k})_{k\geq 0}$.

- **sc-operator**. A bounded linear operator $T : E \to F$ between sc-Banach spaces is called an **sc-operator** if $T(E_m) \subset F_m$ and $T : E_m \to F_m$ is continuous. If, in addition, T is bijective and $T^{-1} : F \to E$ is sc^0, then T is called an **sc-isomorphism**.

- **partial quadrant**. Given an sc-Banach space W, a subset $C \subset W$ is a partial quadrant of W if there is an sc-Banach space Q and an sc-isomorphism $T : W \to \mathbb{R}^n \oplus Q$ such that $T(C) = [0,\infty)^n \oplus Q$.

- **induced sc-structure**. If U is a relatively open subset of a partial quadrant C in the sc-smooth Banach space E, then the nested sequence of sets $U_m = U \cap E_m$ is called the **induced sc-structure** on U. The set U_m inherits sc-smooth structure defined by $(U_{m+k})_{k\geq 0}$. The set U_m equipped with this induced sc-structure is denoted by U^m.

- **direct sum and triangle-sum**. If $(E_n)_{n\geq 0}$ and $(F_n)_{n\geq 0}$ are sc-smooth structures of E and F, then $E \oplus F$ carries the sc-structure defined by $(E \oplus F)_n = E_n \oplus F_n$. By $E \triangleleft F$, the triangle sum, we denote the Banach space $E \oplus F$ equipped with the bi-filtration $(E \triangleleft F)_{m,k} = E_m \oplus F_k$ for pairs (m,k) satisfying $m \geq 0$ and $0 \leq k \leq m+1$.

- **sc^0-map**. A map $\varphi : U \to V$ between relatively open subsets of partial quadrants in sc-Banach spaces is said to be class sc^0 or sc^0 if $\varphi(U_m) \subset V_m$ and the induced maps $\varphi : U_m \to V_m$ are continuous for all $m \geq 0$.

- **tangent bundle TU**. Given a relatively open subset U of the partial quadrant C in the sc-smooth Banach space E, the tangent bundle TU of U is defined as $TU = U^1 \oplus E$. That is, the sc-smooth structure of TU, is given by the nested sequence $(TU)_m := U_{m+1} \oplus E_m$ for all $m \geq 0$. The canonical projection $p : TU \to U^1$ is of class sc^0. The higher order tangent bundles T^kU are defined iteratively as $T^1U = TU$ and $T^kU = T(T^{k-1}U)$ for $k \geq 2$.

- **sc-subspace**. If E is an sc-Banach space, then a closed subspace F of E is called sc-subspace of E if the nested sequence $F_m = F \cap E_m$ is an sc-structure for F. An sc-subspace $F \subset E$ **splits** E if there exists another sc-subspace G of E so that $E_m = F_m \oplus G_m$ for all $m \geq 0$.

- **Fredholm operator**. An sc-operator $T : E \to Y$ is called Fredholm provided that there are sc-splittings $E = K \oplus X$ and $F = Y \oplus C$ having the following properties.

  (1) $K = \text{kernel}(T)$ is finite dimensional.

  (2) C is finite dimensional.

  (3) $Y = T(X)$ and $T : X \to Y$ is an sc-isomorphism.

- **sc$^+$-operator**. An sc-operator $T : E \to F$ is called an sc$^+$-operator if $T(E_m) \subset E_{m+1}$ for every $m \geq 0$ and $T : E \to E^1$ is of class sc^0.

- **sc^1-map** Let E, F be sc-smooth Banach spaces and let $U \subset C \subset E$ be a relatively open subset of a partial quadrant. An sc^0-map $f : U \to F$ is said to be **sc^1** or of **class sc^1** if the following holds.

  (1) For every $x \in U_1$, there exists a bounded linear map $Df(x) \in \mathcal{L}(E_0, F_0)$ satisfying (with $x + h \in U_1$)

  $$\frac{1}{\|h\|_1}\|f(x+h) - f(x) - Df(x)h\|_0 \to 0 \quad \text{as } \|h\|_1 \to 0.$$

  (2) The **tangent map** $Tf : TU \to TF$, defined by

  $$Tf(x,h) = (f(x), Df(x)h)$$

  is an sc^0-map.

