TAKE-HOME PART OF THE FINAL EXAM FOR PROBABILITY: LIMIT THEOREMS I, FALL 2014

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Instructions: Please write down all the solutions of the problems stated below and bring them to the final exam on Wednesday Dec 17. Please staple all your sheets together if possible. Submit them along with the in-class part of the exam.

You are allowed to use any literature to help you, but you are not allowed to receive help from any other person. Please give detailed answers and justify all your claims. Even though two last problems are freely worded, I expect rigorous mathematical reasoning. Unsupported answers will receive no credit.

On the first page of your work you must put in handwriting the following sentence: *"I pledge on my honor that I have not given or received any unauthorized assistance on this examination."* and your signature.

Please let me know if you find any mistakes/misprints.

1. Let $X_1, X_2, \ldots$ be an i.i.d. sequence of random variables such that

$$\mathsf{P}\{X_k = 0\} = \mathsf{P}\{X_k = 2\} = 1/2, \quad k \in \mathbb{N}.$$

Let

$$Y_n = \prod_{k=1}^{n} X_k, \quad n \in \mathbb{N}.$$

Show that it is not possible to find a random variable Z with finite first absolute moment and a filtration $(\mathcal{F}_n)_{n\in\mathbb{N}}$ such that

$$Y_n = \mathsf{E}[Z|\mathcal{F}_n].$$

2. Let $(p_{ij})_{i,j\in E}$ be transition probabilities of a Markov chain $(X_n)_{n\geq 0}$ on a countable state space E. Let $V : E \to \mathbb{R}$ be a bounded function and λ a positive number such that for all $i \in E$,

$$\sum_{j\in E} p_{ij} V(j) \leq \lambda V(i) \tag{1}$$

Prove that the process $(Y_n)_{n\geq 0}$ defined by

$$Y_n = \lambda^{-n} V(X_n), \quad n \geq 0,$$

is a supermartingale with respect to the natural filtration of $(X_n)_{n\geq 0}$.

3. Use the previous problem to prove the following transience condition for a Markov chain. Let (p_{ij}) be a transition probability of a Markov chain (X_n) on $E = \mathbb{Z}^d$ for some $d \geq 0$. Let V be a positive function on $\mathbb{Z}^d$ satisfying $V(i) \to 0$ as $|i| \to \infty$. Suppose that there is a number $R > 0$ such that for $|i| > R$, V satisfies inequality (1) from the statement of the previous problem with $\lambda = 1$. Then

$$\mathsf{P}_i\{\tau_R < \infty\} < 1$$

for large values of $|i|$, where

$$\tau_R = \min\{n \in \mathbb{N} : \ |X_n| \leq R\}.$$

Hint: consider a sequence of stopping times $\tau_R \wedge N$, $N \in \mathbb{N}$, and use the optional stopping theorem.

4. In this course, we obtained Gaussian scaling limit (CLT) for sums of i.i.d. r.v.'s. It turned out that the correct way to rescale is to mutiply the n-th partial sum by $n^{-1/2}$. Summation of r.v.'s is an important model, but there are numerous other examples of various probabilistic models where distributional scaling limits exist. The following is one such example that allows to compute the limiting distribution function directly, without using integral transforms such as characteristic functions.

Suppose there are infinitely many gold mines along a road. You are at the beginning of the road, and it takes k weeks to reach k-th gold mine. When you reach the k-th gold mine you may decide to settle there and then it steadily will produce X_k oz of gold per week for you, where $(X_k)_{k=1}^{\infty}$ is a sequence of i.i.d. r.v.'s each uniformly distributed on $[0, 1]$. You must get back with all that gold back to the start, and it will take you another k weeks. Before starting your journey, you know the values of all X_k's and the total time n (weeks) that you will be traveling and/or exploiting one of the mines. You need to maximize the total gold that you will get. In more mathematical terms, we set

$$Y_n = \max_{1 \leq k \leq n/2} \{(n - 2k)X_k\}, \quad n \in \mathbb{N}.$$

On the one hand, you want to get to better values of X_k, and if k is too small, you may miss best gold mines within reach. On the other hand, if k is too large, you will spend a lot of time traveling and only short time collecting gold.

(a) Prove that $Y_n \overset{a.s.}{\leq} n$, and as $n \to \infty$, $Y_n/n \xrightarrow{\mathsf{P}} 1$.

(b) Find α such that $n^{\alpha}(Y_n - n)$ converges in distribution to a nonconstant random variable, and find the limiting distribution. What can be said about the behavior of the optimal location $k = k_n$ providing the maximum value in the definition of Y_n?

5. A car moves on an infinite road from left to right with constant speed and constant fuel consumption per mile. At mile 0, it has enough fuel to drive for 500 miles. At random locations X_k, $k \in \mathbb{N}$, along the road there are canisters (one canister per location) each containing enough fuel to drive for 100 miles. When the car reaches such a location, the driver pours all the fuel from the canister into the tank of the car. We assume that the tank has infinite volume, the fuel consumption does not depend on the amount of fuel in the tank, and no loss of fuel occurs when the canisters are used to refill the tank. The random locations X_k are such that the spacings

$$Y_k = \begin{cases} X_k - X_{k-1}, & k \geq 2, \\ X_1, & k = 1. \end{cases}$$

between them are i.i.d. positive nonconstant random variables. Let

$$I = \{\text{the car idefinitely drives to the right and never runs out of gas}\}.$$

Find a necessary and sufficient conditions in terms of $\mathsf{E}Y_1$ for $\mathsf{P}(I) > 0$.