PROBABILITY: LIMIT THEOREMS II, SPRING 2015. HOMEWORK PROBLEMS

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Instructions. You are allowed to work on solutions in groups, but you are required to write up solutions on your own. Please give complete solutions, all claims need to be justified. Late homework will not be accepted.

Please let me know if you find any misprints or mistakes.

1. Due by March 4, 11:00am

1. Let $(X_n)_{n\in\mathbb{N}}$ be an i.i.d. positive sequence, and $S_n = X_1 + \ldots + X_n$. Let $N_t = \sup\{n : S_n \le t\}$. Prove that $(N_t)_{t\in\mathbb{R}_+}$ is a stochastic process.
2. Let $(W_t)_{t\in\mathbb{R}_+}$ be a Wiener process. Find $\mathsf{cov}(W_s, W_t)$.
3. Prove that every Borel set B in $\mathbb{R}^d$ is regular, i.e., for every probability Borel measure μ, every $\varepsilon > 0$, there is a compact set K and open set U such that $K \subset B \subset U$ and $\mu(U \setminus K) < \varepsilon$.
4. Prove that cylinders $C(t_1, \ldots, t_n, B)$, $B \in \mathcal{B}(\mathbb{R}^n)$, $t_1, \ldots, t_n \ge 0$, $n \in \mathbb{N}$ form an algebra.
5. We defined the cylindrical σ-algebra as the σ-algebra generated by elementary cylinders. Prove that if we replace "elementary cylinders" by "cylinders" we obtain an equivalent definition.
6. Let $\mathcal{F}_T = \sigma\{C(t_1, \ldots, t_n, B) : t_1, \ldots, t_n \in T\}$ for $T \subset \mathbb{T}$.

   Prove that
$$\mathcal{B}(\mathbb{R}^{\mathbb{T}}) = \bigcup_{\text{countable } T\subset\mathbb{T}} \mathcal{F}_T.$$
7. Use characteristic functions to prove the existence of a Wiener process (up to continuity of paths).
8. Let $(X_t)_{t\in[0,1]}$ be an (uncountable) family of i.i.d. r.v.'s with nondegenerate distribution. Prove that no modification of this process can be continuous.
9. A multidimensional version of the Kolmogorov–Chentsov theorem. Suppose $d \ge 1$, and there is a stochastic field $X : [0,1]^d \times \Omega \to \mathbb{R}$ that satisfies $\mathsf{E}|X(s) - X(t)|^\alpha \le C|s - t|^{d+\beta}$ for some $\alpha, \beta, C > 0$ and all $t, s \in [0,1]^d$. Prove that there is a continuous modification of X on $[0,1]^d$.
10. Prove the following statement. Suppose there is a family of ch.f. $(\phi_{s,t}(\cdot))_{0\le s<t}$ such that for all $\lambda \in \mathbb{R}$ and all $t_1 < t_2 < t_3$,
$$\phi_{t_1,t_2}(\lambda)\phi_{t_2,t_3}(\lambda) = \phi_{t_1,t_3}(\lambda).$$

Then for every distribution function F, there is a stochastic process $(X_t)_{t\in\mathbb{R}_+}$ with independent increments such that $X_0 \sim F$ and $\mathsf{E}e^{i\lambda(X_t-X_s)} = \phi_{s,t}(\lambda)$ for all $s < t$ and $\lambda \in \mathbb{R}$.