- **sc^k-map**. $f : U \subset C \subset E \to F$ is an sc^k-map or of class sc^k if the sc^0-map $T^{k-1}f : T^{k-1}U \to T^{k-1}F$ is of class sc^1. In this case the tangent map $T(T^{k-1}f) : T(T^{k-1}U) \to T(T^{k-1}F)$ is denoted by $T^k f$. If $f : U \subset C \subset E \to F$ is of class sc^k for every $k \geq 0$, then it is called **sc-smooth** or of **class sc$^\infty$**.

- **sc-diffeomorphism**. A homeomorphism $f : U \to V$ between relatively open subsets of partial quadrants in sc-smooth Banach spaces is called an sc-diffeomorphism if f and f^{-1} are sc-smooth.

- **sc-smooth splicing**. Let V be an open subset of a partial quadrant $C \subset W$, let E be an sc-smooth Banach space and let $\pi_v : E \to E$ be a bounded linear projection for every $v \in V$ such that the map

$$\pi : V \oplus E \to E, \quad (v, e) \mapsto \pi_v(e)$$

is sc-smooth. Then the triple $\mathcal{S} = (\pi, E, V)$ is called an sc-smooth splicing.

- **splicing core**. Let $\mathcal{S} = (\pi, E, V)$ be an sc-smooth splicing. The associated splicing core is the subset of $V \oplus E$ defined by

$$K^{\mathcal{S}} = \{(v, e) \in V \oplus E | \, \pi_v(e) = e\}.$$

- **tangent splicing of $\mathcal{S}$**. Given a splicing $\mathcal{S} = (\pi, E, V)$, the tangent splicing of $\mathcal{S}$ is the triple defined by

$$T\mathcal{S} = (T\pi, TE, TV).$$

- The **splicing core of the tangent splicing** $T\mathcal{S}$ is the set

$$K^{T\mathcal{S}} = \{(v, \delta v, e, \delta e) \in TV \oplus TE | \, (T\pi)_{(v,\delta v)}(e, \delta e) = (e, \delta e)\}.$$

- A **local M-polyfold model** consists of a pair $(O, \mathcal{S})$ in which O is an open subset of the splicing core $K^{\mathcal{S}}$ associated with the sc-smooth splicing $\mathcal{S} = (\pi, E, V)$. The **tangent of the local M-polyfold model** $(O, \mathcal{S})$ is the object defined by

$$T(O, \mathcal{S}) = (K^{T\mathcal{S}} | O^1, T\mathcal{S})$$

where $K^{T\mathcal{S}}|O^1$ denotes the collection of all points in $K^{T\mathcal{S}}$ which project under the canonical projection $K^{T\mathcal{S}} \to (K^{\mathcal{S}})^1$ onto the points in O^1.

- **smooth maps between splicing cores**. Given open subsets O, O' of splicing cores $K^{\mathcal{S}} \subset V \oplus E$ and $K^{\mathcal{S}'} \subset V' \oplus E'$, where V and V' are open subsets of partial quadrants in the sc-Banach spaces W and W', define the open set $\widehat{O} \subset V \oplus E$ by $\widehat{O} = \{(v, e) \in V \oplus E | (v, \pi_v(e)) \in O\}$. An sc^0-map $f : O \to O'$ is of class sc^1 provided the map

$$\widehat{f} : \widehat{O} \subset V \oplus E \to W' \oplus E', \quad \widehat{f}(v, e) = f(v, \pi_v(e))$$

is of class sc^1. The tangent map $T\widehat{f}$ associated with the sc^1-map $\widehat{f}$ satisfies $T\widehat{f}(K^{T\mathcal{S}}|O^1) \subset K^{T\mathcal{S}'}|O'$ and induces a map $TO \to TO'$ which is denoted by Tf and called the **tangent map** of f. The tangents TO and TO' are open subsets of the splicing cores $K^{T\mathcal{S}}$ and $K^{T\mathcal{S}'}$, and the notion of f to be of **class sc^k** is defined iteratively.

