Singularity Formation in Hele-Shaw Bubbles

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Abstract
We provide numerical and analytic evidence for the formation of a singularity driven only by surface tension in the mathematical model describing a two-dimensional Hele-Shaw cell with no air injection. Constantin and Pugh have proved that no such singularity is possible if the initial shape is close to a circle; thus we show that their result is not true in general. Our evidence takes the form of direct numerical simulation of the full problem, including a careful assessment of the effects of limited spatial resolution, and comparison of the full problem with the lubrication approximation.

1 Introduction

The Hele-Shaw problem is one of the richest in fluid mechanics. As discussed in numerous review articles [1, 2], beyond its intrinsic interest it has served as a paradigm for pattern formation in many diverse fields such as dendritic solidification and porous media flow.

The experimental geometry can take many forms: an open plane or a confined channel, with injection of air or without; the common defining feature is that a viscous fluid is pushed by pressure gradients in the thin gap between two narrowly spaced plates. In Hele-Shaw's original version, the gap was everywhere filled with fluid, and by injecting colored dye, the cell served as a means to visualize solutions of the Laplace equation. Modern interest is in the dynamics of the interface between two immiscible fluids.

We are interested in the geometry illustrated in Figure 1. The gap is everywhere filled with a fluid of negligible viscosity, such as air, except for a “bubble” of viscous fluid occupying a finite area in the gap. There is no pumping or injection of air or of fluid. The only driving force is the surface tension of the air/fluid interface; this force is resisted by the viscosity of the fluid, since motion of the interface requires pushing the fluid through the narrow gap. The surface tension would like to push the bubble to a circular shape (the only stationary shape), and we pose a simple question: are there smooth simply connected initial bubble shapes for which a singularity forms before the circle is reached? By

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Figure 1: A Hele-Shaw “bubble.” The gap between two narrowly-spaced glass plates is filled with air, except for a bounded region covered by a viscous fluid. Within this region, the fluid moves in response to pressure variations generated by surface tension on the air/fluid interface.

singularity, we mean not only a change of topology but also loss of smoothness of the boundary curve.

The mathematical model system is quite well-known. By Darcy’s law, the gap-averaged fluid velocity $\mathbf{V}(x,t)$ has a potential $\phi(x,t)$, so $\mathbf{V} = \nabla \phi$ where $\phi$ is proportional to negative pressure (relative to the constant air pressure). Incompressibility requires $\phi$ to be harmonic:

$$\Delta \phi = 0 \text{ in } \Omega,$$

where $\Omega(t)$ is the two-dimensional region occupied by fluid. At the fluid-air interface $\Gamma(t) = \partial \Omega$, $\phi$ has boundary values given by the Laplace pressure condition

$$\phi = \mathcal{K} \text{ on } \Gamma,$$

where $\mathcal{K}$ denotes the geometric curvature of the boundary curve, positive where the liquid region is concave outwards (negative for a circle). This boundary condition neglects curvature of the interface in the direction normal to the plates. At each time $t$, $\phi$ is thus determined from the instantaneous geometry by solving an interior Dirichlet problem.

The exterior problem is more commonly studied; in that case $\phi$ is defined and harmonic in the exterior region. Note that our “bubble” is the liquid region, surrounded by air.

Finally, the boundary curve moves with the fluid: its normal velocity $V_n$ is the normal component of the fluid velocity

$$V_n = \frac{\partial \phi}{\partial n}.$$  \hfill (3)

Here, $V_n$ is the outward normal velocity, and $\partial \phi / \partial n$ is the product of $\nabla \phi$ with the outward normal to $\Gamma$. Equations (1–3) constitute a well-defined closed system for the time evolution of the interface shape $\Gamma(t)$, given an initial shape
We ask whether there exist $\Gamma_0$ so that this system forms a mathematical singularity in finite time.

Of course, as a singularity forms, some of the physical assumptions embodied in the model break down; the primary assumption is that the gap spacing is much smaller than any horizontal dimension of the problem. In the singularity, three-dimensional effects will become important.

This mathematical model can easily be extended to describe the motion of a surface bounding a three-dimensional volume. This model is sometimes used to describe flow in porous media; the same model describes diffusion-controlled solidification in the “quasi-static” limit of low undercooling (see [2]).

Our mathematical question could equally well be formulated in terms of the evolution of a two-dimensional bubble of salty water in the interior of a block of ice. Our singularities are the same as those that would occur when two solid seeds grow together in two-dimensional Ostwald ripening [3].

We may quickly identify a few mathematical properties of this system. Clearly, the interface is at rest if and only if $\phi$ is constant in $\Omega$, which requires that the boundary have constant curvature, that is, it must be a circle (or an infinite straight strip). It is also easy to see that the area of $\Omega$ remains constant, while its length is non-increasing. It is thus natural to expect the interface to relax towards the configuration of minimum interface length for the given initial area, which is the circle.