11. Show that the Kolmogorov–Chentsov theorem cannot be relaxed: inequality $\mathsf{E}|X_t - X_s| \le C|t-s|$ is not sufficient for existence of a continuous modification. Hint: consider the following process: let τ be a r.v. with exponential distribution, and define $X_t = \mathbb{1}_{\{\tau\le t\}}$.
12. Prove that there exists a Poisson process such that:
    (a) its realizations are nondecreasing, taking only whole values a.s.
    (b) its realizations are continuous on the right a.s.
    (c) all the jumps of the realizations are equal to 1 a.s.
13. Give an example of a non-Gaussian 2-dimensional random vector with Gaussian marginal distributions.
14. Let $Y \sim \mathcal{N}(a, C)$ be a d-dimensional random vector. Let $Z = AY$ where A is an $n \times d$ matrix. Prove that Z is Gaussian and find its mean and covariance matrix.
15. Prove that an $\mathbb{R}^d$-valued random vector X is Gaussian iff for every vector $b \in \mathbb{R}^d$, the r.v. $\langle b, X\rangle$ is Gaussian.
16. Prove that $(s,t) \mapsto t\wedge s$ defined for $s,t \ge 0$ is positive semi-definite. Hint:
$$\langle \mathbb{1}_{[0,t]}, \mathbb{1}_{[0,s]}\rangle_{L^2(\mathbb{R}_+)} = t \wedge s.$$
17. Prove that $(s,t) \mapsto e^{-|t-s|}$ is positive semi-definite.
18. Prove that if X is a Gaussian vector in $\mathbb{R}^d$ with parameters (a, C) and C is non-degenerate, then the distribution of X is absolutely continuous w.r.t. Lebesgue measure and the density is
$$p_X(x) = \frac{1}{\det(C)^{1/2}(2\pi)^{d/2}} e^{-\frac{1}{2}\langle C^{-1}(x-a),(x-a)\rangle}.$$
19. Find a condition on the mean $a(t)$ and covariance function $r(s,t)$ that guarantees existence of a continuous Gaussian process with these parameters.
20. Suppose $(X_0, X_1, \ldots, X_n)$ is a (not necessarily centered) Gaussian vector. Show that there are constants $c_0, c_1, \ldots, c_n$ such that
$$\mathsf{E}(X_0|X_1,\ldots,X_n) = c_0 + c_1X_1 + \ldots + c_nX_n.$$
Your proof should be valid even if the covariance matrix of $(X_1,\ldots,X_n)$ is degenerate.
21. Consider the standard Ornstein–Uhlenbeck process X (Gaussian process with mean 0 and covariance function $r(s,t) = e^{-|t-s|}$).
    (a) Prove that X has a continuous modification.
    (b) Find $\mathsf{E}(X_4|\ X_1, X_2, X_3)$.
22. Prove that for every centered Gaussian process X with independent increments on $\mathbb{R}_+ = [0,\infty)$, there is a nondecreasing nonrandom function $f : \mathbb{R}_+ \to \mathbb{R}_+$ such that X has the same f.d.d.'s as Y defined by $Y(t) = W(f(t))$, for a Wiener process W.

2. Due by March 25, 11:00am

1. Suppose the process X_t is a stationary Gaussian process, and let H be the Hilbert space generated by $(X_t)_{t\in\mathbb{R}}$, i.e., the space consisting of L^2-limits of linear combinations of values of X_t. Prove that every element in H is a Gaussian r.v.
2. Find the covariance function of a stationary process such that its spectral measure is $\rho(dx) = \frac{dx}{1+x^2}$.
3. Give an example of a weakly stationary stochastic process $(X_n)_{n\in\mathbb{N}}$ such that $(X_1 + \ldots + X_n)/n$ converges in L^2 to a limit that is not a constant.
4. Let $(X_t)_{t\in\mathbb{R}}$ be a weakly stationary centered process with covariance function C and spectral measure ρ. Find the covariance function and spectral measure for process $(Y_t)_{t\in\mathbb{R}}$ defined by $Y_t = X_{2t}$.
5. Let $(X_n)_{n\in\mathbb{Z}}$ be a weakly stationary process. Prove that for any $K \in \mathbb{N}$ and any numbers $a_{-K}, a_{-K+1}, \ldots, a_{K-1}, a_K$, the process $(Y_n)_{n\in\mathbb{Z}}$ defined by

$$Y_n = \sum_{k=-K}^{K} a_k X_{n+k}$$

is weakly stationary. Express the spectral measure of Y in terms of the spectral measure for X.
6. Let stationary process $(X_n)_{n\in\mathbb{Z}}$ satisfy $\mathsf{E}|X_0| < \infty$. Prove that with probability 1, $\lim_{n\to\infty}(X_n/n) = 0$.
7. Consider a map $\theta : \Omega \to \Omega$. A set A is called *(backward) invariant* if $\theta^{-1}A = A$, *forward invariant* if $\theta A = A$. Prove that the collection of backward invariant sets forms a σ-algebra. Give an example of Ω and θ such that the collection of forward invariant sets does not form a σ-algebra.
8. Consider the transformation $\theta : \omega \mapsto \{\omega + \lambda\}$ on $[0, 1)$ equipped with Lebesgue measure. Here $\{\ldots\}$ denotes fractional part of a number. This map can be interpreted as rotation of the circle parametrized by $[0, 1)$ with endpoints 0 and 1 identified. Prove that this dynamical system is ergodic if and only if $\lambda \notin \mathbb{Q}$. Hint: take the indicator of an invariant set and write down the Fourier series for it (w.r.t. $e^{2\pi inx}$). What happens to this expansion under θ?
9. Prove that every Gaussian martingale is a process with independent increments.
10. Let $(X_t, \mathcal{F}_t)_{t\geq 0}$ be a continuous process. Let $a > 0$, and let

