PROBABILITY II, SPRING 2022.
HOMEWORK PROBLEMS

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Instructions. Please submit your solutions via Brightspace. Late homework will not be accepted.

You must type/write on the first page of the assignment paper:
I have neither given nor received unauthorized help when working on this assignment.

You are allowed to use books or existing electronic/online materials to solve the problems. You are allowed to work on solutions in groups, but you are required to write up solutions on your own. You are not allowed to seek any other external help for specific problems. You are also not allowed to post your solutions online.

You must give complete solutions, all claims need to be justified.

Please let me know if you find any misprints or mistakes.

1. Due by Friday Feb 25, 2:00pm

1. Let $(X_n)_{n\in\mathbb{N}}$ be an i.i.d. positive sequence on some probability space $(\Omega,\mathcal{F},\mathsf{P})$, and $S_n = X_1+\ldots+X_n$. Let $N_t=\sup\{n : S_n\le t\}$. Prove that $S_t=S_{[t]}$ and $(N_t)_{t\in\mathbb{R}_+}$ are stochastic processes.
2. Let $v_1, v_2 : \mathbb{R}^2 \to \mathbb{R}^2$ be C^∞ bounded vector fields on $\mathbb{R}^2$. Let N_t be defined as in the previous problem. Let $i(n) = 1$ if n is odd and $i(n) = 2$ if n is even.

   Let Z_t denote the solution of the ODE
   $$\frac{d}{dt}Z_t = v_{i(N_t)}(Z_t), \quad Z_0 = x.$$
   Prove that $(Z_t)_{t\ge 0}$ is a stochastic process.
3. Prove that cylinders $C(t_1,\ldots,t_n,B)$, $B\in\mathcal{B}(\mathbb{R}^n)$, $t_1,\ldots,t_n\ge 0$, $n\in\mathbb{N}$ form an algebra.
4. Prove that elementary cylinders $C(t_1,\ldots,t_n,B_1\times\ldots\times B_n)$, $B_1,\ldots,B_n\in\mathcal{B}(\mathbb{R})$, $t_1,\ldots,t_n\ge 0$, $n\in\mathbb{N}$ form a semi-ring $\mathcal{C}$ (i.e., (i) $\emptyset\in\mathcal{C}$, (ii) if $A,B\in\mathcal{C}$, then $A\cap B\in\mathcal{C}$, (iii) if $A,B\in\mathcal{C}$, then there is n and disjoint $C_1,\ldots,C_n\in\mathcal{C}$ such that $A\setminus B = C_1\cup\cdots\cup C_n$). [*A reminder that does not help to solve the problem but just fyi: when extending measures onto the entire σ-algebra one can start with an algebra or just with a semi-ring.*]
5. We can define the cylindrical σ-algebra as the σ-algebra generated by elementary cylinders or by cylinders. Prove that these definitions are equivalent.