Constantin and Pugh [4] have rigorously proved the following statement: If the initial curve $\Gamma_0$ is sufficiently close to a circle, then it remains smooth for all time, and as $t \to \infty$, the shape approaches a circle. (Chen [5] has proved the same statement for a similar two-sided problem arising from solidification.) The purpose of this paper is to present evidence that their restriction on the initial data is not due merely to technical reasons, but that the statement is false for some initial curves which are not close to a circle, but which are smooth and non-self-intersecting. For such initial data, not only does the curve eventually self-intersect, but its shape forms a genuine singularity, in which the curvature becomes discontinuous as a function of arc-length.

We emphasize that this singularity is driven only by surface tension; in fact, with a suitable choice of inner product, Hele-Shaw flow is gradient flow for the surface energy functional

$$E(\Gamma) = \int_{\Gamma} ds.$$  \hfill (4)

This structure of the system was apparently initially observed by Fife [6]. It means more than simply that $E(\Gamma(t))$ decreases in time, as is well known even for more general systems of Hele-Shaw type [7]; it means that the dynamics is completely specified by the objective function $E$ and the inner product. In particular, Hele-Shaw flow with no injection reduces surface energy as rapidly as possible among all normal velocities with a prescribed maximum value of the norm associated with that inner product [8, 9].

Let us consider functions $u(s)$ and $v(s)$ defined on $\Gamma$, to be thought of as candidate normal velocities. Since interior area is conserved, we require that $u$
and \( v \) have vanishing line integrals. For any such \( u \), let us construct a function \( \psi_u(x) \), defined for \( x \) in the interior \( \Omega \), by solving the Neumann problem

\[
\Delta \psi_u = 0 \quad \text{in} \quad \Omega, \quad \text{with} \quad \frac{\partial \psi_u}{\partial n} = u \quad \text{on} \quad \Gamma,
\]

and similarly for \( \psi_v \). The functions \( \psi_u \) and \( \psi_v \) exist since \( u \) and \( v \) have vanishing integral, and they are unique up to a constant. (\( \partial \psi / \partial n \) denotes the product of \( \nabla \psi \) with the outward normal.) We define an inner product between \( u \) and \( v \) by

\[
\langle u, v \rangle = \int_\Omega \nabla \psi_u \cdot \nabla \psi_v \, dx.
\]

We emphasize that this is an inner product between functions defined on the interface having mean zero, though the integral is taken over the interior. In the notation of Fife [6], this inner product may also be written \( \int_{\partial \Omega} v B^{-1}(u) \, ds \), where \( B \) denotes the Dirichlet-to-Neumann map.

We now show that the Hele-Shaw flow is gradient flow for surface energy in this inner product. This means that the normal velocity \( V \) determined by Equations (1,2,3) may be written as \( V = -\delta \mathcal{E} / \delta \Gamma \), in the sense that

\[
\langle V, w \rangle = -\frac{d}{dt} \mathcal{E}(\Gamma + tw) \bigg|_{t=0}
\]

for every function \( w \) on \( \Gamma \) with mean zero. Here \( \Gamma + tw \) denotes the curve \( \Gamma \) displaced normally by distance \( tw \).

By Equation (3), \( V = \partial \phi / \partial n \), where the potential \( \phi \) is determined by solving the Dirichlet problem (1,2). It is therefore evident that, up to a constant, \( \psi_V = \phi \), and on integrating by parts we immediately see that

\[
\langle V, w \rangle = \int_\Omega \nabla \psi_V \cdot \nabla \psi_w \, dx = \int_\Gamma \psi_V \frac{\partial \psi_w}{\partial n} \, ds = \int_\Gamma \mathcal{K}(s) \, w(s) \, ds.
\]

The constant in \( \psi_V \) disappears since \( w \) has vanishing integral.

The first variation of isotropic surface energy is

\[
\frac{d}{dt} \mathcal{E}(\Gamma + tw) \bigg|_{t=0} = -\int_\Gamma \mathcal{K}(s) \, w(s) \, ds;
\]

the minus sign appears because of our choice of sign of \( \mathcal{K} \) in Section 1; for anisotropic surface energy, this relation would define the appropriate notion of “weighted mean curvature” [10]. Equations (8) and (9) establish (7).

The most familiar example of gradient flow for surface energy is motion by mean curvature, obtained by replacing (6) by the \( L^2 \) product \( \langle u, v \rangle = \int_{\partial \Omega} u \, v \, ds \), without conservation interior volume. For motion by mean curvature of a curve in the plane, any embedded initial curve becomes convex and approaches a circle as it shrinks to a point [11, 12], with no singularity formation. In two dimensions, surface tension normally acts against singularity formation.

In three dimensions, surface tension acting in the circumferential direction can drive a cylindrical interface towards singularity. For example, a fluid column
is subject to the Rayleigh instability, and motion by mean curvature can pinch a neck to zero width in finite time [13]. Our example shows that, acting in conjunction with nonlocal fluid dynamics, and despite the linear stability of a straight interface, surface tension can play a similar role in two dimensions.

