

# LAGRANGIAN HYPERPLANES IN HOLOMORPHIC SYMPLECTIC VARIETIES: COMPUTATIONAL APPENDIX

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ABSTRACT. We collect here the computations from [BJ11]. In that paper, we classify the cohomology classes of Lagrangian hyperplanes  $\mathbb{P}^4$  in a smooth manifold  $X$  deformation equivalent to a Hilbert scheme of 4 points on a K3 surface, up to the monodromy action. Classically, the cone of effective curves on a K3 surface  $S$  is generated by nonnegative classes  $C$ , for which  $(C, C) \geq 0$ , and nodal classes  $C$ , for which  $(C, C) = -2$ ; Hassett and Tschinkel conjecture that the cone of effective curves on a holomorphic symplectic variety  $X$  is similarly controlled by “nodal” classes  $C$  such that  $(C, C) = -\gamma$ , for  $(\cdot, \cdot)$  now the Beauville-Bogomolov form, where  $\gamma$  classifies the geometry of the extremal contraction associated to  $C$ . In particular, they conjecture that for  $X$  deformation equivalent to a Hilbert scheme of  $n$  points on a K3 surface, the class  $C = \ell$  of a line in a smooth Lagrangian  $n$ -plane  $\mathbb{P}^n$  must satisfy  $(\ell, \ell) = -\frac{n+3}{2}$ . We prove the conjecture for  $n = 4$  by computing the ring of monodromy invariants on  $X$ , and showing there is a unique monodromy orbit of Lagrangian hyperplanes.

## 1. INVARIANT CLASSES

(1.1) By [BJ11, §1.5],

$$\begin{array}{ll}
 \dim H^2(S^{[4]}, \mathbb{Q})^{G_S} = 1 & \dim H^2(S^{[4]}, \mathbb{Q})^{G_X} = 0 \\
 \dim H^4(S^{[4]}, \mathbb{Q})^{G_S} = 4 & \dim H^4(S^{[4]}, \mathbb{Q})^{G_X} = 2 \\
 \dim H^6(S^{[4]}, \mathbb{Q})^{G_S} = 5 & \dim H^6(S^{[4]}, \mathbb{Q})^{G_X} = 1 \\
 \dim H^8(S^{[4]}, \mathbb{Q})^{G_S} = 8 & \dim H^8(S^{[4]}, \mathbb{Q})^{G_X} = 3
 \end{array}$$

(1.2) In the notation of [BJ11, §1.8], the invariant classes for  $H^2(S^{[4]}, \mathbb{Q})$  are:

$$\delta = I(\{1\}_2, \{1, 1\}_1) = \sum_{(12)} 1_{12} \otimes 1_3 \otimes 1_4(12)$$

(1.3) The invariant classes for  $H^4(S^{[4]}, \mathbb{Q})$  are:

$$\begin{aligned}
W &= I(\{1\}_3, \{1\}_1) = \sum_{(123)} 1_{123} \otimes 1_4(123) \\
X &= I(\{1, 1\}_2) = \sum_{(12)(34)} 1_{12} \otimes 1_{34}(12)(34) \\
Y &= I(\{1, 1, 1, [\text{pt}]\}_1) = \sum_1 [\text{pt}]_1 \otimes 1_2 \otimes 1_3 \otimes 1_4(\text{id}) \\
Z &= I(\{1, 1, e, e^\vee\}_1) = \sum_{j, (12)} (e_j)_1 \otimes (e_j^\vee)_2 \otimes 1_3 \otimes 1_4(\text{id})
\end{aligned}$$

(1.4) The invariant classes for  $H^6(S^{[4]}, \mathbb{Q})$  are:

$$\begin{aligned}
P &= I(\{1\}_4) = \sum_{(1234)} 1_{1234}(1234) \\
Q &= I(\{[\text{pt}]\}_2, \{1, 1\}_1) = \sum_{(12)} [\text{pt}]_{12} \otimes 1_3 \otimes 1_4(12) \\
R &= I(\{1\}_2, \{1, [\text{pt}]\}_1) = \sum_{(12), 3} 1_{12} \otimes [\text{pt}]_3 \otimes 1_4(12) \\
S &= I(\{e^\vee\}_2, \{e, 1\}_1) = \sum_{j, 1, (23)} (e_j)_1 \otimes (e_j^\vee)_{23} \otimes 1_4(23) \\
T &= I(\{1\}_2, \{e, e^\vee\}_1) = \sum_{(12)} 1_{12} \otimes (e_j)_3 \otimes (e_j^\vee)_4(12)
\end{aligned}$$