- **M-polyfold**. Let X be a second countable Hausdorff space. An **M-polyfold chart** for X is a triple $(U, \varphi, \mathcal{S})$ in which U is an open subset of X, $\mathcal{S} = (\pi, E, V)$ is an sc-smooth splicing and $\varphi : U \to K^{\mathcal{S}}$ is a homeomorphism onto an open subset of the splicing core $K^{\mathcal{S}} = \{(v,e) \in V \oplus E |\, \pi_v(e) = 0\}$. Two such charts are compatible if the transition maps between open subsets of splicing cores are sc-smooth. A maximal atlas of sc-smoothly compatible M-polyfold charts is called an **M-polyfold structure** on X and X is equipped with such a structure is called a **M-polyfold of type 0**. By definition a M-polyfold looks locally like an open subset of a splicing core.

- **sc-smooth map between M-polyfolds**. A map $f : X \to X'$ is called of class sc^0, resp. sc^k or sc-smooth if for every point $x \in X$ there exist a chart $(U, \varphi, \mathcal{S})$ around x and a chart $(U', \varphi', \mathcal{S}')$ around $f(x)$ so that $f(U) \subset U'$ and
$$\varphi' \circ f \circ \varphi(U) \to \varphi'(U')$$
is of class sc^0, resp. sc^k or sc-smooth.

- A **general sc-smooth splicing** is a triple $\mathcal{R} = (\rho, F, (O, \mathcal{S}))$ in which $(O, \mathcal{S})$ is a local M-polyfold model associated with the sc-smooth splicing $\mathcal{S} = (\pi, E, V)$ and O is an open subset of the splicing core $K^{\mathcal{S}} = \{(v,e) \in V \oplus E |\, \pi_v(e) = e\}$. The space F is an sc-smooth Banach space and the map
$$\rho : O \oplus F \to F, \quad ((v,e),u) \mapsto \rho(v,e,u)$$
is sc-smooth. Moreover, for every $(v,e) \in O$, the map $\rho_{(v,e)} = \rho(v,e,\cdot) : F \to F$ is a bounded linear projection. A second countable Hausdorff space equipped with a maximal atlas where the local models are open subsets of splicing cores of general splicings are called **M-polyfolds of type 1**.

- The **tangent of a general splicing** $\mathcal{R} = (\rho, F, (O, \mathcal{S}))$ is the triple
$$T\mathcal{R} = (T\rho, TF, (TO, T\mathcal{S})).$$

- A **strong bundle splicing** is a general sc-smooth splicing

$$\mathcal{R} = (\rho, F, (O, \mathcal{S}))$$

having the following **additional property**. If $(v, e) \in O_m$ and $u \in F_{m+1}$, then $\rho((v, e), u) \in F_{m+1}$, and the triple $\mathcal{R}^1 = (\rho, F^1, (O, \mathcal{S}))$ is also a general sc-smooth splicing. The **complementary strong bundle splicing** $\mathcal{R}^c$ is defined by $\mathcal{R}^c = (1 - \rho, F, (O, \mathcal{S}))$.

- **splicing core of the strong bundle splicing**. Given a strong bundle splicing $\mathcal{R} = (\rho, F, (O, \mathcal{S})$, the set

$$K^{\mathcal{R}} = \{(w, u) \in O \oplus F |\, \rho(w, u) = u\}$$

is called the splicing core of the strong bundle splicing $\mathcal{R}$. The splicing core $K^{\mathcal{R}}$ has the bi-filtration

$$K^{\mathcal{R}}_{m,k} = \{(w, u) \in K^{\mathcal{R}} |\, w \in O_m, u \in F_k\}$$