The simplest surface tension-driven motion which conserves interior volume is motion by mean curvature minus its average, obtained by using the $L^2$ inner product with the additional prescription that candidate motions $u$ and $v$ have zero integral. For curves in the plane, initially convex shapes remain convex and approach a circle [14]; nonconvex curves certainly can develop self-intersections.

Cahn and Taylor [15] have constructed models for evolution of a crystal shape by surface diffusion, which give a normal velocity depending on Laplacian of surface curvature, and which conserve interior volume (or area) while reducing surface area (or length); these models include motion by curvature minus its average as a limiting case. Elliott and Garcke [16] have shown local existence, and existence of solutions for all time if the initial data is close to a circle.

We shall present our example in the following way.

First, we present the initial data which contains the necessary mechanisms for singularity formation, and show numerical computations of the full problem which indicate singularity formation. These computations establish that the global structure of the flow can act on the local neighborhood of the singularity so as to force together two pieces of interface.

Second, we consider the well-known “lubrication approximation” [17], accurate for thin necks. Goldstein, Pesci, and Shelley [18] have derived this model for two-fluid flows, and demonstrated formation of a singularity driven by gravity. Constantin et al. [19] have studied the single-fluid model without gravity and demonstrated the existence of singularities in infinite time; Dupont et al. [20] demonstrated the existence of singularities in finite time, driven by the boundary conditions. From this work, we know that the local dynamics does not preclude formation of a singularity. We compare solutions of the lubrication model with solutions of the full model, and demonstrate close agreement.

Elsewhere [21], we shall present new results on singularity formation in the lubrication model in periodic geometry, arising from the same mechanisms as in our full geometry, without forcing from the boundary conditions. There, the singularity can be very precisely studied and characterized, and its existence provides strong evidence for existence of the singularity in the full Hele-Shaw problem.

2 Full Hele-Shaw Problem

We take the initial bubble to be in the form of a dumbbell with a thin neck (Figure 2). Near the center of the neck, the interface is very flat; the curvature, and its first two derivatives with respect to arclength, vanish at the midpoint. The maximum of curvature is at the “shoulders” of the neck, where the shape bulges into the lobes of the dumbbell.

Near the middle of the neck, the velocity potential $\phi$ is close to zero, because
Figure 2: Dumbbell initial shape. The curvature is zero at the midpoint of the thin neck, and the concavities at the end of the neck may suck enough fluid from the neck to make it break.

the curvature is very small. Where the neck joins the large lobes, the curvature and hence $\phi$ are positive. Thus the variation of curvature generates a flow outwards from the center of the neck. The neck will certainly become thinner than its initial width, and if it is initially thin enough its width may reach zero in finite time. Simple estimates based on the lubrication approximation [22, 20] show that if the neck width goes to zero, derivatives of the curvature must become infinite; thus this is a true singularity and not simply loss of embedding of the boundary curve.

The real question is whether or not purely local mechanisms will intervene as the neck width approaches zero to prevent singularity formation; the rest of this paper is devoted to assessing this possibility. Our first tool is direct numerical simulation of the full two-dimensional Hele-Shaw problem.

2.1 Numerical Method

Numerical methods for the Hele-Shaw problem are quite highly developed. The key ingredients are the following:

1. representation of the harmonic field in terms of a boundary integral over a vortex sheet as in [23, 24, 25]. We use fast multipole summation [26], with small modifications to take advantage of bilateral symmetry, to evaluate the sums in time $O(N)$.

2. a fixed arc-length discretization of the bounding curve, permitting local concentration of node points near the suspected singularity. The arc length $s$ is related to the parameterization variable $p$ by

$$s_p(p, t) = f(p) \frac{L(t)}{2\pi},$$

where $L(t)$ is the total length of the interface. We choose $f$ explicitly before the computation begins, and keep it fixed throughout the time evolution. With $f(p) \equiv 1$, this is the equal arclength parameterization of
Brower et al. [27]; this extension to fixed but non-constant arclength was proposed by Hou et al. [28].

3. stiffness reduction by a semi-implicit discretization of the evolution equation [28], based on an approximate linearization of the dynamics valid for large wave numbers. The resulting linear systems are solved with GMRES [29]; this becomes more difficult as \( f(p) \) is further from 1, just as for anisotropic surface tension [30].

We use a constant time step, fixed for each run. Time discretization is not the main source of error, as we have determined by repeating runs with different time steps.

2.2 Numerical Results

Our first task is to represent the dumbbell shape described above in terms of the normal angle \( \theta(p) \). We choose a simple smooth function \( \theta(p) = p + a_1 \sin 2p + a_2 \sin 4p + a_3 \sin 6p \), so that the difference from the linear form is odd in \( p \) and of period \( \pi \) (a circle corresponds to \( \theta(p) = p \) and \( f \equiv 1 \)). This corresponds to reflection symmetry of the dumbbell about the \( x \) and \( y \) axes; \( p = 0 \) corresponds to the right end of the dumbbell, \( p = \pi/2 \) to the upper midpoint of the neck, and so on.