(1.5) The invariant classes for  $H^8(S^{[4]}, \mathbb{Q})$  are:

$$\begin{aligned}
A &= I(\{e\}_3, \{e^\vee\}_1) = \sum_{j, (123)} (e_j)_{123} \otimes (e_j^\vee)_4(123) \\
B &= I(\{1\}_3, \{[\text{pt}]\}_1) = \sum_{(123)} 1_{123} \otimes [\text{pt}]_4(123) \\
C &= I(\{[\text{pt}]\}_3, \{1\}_1) = \sum_{(123)} [\text{pt}]_{123} \otimes 1_4(123) \\
D &= I(\{1, [\text{pt}]\}_2) = \sum_{(12)} [\text{pt}]_{12} \otimes 1_{34}(12)(34) \\
E &= I(\{e, e^\vee\}_2) = \sum_{j, (12)(34)} (e_j)_{12} \otimes (e_j^\vee)_{34}(12)(34) \\
F &= I(\{1, 1, [\text{pt}], [\text{pt}]\}_1) = \sum_{(12)} [[\text{pt}]]_1 \otimes [[\text{pt}]]_2 \otimes 1_3 \otimes 1_4(\text{id}) \\
G &= I(\{1, e, e^\vee, [\text{pt}]\}_1) = \sum_{j, 1, (23)} [[\text{pt}]]_1 \otimes (e_j)_2 \otimes (e_j^\vee)_3(\text{id}) \\
H &= I(\{e, e, e^\vee, e^\vee\}_1) = \sum_{j, k, (12)(34)} (e_j)_1 \otimes (e_j^\vee)_2 \otimes (e_k)_3 \otimes (e_k^\vee)_4 \cdot \text{id}
\end{aligned}$$

## 2. THE RING $A\{S_4\}$

(2.1) Below, for various pairs of  $\pi$  and  $\sigma$  we give the relevant orbits and defect, *cf.* [BJ11, §1.1]:

|       |          |                    |                                     |  |   |  |                  |
|-------|----------|--------------------|-------------------------------------|--|---|--|------------------|
| $\pi$ | $\sigma$ | $\pi \cdot \sigma$ | $\langle \pi \rangle \setminus [n]$ | $\langle \sigma \rangle \setminus [n]$ | $\langle \pi, \sigma \rangle \setminus [n]$ | $\langle \pi \cdot \sigma \rangle \setminus [n]$ | $g(\pi, \sigma)$ |
| (id)  | $\sigma$ | $\sigma$           | 1, 2, 3, 4                          | *                                      | *   | *  | 1                |