where $m \geq 0$ and $0 \leq k \leq m+1$. Clearly $K^{\mathcal{R}}$ can be viewed as a subset of $O \triangleleft F$ and its bi-filtration is the induced one. The bundle $K^{\mathcal{R}} \to O$ is called a **local strong bundle**. With the strong bundle splicing $\mathcal{R}$ there are associated two splicing cores $K^{\mathcal{R}^0}$ and $K^{\mathcal{R}^1}$, denoted by $K^{\mathcal{R}}(0)$ and $K^{\mathcal{R}}(1)$, and equipped with the filtrations

$$K^{\mathcal{R}}(0)_m = K^{\mathcal{R}}_{m,m} \quad \text{and} \quad K^{\mathcal{R}}(1)_m = K^{\mathcal{R}}_{m,m+1}$$

for $m \geq 0$.

- **$\mathbf{sc}^1_\triangleleft$-maps**. Let $\mathcal{R} = (\rho, F, (O, \mathcal{S})$ and $\mathcal{R}' = (\rho', F', (O', \mathcal{S}')$ be general sc-smooth splicing with associated splicing cores $K^{\mathcal{R}} \subset O \oplus F$ and $K^{\mathcal{R}'} \subset O' \oplus F'$. Let the bundle map $f : K^{\mathcal{R}} \to K^{\mathcal{R}'}$ be of the form

$$f(w, u) = (\varphi(w), \Phi(w, u))$$

where $\varphi : O \to O'$ and $\Phi : O \oplus F \to F'$. Then

  (1) f is of **class $\mathbf{sc}^0_\triangleleft$** if it induces sc^0-maps $K^{\mathcal{R}}(i) \to K^{\mathcal{R}'}(i)$ for $i = 0$ and $i = 1$.

  (2) f is of **class $\mathbf{sc}^1_\triangleleft$** it it is $\mathrm{sc}^0_\triangleleft$ and induces sc^1-maps $K^{\mathcal{R}}(i) \to K^{\mathcal{R}'}(i)$ for $i = 0$ and $i = 1$.

If $f: K^{\mathcal{R}} \to K^{\mathcal{R}'}$ is a map of class $\mathrm{sc}^1_\triangleleft$, then tangent map $Tf: TK^{\mathcal{R}} \to TK^{\mathcal{R}'}$ is of class $\mathrm{sc}^0_\triangleleft$. If the tangent map Tf is of class $\mathrm{sc}^1_\triangleleft$, then f is said to be of **class $\mathbf{sc}^2_\triangleleft$**. The $\mathrm{sc}^k_\triangleleft$-classes are defined inductively. The map $f: K^{\mathcal{R}} \to K^{\mathcal{R}'}$ is of **class $\mathbf{sc}^\infty_\triangleleft$** or $\mathrm{sc}_\triangleleft$-smooth if it is of class $\mathrm{sc}^k_\triangleleft$ for every k.

- **sc-smooth section of a local strong bundle $p: K^{\mathcal{R}} \to O$.** Given a local strong bundle $p: K^{\mathcal{R}} \to O$, a section f of p is called sc-smooth, if f is an sc-smooth section of the bundle $K^{\mathcal{R}}(0) \to O$. The section f is called an **sc$^+$-smooth section**, if it defines an sc-smooth section of the bundle $K^{\mathcal{R}}(1) \to O$.

- **strong M-polyfold bundle.** Let Y be an M-polyfold of type 1, let X an M-polyfold of type 0, and let $p: Y \to X$ be a surjective sc-smooth map. It is assumed that each fiber $p^{-1}(x) = Y_x$ is a Banach space. A **strong M-polyfold bundle chart** for the $p: Y \to X$ is a triple $(U, \Phi, (K^{\mathcal{R}}, \mathcal{R}))$ in which $U \subset X$ is an open set and $\mathcal{R} = (\rho, F, (O, \mathcal{S}))$ a strong bundle splicing with the local model $(O, \mathcal{S})$ of the M-polyfold X. The map Φ is an sc-diffeomorphism $p^{-1}(U) \to K^{\mathcal{R}}$ which is linear on the fibers and which covers the sc-diffeomorphism $\varphi: U \to O$ so that the following diagram commutes,