Vanishing of the curvature \( K \) at the midpoint \( p = \pi/2 \) requires \( \theta'(\pi/2) = 0 \). The first \( s \)-derivative \( K' \propto \theta' \) vanishes by symmetry. We may further require that the second \( s \)-derivative of curvature vanish, that is, \( \theta''(\pi/2) = 0 \). We may then eliminate \( a_1 \) and \( a_2 \) and write the complete form as

\[
\theta(p) = p + \left( \frac{5}{3} + 5a_3 \right) \sin 2p + \left( \frac{17}{15} + 4a_3 \right) \sin 4p + a_3 \sin 6p, \tag{11}
\]

in which there remains one free parameter \( a_3 \). We have thus defined a one-parameter family of initial data, in which we may pick the initial width of the neck by choosing \( a_3 \); all members of this family are very flat in the center. (The circle is not a member of this family.) We choose the initial interface length \( L = 2\pi \).

The only quantity which affects the nature of the solution is the shape of the initial bubble: scaling its size, or introducing a surface tension parameter different from one, only changes the time scale. This family of initial shapes is therefore likely to be adequate to explore the whole range of solution behaviors.

We express the local refinement function \( f(p) \) in terms of a mapping \( p_0(p) \), which gives us the parameter value in the equal arc-length representation in terms of the parameter value in the actual parameterization. We write this function as \( p_0(p) = p + a \sin 2p + b \sin 4p \), and determine \( a \) and \( b \) so that \( p_0(p) \) takes on its minimum value of \( 1/K \) when \( p_0 = p_0^* \). Here, the first parameter \( K \geq 1 \) is the “concentration factor,” the spacing of node points in the neighborhood of the singularity relative to their average spacing around the interface. The second parameter \( p_0^* \) represents our best guess for the location of the singularity.
Figure 3: Computations of full problem, with two slightly different initial widths, \( h_{m,n}^0 = 0.02 \) and \( h_{m,n}^0 = 0.0156 \) respectively. With width just above the critical value, it is easy to compute a smooth solution which tends toward a circle. With width just below the critical value, no amount of refinement permits us to compute beyond the apparent singularity.

...in the equal arclength frame, based on earlier computations. The coefficients \( a \) and \( b \) are determined numerically for given \( K, p_0 \).

Two sample computations are shown in Figure 3, where we take two initial dumbbells with slightly different values of the neck width, one just above and one just below a critical value. In the first computation, the neck becomes thinner due to the mechanisms outlined at the beginning of this section. Its width reaches a finite minimum, before spreading out towards a convex shape, which will eventually relax toward a circle. With increased resolution, the shape is better and better resolved; the computation clearly indicates that the solution trajectory is well-defined and smooth throughout its lifetime.

The second computation in Figure 3 has initial neck width just thinner than the critical value. The neck width becomes thinner and thinner, and quickly becomes too small for our numerics to resolve. Regardless of how much we refine the discretization, we are unable to compute a trajectory which recovers and widens out again. This behavior alone is not sufficient to claim existence of a singularity, but it is strongly suggestive.

To study the shape in more detail, we consider the neck half-width \( h(x, t) \), viewed as a function of position \( x \) relative to the center of symmetry. In particular, we display \( h_{m,n}(t) \), the minimum over \( x \) of \( h(x, t) \) in the neck region (clearly, \( h = 0 \) at the ends of the lobes), and \( x_{m,n}(t) \), the \( x \)-value where \( h \) takes its minimum. These two quantities are shown in Figures 4 and 5 as functions...
Figure 4: Minimum neck half-thickness $h_{m, \text{in}}$ as a function of time $t$, for initial thickness $h_{m, \text{in}}^0 = 0.0106, 0.0156, 0.02, \ldots, 0.07$. The lower two trajectories, drawn with dashed lines, reach $h = 0$ in finite time.

Figure 5: Position of minimum neck thickness $x_{m, \text{in}}(t)$, for the same cases as in Figure 4. In all cases, the single initial minimum at $x = 0$ splits into two asymmetric minima located at $\pm x_{m, \text{in}}(t)$. For $h_{m, \text{in}}^0 = 0.0156$ the minimum remains at the origin for as long as our resolution permits us to follow it. For $h_{m, \text{in}}^0 = 0.0106$, the minimum splits again into two asymmetric minima.
of time, for various values of the initial neck width, obtained by varying the parameter $\alpha_3$ above.

The two dashed lines indicate trajectories which, to the limit of our resolution capability, appear to become singular in finite time.

At $t = 0$, the minimum is always in the center $x = 0$. At positive times, the initial minimum splits into two equal minima located at $\pm x_{m,n}(t)$; as the thickness decreases, the two minima move back toward the origin, and eventually coalesce as the thickness continues to decrease. For initial thickness above the critical value, the decrease reverses itself, remaining at the origin.

For the smaller of the two neck widths we have shown, the single minimum eventually splits apart as it continues to decrease, rather than remaining a symmetric minimum at the origin. Because symmetric singularities are almost never observed in computations of the lubrication model corresponding to Hele-Shaw flow [22], we believe that the other case, with $h_{m,n}^0 = 0.0156$, would eventually split if we were able to follow it to small enough $h_{m,n}$.