6. Let $\mathcal{F}_T = \sigma\{C(t_1,\ldots,t_n,B):\ t_1,\ldots,t_n \in T\}$ for $T \subset \mathbb{T}$.
   Prove that
$$\mathcal{B}(\mathbb{R}^{\mathbb{T}}) = \bigcup_{\text{countable } T\subset\mathbb{T}} \mathcal{F}_T.$$
7. Prove that $X : \mathbb{T}\times\Omega \to \mathbb{R}$ is a stochastic process iff X seen as $X : \Omega \to \mathbb{R}^{\mathbb{T}}$ is $(\Omega,\mathcal{F}) \to (\mathbb{R}^{\mathbb{T}},\mathcal{B}(\mathbb{R}^{\mathbb{T}}))$ measurable.
8. Use characteristic functions to prove the existence of a Wiener process (up to continuity of paths).
9. Let $(X_t)_{t\in[0,1]}$ be an (uncountable) family of i.i.d. r.v.'s with nondegenerate distribution. Prove that no modification of this process can be continuous.
10. A multidimensional version of the Kolmogorov–Chentsov theorem. Suppose $d \ge 1$, and there is a stochastic field $X : [0,1]^d \times\Omega \to \mathbb{R}$ that satisfies $\mathsf{E}|X(s) - X(t)|^\alpha \le C|s-t|^{d+\beta}$ for some $\alpha, \beta, C > 0$ and all $t, s \in [0,1]^d$. Prove that there is a continuous modification of X on $[0,1]^d$.
11. Show that the Kolmogorov–Chentsov theorem cannot be relaxed: inequality $\mathsf{E}|X_t - X_s|^\alpha \le C|t-s|$ holding for some $\alpha > 0$ and all t, s is not sufficient for existence of a continuous modification. Hint: consider the following process: let τ be a r.v. with exponential distribution, and define $X_t = \mathbb{1}_{\{\tau\le t\}}$.
12. Prove that there exists a Poisson process (a process with independent increments that have Poisson distribution with parameter proportional to the length of time increments) such that:
    (a) its realizations are nondecreasing, taking only whole values a.s.
    (b) its realizations are continuous on the right a.s.
    (c) all the jumps of the realizations are equal to 1 a.s.
13. Give an example of a non-Gaussian 2-dimensional random vector with Gaussian marginal distributions.
14. Let $Y \sim \mathcal{N}(a, C)$ be a d-dimensional random vector. Let $Z = AY$ where A is an $n \times d$ matrix. Prove that Z is Gaussian and find its mean and covariance matrix.
15. Prove that an $\mathbb{R}^d$-valued random vector X is Gaussian iff for every vector $b \in \mathbb{R}^d$, the r.v. $\langle b, X\rangle$ is Gaussian.
16. Prove that $(s,t) \mapsto t\wedge s$ defined for $s, t \ge 0$ is positive semi-definite. Hint:
$$\langle \mathbb{1}_{[0,t]}, \mathbb{1}_{[0,s]}\rangle_{L^2(\mathbb{R}_+)} = t \wedge s.$$
17. Prove that $c : (s,t) \mapsto e^{-|t-s|}$ is positive semi-definite on $\mathbb{R}$, finding an auxiliary Hilbert space H and a family $h_t \in H$ such that $\langle h_t, h_s\rangle = c(s,t)$ for all t, s.
18. Prove that if X is a Gaussian vector in $\mathbb{R}^d$ with parameters (a, C) and C is non-degenerate, then the distribution of X is absolutely continuous w.r.t. Lebesgue measure and the density is
$$p_X(x) = \frac{1}{\det(C)^{1/2}(2\pi)^{d/2}} e^{-\frac{1}{2}\langle C^{-1}(x-a),(x-a)\rangle}.$$

19. Use the Chentsov–Kolmogorov theorem to find some condition on the mean $a(t)$ and covariance function $c(s,t)$ that guarantees existence of a continuous Gaussian process with these parameters. This asks for some nontrivial reasonable conditions (do not just say let a and c be identical zero or constant) but it does not have to be very general.
20. Suppose $(X_0, X_1, \ldots, X_n)$ is a (not necessarily centered) Gaussian vector. Show that there are constants $c_0, c_1, \ldots, c_n$ such that

$$\mathsf{E}(X_0|X_1, \ldots, X_n) = c_0 + c_1 X_1 + \ldots + c_n X_n.$$

Your proof should be valid irrespective of whether the covariance matrix of $(X_1, \ldots, X_n)$ is degenerate or not.

21. Consider the standard Ornstein–Uhlenbeck process X (Gaussian process with mean 0 and covariance function $c(s,t) = e^{-|t-s|}$).
    (a) Prove that X has a continuous modification.
    (b) Find $\mathsf{E}(X_4|\ X_1, X_2, X_3)$.
22. Prove that for every centered Gaussian process X with independent increments on $\mathbb{R}_+ = [0, \infty)$ ($X(t) - X(s)$ is required to be independent of $\sigma(X(r), r \in [0,s])$ for $t \geq s \geq 0$), there is a nondecreasing nonrandom function $f : \mathbb{R}_+ \to \mathbb{R}_+$ such that X has the same f.d.d.'s as Y defined by $Y(t) = W(f(t))$, for a Wiener process W.
23. Let μ be a σ-finite Borel measure on $\mathbb{R}^d$. Prove existence of Poisson point process and (Gaussian) white noise with leading measure μ.

    In both cases, the process X we are interested in is indexed by Borel sets A with $\mu(A) < \infty$, with independent values on disjoint sets, with finite additivity property for disjoint sets and such that
    (a) $X(B)$ is Poisson with parameter $\mu(B)$ (for Poisson point process).
    (b) $X(B)$ is centered Gaussian with variance $\mu(B)$ (for white noise).
24. Suppose the process X_t is a Gaussian process, and let H be the Hilbert space generated by $(X_t)_{t \in \mathbb{R}}$, i.e., the space consisting of L^2-limits of linear combinations of values of X_t. Prove that every element in H is a Gaussian r.v.