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow{\Phi} & K^{\mathcal{R}} \\ \big\downarrow p & & \big\downarrow \mathrm{pr}_1 \\ U' & \xrightarrow{\varphi} & O. \end{array}$$

  Moreover, Φ resp. φ are smoothly compatible with the M-polyfold structures on Y and X, respectively.
  Two M-polyfold bundle charts $(U, \Phi, (K^{\mathcal{R}}, \mathcal{R}))$ and $(U', \Psi, (K^{\mathcal{R}'}, \mathcal{R}'))$ with Φ covering the sc-diffeomorphism $\varphi: U \to O$ and Ψ covering the sc-diffeomorphism $\psi: U' \to O'$ are $\mathrm{sc}_\triangleleft$-compatible if the the transition map

$$\Psi \circ \Phi^{-1}: K^{\mathcal{R}}|\varphi(U \cap U') \to K^{\mathcal{R}'}|\psi(U \cap U')$$

  between their splicing cores $K^{\mathcal{R}}$ and $K^{\mathcal{R}'}$ are $\mathrm{sc}_\triangleleft$-smooth.
  An **M-polyfold bundle atlas** consists of a family of M-polyfold bundle charts $(U, \Phi, (K^{\mathcal{R}}, \mathcal{R}))$ so that the underlying open sets cover X

and so that any two charts are $\mathrm{sc}_\triangleleft$-compatible. A maximal atlas of M-polyfold bundle charts is called an M-polyfold bundle structure and the map

$$p: Y \to X$$

is called a **strong M-polyfold bundle**.

- **sc-smooth section**. Given a strong M-polyfold bundle $p: Y \to X$, a section $f: X \to Y$ is called sc-smooth, if its local representations in the strong M-polyfold bundle charts are sc-smooth. It is called an **sc$^+$-smooth section** if its local representations in the strong M-polyfold bundle charts are sc$^+$-smooth sections.

- **linearization of an sc-smooth section**. Given a strong M-polyfold bundle $p: Y \to X$ and an sc-smooth section $f: X \to Y$. If $q \in X$ is a smooth point at which the section f vanishes, the linearization of f at q is defined by

  $$f'(q): T_qX \to Y_q, \quad h \mapsto P_q \circ Tf(q)h$$

  where P_q is the projection $T_qX \oplus Y_q \to Y_q$. If at the smooth point $q \in X$ the section does not vanish, then the linearization of f at q is defined as follows. Take any sc$^+$-section defined near q satisfying $s(q) = f(q)$. Then the section $f - s$ vanishes at the smooth point q and the **linearization of f with respect to s** is defined by

  $$f'_{[s]}(q): T_qX \to Y_q, \quad h \mapsto P_q \circ T(f-s)(q)h.$$

  If s and t are two sc$^+$-sections such that $s(q) = t(q) = f(q)$, then the linearizations $f'_{[s]}(q)$ and $f'_{[t]}(q)$ differ by an sc$^+$-operator. In particular, if one linearization is an sc-Fredholm operator, then the same holds for the other in view of Proposition 2.11. in [1].

- **linearized Fredholm section**. An sc-smooth section f of the strong M-polyfold bundle $p: Y \to X$ is called linearized Fredholm at the smooth point $q \in X$ if the linearization of f at q is an sc-Fredholm operator. The section is called linearized Fredholm, if this holds true at all smooth points q.

References

[1] H. Hofer, K. Wysocki and E. Zehnder, A General Fredholm Theory I: A Splicing-Based Differential Geometry, JEMS, Vol. 4, Issue 4, 2007, 841-876.

[2] H. Hofer, K. Wysocki and E. Zehnder, A General Fredholm Theory II: Implicit Function Theorems, arXiv:0705.1310v1.

[3] H. Hofer, K. Wysocki and E. Zehnder, Integration Theory for Zero Sets of Polyfold Fredholm Sections, arXiv:0711.0781.