2.3 Resolution Assessment

As mentioned above, the computational discretization is breaking down as the singularity is approached, since one of the assumptions in the construction of the trapezoidal rule is that the point spacing is much smaller than all other scales in the problem—this assumption is violated in the singularity. We now address this problem.

We focus on the second of the two singular cases shown in Figures 4 and 5, that with $h_{m,n}^0 = 0.0106$ (corresponding to $\alpha_3 = 0.097$). We run the same computation many times, taking most, though not all, combinations of total points $N = 1024, 2048, 4096, 8192$, and concentration factor $K = 1, 2, 5, 10$. We are not able to take larger concentration factors because the linear system becomes too ill-behaved to be solved efficiently.

The local point spacing in the neighborhood of the singularity, $\delta s = s_p h$ (recall that $h$ is the discretization interval in $p$), is inversely proportional to the product $KN$. Its value may be simply estimated as the total length $L$, divided by the number of points $N$ times the concentration factor $K$. The length $L = 2\pi$ at $t = 0$, and is monotonically decreasing; at the time of the singularity, $L \approx 4.5$.

Our intuitive belief is that the solution behavior should be controlled by this quantity, and that the computation should fail, i.e., begin to differ substantially from the true solution, when $h_{m,n}$ reaches some value proportional to $\delta s$. This belief is substantiated by Figures 6 and 7.

Figure 6 shows solution trajectories $h_{m,n}(t)$ for a range of values of $\delta s$. As $\delta s$ is increased, the trajectories move monotonically to the right, collapsing onto a curve which we may believe is the true solution of the partial differential equation. This true solution appears to tend to $h_{m,n} = 0$ at some critical time $t^*$, though this plot does not allow us to make any conjectures about the asymptotic nature of the approach. The solutions for small values of $\delta s$ are each represented by two nearly superposed curves, supporting our belief that the local point spacing is more important than the total number of points.
Figure 6: The minimum neck thickness $h_{\text{min}}(t)$ for different resolutions, arranged by the grid point spacing $\delta s$ in the neighborhood of the singularity. As $\delta s$ is reduced, the curves converge toward the trajectory which is their envelope towards the right.

Figure 7: Numerically differentiated $dh_{\text{min}}/dt$ vs. $h_{\text{min}}$. Each collection of trajectories becomes unreliable when $h_{\text{min}}$ is roughly $\delta s/5$. 
In Figure 7 we have plotted $h_{\text{min}}(t)$ on the horizontal axis, and $-h'_{\text{min}}(t)$ obtained by numerical differencing on the vertical axis. This lets us look for self-similar scaling behavior without the need to estimate the critical time $t^*$. Here we see clearly that the series of computations approach what appears to be a straight line in this log-log plot; each computation diverges from the limit behavior at a clear value of $-h_{\text{min}}(t)$. It appears that each computation agrees with the true solution as long as $h_{\text{min}} \geq 0.2\delta s$, which is certainly further that we would have been inclined to believe the computation based on a priori reasoning; the spacing between consecutive node points is 2.5 times the distance separating the two pieces of the interface. We conjecture that the flatness of the interface in the neighborhood of the singularity may explain this rather surprising behavior; as discussed below, the singularity involves blowup only of third spatial derivatives.

If we construct a straight-line approximation to the limit solution in Figure 7, we may measure a slope corresponding to $-h'_{\text{min}}(t) \propto h_{\text{min}}(t)^{0.28}$, which corresponds to the fractional-power behavior $h_{\text{min}}(t) \propto (t^* - t)^{1.4}$. Dupont et al. [20] observed behavior $h_{\text{min}}(t) \propto (t^* - t)^{2.42}$, with logarithmic corrections, in the final asymptotic stages of the singularity in their lubrication model. However, the solution shown in Figure 7 corresponds to the very early stages of the singularity of [20] and is roughly consistent with the behavior shown there for $h_{\text{min}} \geq 10^{-5}$. As we explain at the end of Section 3, we believe that the limiting trajectory seen in Figure 7 is still far from the final asymptotic solution behavior.

Finally, we examine the spatial structure of the solution in the neighborhood of the singularity. Figure 8 shows graphs of $h(x,t)$ and its second spatial derivative $h_{xx}(x,t)$ for the same case as above, $\alpha_3 = 0.097$, $h_{\text{min}}^0 = 0.0106$, for the finest spatial resolution we have computed. From Figure 7, this computation should be reliable down to the neighborhood of $h_{\text{min}} \sim 10^{-5}$. From the profile of $h_{xx}$, we see that the solution is developing the step discontinuity in curvature which is typical of singularities observed in lubrication computations [20].

Using this boundary-integral method for the outer flow, we are unable to resolve the solution well enough to attain the final asymptotic regime. Nonetheless, the behavior is strongly suggestive of singularity formation in finite time.