$$\tau = \inf\{t : X(t) > a\}.$$

Show that τ is a stopping time w.r.t $(\mathcal{F}_{t+})_{t\geq 0}$, where $\mathcal{F}_{t+} = \bigcap_{\varepsilon>0} \mathcal{F}_{t+\varepsilon}$.
11. Show that if $\tau_1 \leq \tau_2 \leq \ldots$ are stopping times w.r.t. to a filtration $(\mathcal{F}_t)$, then $\tau = \lim_{n\to\infty} \tau_n$ is also a stopping time w.r.t. to $(\mathcal{F}_t)$.
12. Let $\mathcal{F}_\tau = \{A : \ A \cap \{\tau \leq t\} \in \mathcal{F}_t\}$ for a filtration $(\mathcal{F}_t)$ and a stopping time τ. Show that $\mathcal{F}_\tau$ is a σ-algebra.

13. Give an example of the following: a random variable $\tau \geq 0$ is not a stopping time, $\mathcal{F}_\tau$ is not a σ-algebra.
14. Suppose τ is a stopping time w.r.t. $(\mathcal{F}_{t+})$. Let us define

$$\sout{\tau_n = \frac{\lceil 2^n \tau \rceil}{2^n} = \sum_{k\in\mathbb{N}} \frac{k}{2^n} \mathbb{1}_{\left\{\tau \in \left(\frac{k-1}{2^n}, \frac{k}{2^n}\right]\right\}}, \quad n \in \mathbb{N}.}$$

$$\tau_n = \frac{[2^n \tau] + 1}{2^n} = \sum_{k\in\mathbb{N}} \frac{k}{2^n} \mathbb{1}_{\left\{\tau \in \left[\frac{k-1}{2^n}, \frac{k}{2^n}\right)\right\}}, \quad n \in \mathbb{N}.$$

Prove that for every $n \in \mathbb{N}$, τ_n is a stopping time w.r.t. $(\mathcal{F}_t)_{t\geq 0}$, $\mathcal{F}_{\tau_n} \supset \mathcal{F}_{\tau+}$, and $\tau_n \downarrow \tau$.
15. Prove: if $(X_t, \mathcal{F}_t)_{t\geq 0}$ is a continuous process, then for any stopping time τ, X_τ is a r.v. measurable w.r.t. $\mathcal{F}_\tau$.
16. Let $(X_t, \mathcal{F}_t)$ be a continuous martingale and let τ be a stopping time w.r.t. $\mathcal{F}_t$. Prove that the "stopped" process $(X_t^\tau, \mathcal{F}_t)_{t\geq 0}$, where $X_t^\tau = X_{\tau\wedge t}$, is also a martingale.

3. Due by April 15, 11:00am

1. Find the density of the distribution of
$$\tau_b = \inf\{t \ge 0 : W(t) \ge b\},$$
where $b > 0$, and W is the standard Wiener process. Hint: use the reflection principle to find $\mathsf{P}\{\tau_b \le x\}$ first.

   (Optional: prove that $(\tau_b)_{b\ge 0}$ is a process with independent increments.)
2. Let W^1 and W^2 be two independent Wiener processes w.r.t. a filtration $(\mathcal{F}_t)_{t\ge 0}$, and let X be a bounded process adapted to $(\mathcal{F}_t)_{t\ge 0}$.