3 The Lubrication Approximation

In the neighborhood of the singularity, the neck is uniformly thin and flat, that is, $h(x,t) \ll L$ and $|h_{x}| \ll 1$, where $L$ is a typical length scale in the $x$-direction. The lubrication model is a multiple-scales asymptotic expansion valid in this regime, leading to a great simplification in the governing equations: $h(x,t)$ is governed by a local partial differential equation (PDE) rather than the nonlocal integral equation of Section 2.

The PDE can be solved to much higher accuracy than the full nonlocal model. Thus we are able to obtain very highly detailed numerical simulations, which we shall compare to the full simulations of Section 2. This will both
establish the validity of the lubrication model, and support our case for finite-time singularity formation in the full system.

3.1 Thin-Neck Asymptotics

The lubrication model is usually constructed using rather heuristic arguments (see, e.g., [19]); we are aware of only one systematic derivation [18], based on a boundary-integral formulation. We shall present a different derivation based on a standard multiple-scales expansion. Formally, this expansion is based on assumptions about derivatives of the solution which cease to be valid in the singularity, and this is one of its main limitations. We shall subsequently explain why we believe the leading-order equation nonetheless remains uniformly valid in the singularity.

We make the following Ansatz for the neck width in terms of a small para-
meter $\epsilon$:

$$h(x,t) = \epsilon H_1(x,\epsilon t) + \epsilon^2 H_2(x,\epsilon t) + \epsilon^3 H_3(x,\epsilon t) + \ldots$$  \hspace{1cm} (12)

By writing this expression, we assume, first, that the typical scale length $L$ in the $x$-direction is of order one, and, second, that all spatial derivatives of $h$ are of the same asymptotic order as $h$ itself. We are to determine governing equations for $H_1(x,\tau)$, $H_2(x,\tau)$, etc., so that the complete expansion solves the original problem at each successive order in $\epsilon$.

However, as the singularity forms, the spatial derivatives of $h$ become infinite, and the expansion formally breaks down. The reasoning is somewhat analogous to construction of a numerical method: we assume that everything is well-behaved to construct an approximate system, drive the approximation until it crashes, and then post facto assess what we may learn from the wreckage. For reasons which we present below, we believe that the lubrication model is more believable than this pessimistic description might suggest.

Continuing the multiple-scales construction, we expand the potential field as

$$\phi(x,y,t) = \epsilon \Phi_1(x, \frac{y}{\epsilon}, \epsilon t) + \epsilon^2 \Phi_2(x, \frac{y}{\epsilon}, \epsilon t) + \epsilon^3 \Phi_3(x, \frac{y}{\epsilon}, \epsilon t) + \ldots$$  \hspace{1cm} (13)

where we must determine consistent forms for $\Phi_1(x,\eta,\tau), \Phi_2(x,\eta,\tau), \ldots$ by matching terms in the expansion of the system (1,2,3) in powers of $\epsilon$.

The curvature is readily expanded as

$$K(x,t) = \left(1 + h_x^2\right)^{-3/2} h_{xx}$$

$$= \epsilon H_{1,xx} + \epsilon^2 H_{2,xx} + \epsilon^3 \left\{ H_{3,xx} - \frac{3}{2} H_{1,xx} H_{1,x}^2 \right\} + \ldots$$  \hspace{1cm} (14)

and the Laplacian is

$$\Delta \phi = \frac{1}{\epsilon} \Phi_{1,\eta\eta} + \Phi_{2,\eta\eta} + \epsilon \left\{ \Phi_{3,\eta\eta} + 2 \Phi_{1,xx} \right\} + \ldots$$  \hspace{1cm} (15)

Assuming that $\Phi_1, \Phi_2, \ldots$ are regular in $\eta$ near $\eta = H_1$, we may expand the boundary value as

$$\phi(x,h) = \epsilon \Phi_1 + \epsilon^2 \left\{ \Phi_2 + H_2 \Phi_1,\eta \right\}$$

$$+ \epsilon^3 \left\{ \Phi_3 + H_3 \Phi_2 + H_3 \Phi_1,\eta + \frac{1}{2} H_2 \Phi_1,\eta\eta \right\} + \ldots$$  \hspace{1cm} (16)

in which $\Phi_1, \Phi_2, \Phi_1,\eta, \ldots$ are evaluated at $(x,H_1(x))$ and we have temporarily suppressed the time dependence.

The Dirichlet problem (1,2) thus reduces to the series of ordinary differential
equations in $|\eta| < H_1(x,t)$ with boundary conditions at $\eta = H_1$

$$\Phi_{1,\eta} = 0, \quad \Phi_1(x,H_1) = H_{1,xx}; \quad (17)$$

$$\Phi_{2,\eta} = 0, \quad \Phi_2(x,H_1) = -H_2\Phi_{1,\eta} + H_{2,xx}; \quad (18)$$

$$\Phi_{3,\eta} = -\Phi_{1,xx}, \quad \Phi_3(x,H_1) = -H_3\Phi_{2,\eta} - H_3\Phi_{1,\eta} - \frac{1}{2}H_2^2\Phi_{1,\eta\eta}$$

$$+ H_{3,xx} - \frac{3}{2}H_{1,xx}H_{1,xx}^2; \quad (19)$$

\[ \vdots \]

which we may solve in succession for $\Phi_1(x,\eta)$, $\Phi_2(x,\eta)$, \ldots in terms of $H_1(x)$, $H_2(x)$, \ldots. Of course the boundary data are symmetric at $\eta = -H_1$, or we may more simply impose $\Phi_{j,\eta} = 0$ at $\eta = 0$. The solutions are

$$\Phi_1(x,\eta) = H_{1,xx} \quad (20)$$

$$\Phi_2(x,\eta) = H_{2,xx} \quad (21)$$

$$\Phi_3(x,\eta) = H_{3,xx} - \frac{3}{2}H_{1,xx}H_{1,xx}^2 + \frac{1}{2}H_{1,xxx}(H_{1,xx}^2 - \eta^2) \quad (22)$$

\[ \vdots \]

Thus $\Phi_1$ and $\Phi_2$ are functions of $x$ only, independent of $\eta$, and $\Phi_3$ has a simple parabolic profile in $\eta$.

It only remains to determine the time dependence of $H_1, H_2, \ldots$. This comes from the material consistency condition (3), which may be written

$$h_t + uh_x = v \quad (23)$$

in which the fluid velocity components on the interface are (replacing $\Phi_j$ by $\Phi_{j,x}$ and $\epsilon^{-1}\Phi_{j,\eta}$ respectively in (16))

$$u = \phi(x,h) = \epsilon\Phi_{1,x} + \epsilon^2\Phi_{2,x} + \epsilon^3\Phi_{3,x} + \ldots \quad (24)$$

$$v = \phi_y(x,h) = \epsilon^2\Phi_{3,\eta} + \ldots, \quad (25)$$

since the $\eta$-derivatives of $\Phi_1$ and $\Phi_2$ vanish. Thus, the time evolution equation at $O(\epsilon^2)$ requires

$$H_{1,\tau} + H_{1,x}\Phi_{1,x} = \Phi_{3,\eta}, \quad (26)$$

or

$$H_{1,\tau} + (H_1H_{1,xxx})_x = 0. \quad (27)$$

That is, $h(x,t)$ approximately solves

$$h_t + (hh_{xxx})_x = 0, \quad (28)$$

in the sense that both terms appearing are of size $O(h^2)$, and we have neglected terms of size $O(h^3)$. Equation (28) is the well-known “lubrication equation” [17].

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We may carry this procedure to higher order in $\epsilon$. At the next order, we find
\[ H_{2,\gamma} + (H_1 H_{2,xxx} + H_2 H_{1,xxx}) = 0, \]
a homogeneous linear equation in $H_2$. This equation is just a linearization of the lubrication equation, and contains no new information. We may consistently take $H_2 \equiv 0$, and thus the lubrication equation is formally valid to one additional order in $\epsilon$; this is a consequence of reflection symmetry.

At next order (with substantially more work), we obtain
\[ H_{3,\gamma} + (H_1 H_{3,xxx} + H_3 H_{1,xxx}) + (H_2 H_{2,xxx}) + \mathcal{R}[H_1] = 0, \]
in which $\mathcal{R}[H_1]$ is a nonlinear functional of $H_1$ including derivatives up to the sixth order. It is clear that each successive order in the expansion will involve higher and higher spatial derivatives, and this is not likely to be a productive route to assess the validity of the lubrication model in the singularity. Our belief that the lubrication model should be more reliable than indicated by the above remarks rests instead on the following reasoning.

The lubrication approximation rests on two assumptions. The first is that the slope $h_x$ is small, permitting expansion of the curvature $\mathcal{K}$. The second is that the dependence on $y$ of $\phi$ may be neglected. This depends on the horizontal length scale over which $h_{xx}$ is changing, or $h_{xx}/h_{xxx}$, compared with the height $h$; thus we require $h h_{xxx} \ll h_{xx}$ even in the singularity. This appears to be the case for the singularities explored in [20].

### 3.2 Comparison with full problem

Since the lubrication expansion is an expansion in $h$ and $h_x$, we believe that it should be valid in the thin neck region, and that it should continue to be valid as the singularity forms. We now test this belief by comparing the solutions of the full problem with solutions of the lubrication approximation.

Since the lubrication equation (28) is fourth-order in $x$, it requires two boundary conditions at each end of a finite interval. These may be, for example, “pressure” boundary conditions $h$ and $h_{xx}$ [22].

We pick a specific value of $x_0 > 0$, so that in our full computation, the region $|x| < x_0$ is within the thin-neck region. We extract from the full computation the values of $h(x_0, t)$ and $h_{xx}(x_0, t)$. Then we solve the lubrication equation on the domain $|x| < x_0$ with the same initial data, using the extracted values as boundary conditions. Since the full computation breaks down at a finite time, we extrapolate the data slightly in time until the lubrication solution also develops a singularity.