   For a partition t of time interval $[0,T]$ (i.e., a sequence of times $t = (t_0, t_1, \ldots, t_n)$ such that $0 = t_0 < t_1 < t_2 < \ldots < t_n = T$), we define
$$Q(t) = \sum_j A_{t_j}(W^1_{t_{j+1}} - W^1_{t_j})(W^2_{t_{j+1}} - W^2_{t_j}).$$
Prove:
$$\lim_{\max(t_{j+1}-t_j)\to 0} Q(t) = 0 \quad \text{in } L^2.$$
3. Let $(\mathcal{F}_t)$ be a filtration. Suppose that $0 = A_0(t) + A_1(t)W(t)$ for all t, where $(A_0, \mathcal{F}_t)$ and $(A_1, \mathcal{F}_t)$ are C^1 processes, and $W(t)$ is a Wiener process w.r.t. $(\mathcal{F}_t)$. Prove that $A_0 \equiv 0$ and $A_1 \equiv 0$.
4. Show that the function
$$P(s,x,t,\Gamma) = P(t-s,x,\Gamma) = \int_\Gamma \frac{1}{\sqrt{2\pi(t-s)}} e^{-\frac{(x-y)^2}{2(t-s)}}\,dy$$
is a Markov transition probability function for the standard Wiener process.
5. Prove the following theorem (Kolmogorov, 1931) using Taylor expansions of test functions:

   Suppose $(\mathsf{P}_x)_{x\in\mathbb{R}^d}$ is a (homogeneous) Markov family on $\mathbb{R}^d$ with transition probabilities $P(\cdot,\cdot,\cdot)$. Suppose there are continuous functions $a^{ij}(x)$, $b^i(x)$, $i,j = 1,\ldots,d$, such that for every $\varepsilon > 0$, the following relations hold uniformly in x:
$$P(t,x,B^c_\varepsilon(x)) = o(t), \quad t \to 0,$$
$$\int_{B_\varepsilon(x)} (y^i - x^i)P(t,x,dy) = b^i(x)t + o(t), \quad t \to 0,$$
$$\int_{B_\varepsilon(x)} (y^i - x^i)(y^j - x^j)P(t,x,dy) = a^{ij}(x)t + o(t), \quad t \to 0.$$
where $B_\varepsilon(x)$ is the Euclidean ball of radius ε centered at x. Then the infinitesimal generator A of the Markov semigroup associated to the Markov family is defined on all functions f such that f itself and all its partial

derivatives of first and second order are bounded and uniformly continuous. For such functions

$$Af = \frac{1}{2}\sum_{ij} a^{ij}\partial_{ij} f + \sum_i b^i \partial_i f.$$

6. Consider a Markov process X in $\mathbb{R}^2$ given by

$$X_1(t) = X_1(0) + W(t),$$
$$X_2(t) = X_2(0) + \int_0^t X_1(s)ds.$$

   Find its generator on C^2-functions with compact support.

7. Consider the Poisson transition probabilities, i.e., fix a number $\lambda > 0$ and for $i \in \mathbb{Z}$ and $t \geq 0$, let $P(i, t, \cdot)$ be the distribution of $i + \pi_{\lambda t}$, where π_s denotes a random variable with Poisson distribution with parameter $s > 0$. In other words,

$$P(i, t, \{j\}) = e^{-\lambda t}\frac{(\lambda t)^{j-i}}{(j-i)!}, \quad i \in \mathbb{Z}, \quad j \in \{i, i+1, \ldots, \}, \quad t > 0.$$

   Find the generator of the Markov semigroup on all bounded test functions $f : \mathbb{Z} \to \mathbb{R}$.

8. Find the transition probabilities and generator associated to the OU process (see, e.g., the second HW assignment or lecture notes for a definition of OU process).
9. Let W be a standard Wiener process. Prove that $W_t^2 - t$ is a martingale.
10. The so-called Stratonovich stochastic integral may be defined for a broad class of adapted processes X_t via

$$\int_0^T X_t \circ dW_t \stackrel{def}{=} \lim_{\max(t_{j+1}-t_j)\to 0} \sum_j \frac{X_{t_{j+1}} + X_{t_j}}{2}(W_{t_{j+1}} - W_{t_j}) \quad \text{in } L^2.$$

   Impose any conditions you need on X and express the difference between the Itô and Stratotovich integrals in terms of quadratic covariation between X and W. Compute $\int_0^T W_t \circ dW_t$. Is the answer a martingale?

4. Due by May 6, 11:00am

$\mathcal{M}_c^2 = \{$square-integrable martingales with continuous paths$\}$

1. Prove that if $(M_t, \mathcal{F}_t) \in \mathcal{M}_c^2$, then
$$\mathsf{E}[(M_t - M_s)^2|\mathcal{F}_s] = \mathsf{E}[M_t^2 - M_s^2|\mathcal{F}_s] = \mathsf{E}[\langle M\rangle_t - \langle M\rangle_s|\mathcal{F}_s], \quad s < t.$$
2. Suppose $(M_t, \mathcal{F}_t) \in \mathcal{M}_c^2$, X is a simple process, and $(X\cdot M)_t = \int_0^t X_s dM_s$. Prove that
$$\mathsf{E}[((X\cdot M)_t - (X\cdot M)_s)^2|\mathcal{F}_s] = \mathsf{E}\left[\int_s^t X_r^2 d\langle M\rangle_r|\mathcal{F}_s\right], \quad s < t.$$
3. Let $M \in \mathcal{M}_c^2$. Prove that
$$Y\cdot(X\cdot M) = (YX)\cdot M$$
for simple processes X, Y. Find reasonable weaker conditions on X and Y guaranteeing the correctness of this identity in the sense of square integrable martingales.
4. Suppose $(M_t, \mathcal{F}_t) \in \mathcal{M}_c^2$, and X, Y are bounded processes. Prove that
$$\langle X\cdot M, Y\cdot M\rangle_t = \int_0^t X_s Y_s d\langle M\rangle_s.$$
Here, for two processes $M, N \in \mathcal{M}_c^2$ the cross-variation $\langle M, N\rangle_t$ is defined by
$$\langle M, N\rangle_t = \frac{\langle M+N\rangle_t - \langle M-N\rangle_t}{4}.$$
5. Let us define the process X by
$$X_t = e^{\lambda t}X_0 + \varepsilon e^{\lambda t}\int_0^t e^{-\lambda s}dW_s, \quad t \ge 0.$$
Here $\lambda \in \mathbb{R}$, $\varepsilon > 0$, W is a standard Wiener process, and X_0 is a square-integrable r.v., independent of W. Prove that
$$dX_t = \lambda X_t dt + \varepsilon dW_t.$$
6. Prove that if $f : [0,\infty)$ is a deterministic function, bounded on any interval $[0, t]$, then
$$X_t = \int_0^t f(s)dW_s, \quad t \ge 0,$$
is a Gaussian process. Find its mean and covariance function.
7. In the context of Problem 5, find all the values of λ with the following property: there are a and σ^2 such that if $X_0 \sim \mathcal{N}(a, \sigma^2)$, then X_t is a stationary process.
8. Suppose $u_0 : \mathbb{R} \to [0,\infty)$ and $\phi : [0,\infty)\times\mathbb{R} \to \mathbb{R}$ are smooth bounded functions. Suppose that $u : [0,\infty)\times\mathbb{R} \to \mathbb{R}$ is a smooth function satisfying
$$\partial_t u(t,x) = \frac{1}{2}\partial_{xx}u(t,x) + \phi(t,x)u(t,x),$$
$$u(0,x) = u_0(x).$$

Prove that

$$u(t,x) = \mathsf{E} e^{\int_0^t \phi(t-s, x+W_s)ds} u_0(x+W_t), \quad t>0,\ x\in\mathbb{R},$$

where W is a standard Wiener process.

9. Suppose $a\in\mathbb{R}$, $\sigma>0$, $x_0>0$, and W is the standard Wiener process. Find constants $A, B\in\mathbb{R}$ such that the process S defined for all $t\ge 0$ by $S_t = x_0\exp(at+\sigma W_t)$ (and often called "the geometric Brownian motion") satisfies the following stochastic equation

$$dS_t = AS_t dt + BS_t dW_t, \quad t\ge 0.$$

Find necessary and sufficient conditions on a and σ for (S_t) to be a martingale.

10. Suppose $(W_t, \mathcal{F}_t)$ is a Wiener process and $(X_t, \mathcal{F}_t)$ is a bounded process. Use the Itô formula to prove that

$$Z_t = \exp\left[\int_0^t X_s dW_s - \frac{1}{2}\int_0^t X_s^2 ds\right], \quad t\ge 0,$$

is a local martingale w.r.t. $(\mathcal{F}_t)$. (In fact, it is a true martingale)