The information flow is only one-way: the lubrication solution feeds nothing back to the full problem. It might be natural to consider coupling the two solution methods in some way: this would be difficult since we are using a spectral method in the outer region.

We study the case $h_{u,n}^h = 0.0106$ and $a_3 = 0.097$, the second dashed line in Figures 4 and 5. Boundary data from the full solution is extracted as functions of
Figure 9: Boundary data at $x_0 = 0.0625$ as functions of time. We show $H(t) = h(x_0, t)$, $H_2(t) = h_{xx}(x_0, t)$, and $H_0(t)$, the height at $x = 0$ of the parabola which matches the boundary data. The curves stop when the full simulation breaks down due to singularity formation.

time for $x_0 = 0.0625$; this value was determined empirically so that $|h_x| < 0.01$ for $|x| < x_0$. Spectral interpolation in space is used to obtain $h$ and $h_{xx}$ at $x_0$ at the time steps of the outer solution, and cubic spline interpolation in time is used to provide boundary data to the lubrication code at arbitrary intermediate times.

The resulting boundary data as functions of time is shown in Figure 9. In addition to $H(t) = h(x_0, t)$ and $H_2(t) = h_{xx}(x_0, t)$, we have plotted $H_0(t) = H - \frac{1}{2}H_{xx}x_0^2$, the height at $x = 0$ of the parabola which matches the given boundary values of $h$ and $h_{xx}$. When $H_0 < 0$, no stationary equilibrium solution exists for the given boundary values; previous simulations of the lubrication model [19] have suggested that this negativity favors singularity formation. In our case, $H_0(t)$ clearly becomes negative and remains so; thus the outer flow is pushing the local behavior in the direction of singularity.

The lubrication code uses the advanced techniques described in [22, 20, 19], including fully implicit time discretization to overcome stiff time step restrictions associated with the fourth-order spatial derivatives, and repeated grid refinement around the singularity. Because the lubrication model is a local PDE, very accurate solutions are possible. We thank both Michael Brenner and Andrea Bertozzi for running several different versions of their codes with our boundary data.

In Figure 10 we compare results of this computation with the full compu-
Figure 10: Comparison of full solution with lubrication solution. The dashed lines are the same data as in Figure 7; the solid line is the result of a highly refined lubrication computation.

As resolution is increased for the full simulation, we converge nearly to the lubrication solution with the same boundary data. Of course, the convergence cannot be exact, since the lubrication model is slightly different from the full one, but the difference is too small to see on this graph. The period shown in Figure 10 is after the separation of the minimum into two minima (see Figure 5). The reproduction of such qualitative features both in the full system and in the lubrication model strengthens our belief that the lubrication model accurately captures the full behavior.

The reader may notice that the slope of the curve in Figure 10 continues to change after the range where we are able to compare values, that is, below $h_{\text{min}} \sim 10^{-6}$. Certainly, we cannot claim to have exhibited a fractional-power behavior $h_{\text{min}} \sim (t^* - t)^\alpha$. In fact, the value of $-dh_{\text{min}}/dt$ is decreasing considerably, and one may ask whether the trajectory $h_{\text{min}}(t)$ may not level out and perhaps even increase, if we were able to pursue the computations to higher resolution.

In fact, in Figure 11 we plot $h_{\text{min}}(t)$ for the lubrication solution near its endpoint, and it is evident that the solution is on the verge of exhibiting a substantial change in behavior just at the time when the outer solution breaks down, and resolution limitations do not permit us to continue any further.

We believe that this is not a numerical artifact, and in our companion paper [21], we consider in great detail the nature of the singularity formed in the lubrication model with periodic boundary conditions, finding quite surprising behavior.

Our conclusion here is that the lubrication model, with boundary data chosen to match the full simulation, does develop a singularity in finite time, although
from the data shown here, we are not able to conclusively establish the nature of the singularity. Since the full simulation matches the lubrication solution over the range where we are able to compare, we believe that the full solution will develop a singularity of the same type as the lubrication model. We explore this topic more fully in [21], in which we discover a surprisingly rich range of singularity structures.

4 Conclusion

We have given numerical evidence that a singularity forms in the Hele-Shaw model in two dimensions, driven only by surface tension, for the initial data we have given above. Our argument has two main components.

First, the global geometry and dynamics can be arranged so as to push the center of the thin neck in the direction of breaking. By fairly simple reasoning, we have understood the nature of the mechanism. We have shown that the minimum of the equilibrium parabola matching the outer solution becomes and stays negative, suggesting singularity formation.

Second, we have argued that purely local mechanisms do not come into play when the neck becomes very thin, so as to prevent an actual singularity. The absence of such mechanisms is well documented within the lubrication approximation; we have argued, and shown numerically within the limits of our resolution, that the lubrication model is a good one near the singularity.

Since the outer flow pushes the inner flow towards a singularity, and since the inner flow possesses no mechanism which prevents singularity, we conclude that the singularity forms in finite time when the neck is thin enough.
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References