| $\pi$ | $\sigma$ | $\pi \cdot \sigma$ | $\langle \pi \rangle \setminus [n]$ | $\langle \sigma \rangle \setminus [n]$ | $\langle \pi, \sigma \rangle \setminus [n]$ | $\langle \pi \cdot \sigma \rangle \setminus [n]$ | $g(\pi, \sigma)$                  |
|-------|----------|--------------------|-------------------------------------|--|---|--|-----------------------------------|
| (12)  | (12)     | id                 | (1, 2), 3, 4                        | (1, 2), 3, 4                           | (1, 2), 3, 4                                | 1, 2, 3, 4                                       | $1_{1,2} \otimes 1_3 \otimes 1_4$ |
| (12)  | (13)     | (132)              | (1, 2), 3, 4                        | (1, 3), 2, 4                           | (1, 2, 3), 4                                | (1, 2, 3), 4                                     | $1_4 \otimes 1_{1,2,3}$           |
| (12)  | (14)     | (142)              | (1, 2), 3, 4                        | (1, 4), 2, 3                           | (1, 2, 4), 3                                | (1, 2, 4), 3                                     | $1_3 \otimes 1_{1,2,4}$           |
| (12)  | (23)     | (123)              | (1, 2), 3, 4                        | 1, 4, (2, 3)                           | (1, 2, 3), 4                                | (1, 2, 3), 4                                     | $1_4 \otimes 1_{1,2,3}$           |
| (12)  | (24)     | (124)              | (1, 2), 3, 4                        | 1, 3, (2, 4)                           | (1, 2, 4), 3                                | (1, 2, 4), 3                                     | $1_3 \otimes 1_{1,2,4}$           |
| (12)  | (34)     | (12)(34)           | (1, 2), 3, 4                        | 1, 2, (3, 4)                           | (1, 2), (3, 4)                              | (1, 2), (3, 4)                                   | $1_{1,2} \otimes 1_{3,4}$         |
| (12)  | (12)(34) | (34)               | (1, 2), 3, 4                        | (1, 2), (3, 4)                         | (1, 2), (3, 4)                              | 1, 2, (3, 4)                                     | $1_{1,2} \otimes 1_{3,4}$         |
| (12)  | (13)(24) | (1324)             | (1, 2), 3, 4                        | (1, 3), (2, 4)                         | (1, 2, 3, 4)                                | (1, 2, 3, 4)                                     | $1_{1,2,3,4}$                     |
| (12)  | (14)(23) | (1423)             | (1, 2), 3, 4                        | (1, 4), (2, 3)                         | (1, 2, 3, 4)                                | (1, 2, 3, 4)                                     | $1_{1,2,3,4}$                     |
| (12)  | (123)    | (23)               | (1, 2), 3, 4                        | (1, 2, 3), 4                           | (1, 2, 3), 4                                | 2, 4, (1, 3)                                     | $1_4 \otimes 1_{1,2,3}$           |
| (12)  | (132)    | (13)               | (1, 2), 3, 4                        | (1, 2, 3), 4                           | (1, 2, 3), 4                                | 1, (2, 3), 4                                     | $1_4 \otimes 1_{1,2,3}$           |
| (12)  | (124)    | (24)               | (1, 2), 3, 4                        | (1, 2, 4), 3                           | (1, 2, 4), 3                                | (1, 4), 2, 3                                     | $1_3 \otimes 1_{1,2,4}$           |
| (12)  | (142)    | (14)               | (1, 2), 3, 4                        | (1, 2, 4), 3                           | (1, 2, 4), 3                                | 1, 3, (2, 4)                                     | $1_3 \otimes 1_{1,2,4}$           |
| (12)  | (134)    | (1342)             | (1, 2), 3, 4                        | (1, 3, 4), 2                           | (1, 2, 3, 4)                                | (1, 2, 3, 4)                                     | $1_{1,2,3,4}$                     |
| (12)  | (143)    | (1432)             | (1, 2), 3, 4                        | (1, 3, 4), 2                           | (1, 2, 3, 4)                                | (1, 2, 3, 4)                                     | $1_{1,2,3,4}$                     |
| (12)  | (234)    | (1234)             | (1, 2), 3, 4                        | 1, (2, 3, 4)                           | (1, 2, 3, 4)                                | (1, 2, 3, 4)                                     | $1_{1,2,3,4}$                     |
| (12)  | (243)    | (1243)             | (1, 2), 3, 4                        | 1, (2, 3, 4)                           | (1, 2, 3, 4)                                | (1, 2, 3, 4)                                     | $1_{1,2,3,4}$                     |
| (12)  | (1234)   | (234)              | (1, 2), 3, 4                        | (1, 2, 3, 4)                           | (1, 2, 3, 4)                                | (1, 3, 4), 2                                     | $1_{1,2,3,4}$                     |
| (12)  | (1243)   | (243)              | (1, 2), 3, 4                        | (1, 2, 3, 4)                           | (1, 2, 3, 4)                                | (1, 3, 4), 2                                     | $1_{1,2,3,4}$                     |
| (12)  | (1324)   | (13)(24)           | (1, 2), 3, 4                        | (1, 2, 3, 4)                           | (1, 2, 3, 4)                                | (1, 4), (2, 3)                                   | $1_{1,2,3,4}$                     |
| (12)  | (1342)   | (134)              | (1, 2), 3, 4                        | (1, 2, 3, 4)                           | (1, 2, 3, 4)                                | 1, (2, 3), 4                                     | $1_{1,2,3,4}$                     |
| (12)  | (1423)   | (14)(23)           | (1, 2), 3, 4                        | (1, 2, 3, 4)                           | (1, 2, 3, 4)                                | (1, 3), (2, 4)                                   | $1_{1,2,3,4}$                     |
| (12)  | (1432)   | (143)              | (1, 2), 3, 4                        | (1, 2, 3, 4)                           | (1, 2, 3, 4)                                | 1, (2, 3, 4)                                     | $1_{1,2,3,4}$                     |

| $\pi$    | $\sigma$ | $\pi \cdot \sigma$ | $\langle \pi \rangle \setminus [n]$ | $\langle \sigma \rangle \setminus [n]$ | $\langle \pi, \sigma \rangle \setminus [n]$ | $\langle \pi \cdot \sigma \rangle \setminus [n]$ | $g(\pi, \sigma)$           |
|----------|----------|--------------------|-------------------------------------|--|---|--|----------------------------|
| (12)(34) | (12)(34) | id                 | (1, 2), (3, 4)                      | (1, 2), (3, 4)                         | (1, 2), (3, 4)                              | 1, 2, 3, 4                                       | $1_{1,2} \otimes 1_{3,4}$  |
| (12)(34) | (13)(24) | (14)(23)           | (1, 2), (3, 4)                      | (1, 3), (2, 4)                         | (1, 2, 3, 4)                                | (1, 4), (2, 3)                                   | $1_{1,2,3,4}$              |
| (12)(34) | (14)(23) | (13)(24)           | (1, 2), (3, 4)                      | (1, 4), (2, 3)                         | (1, 2, 3, 4)                                | (1, 3), (2, 4)                                   | $1_{1,2,3,4}$              |
| (12)(34) | (123)    | (243)              | (1, 2), (3, 4)                      | 4, (1, 2, 3)                           | (1, 2, 3, 4)                                | 2, (1, 3, 4)                                     | $1_{1,2,3,4}$              |
| (12)(34) | (132)    | (143)              | (1, 2), (3, 4)                      | 4, (1, 2, 3)                           | (1, 2, 3, 4)                                | 1, (2, 3, 4)                                     | $1_{1,2,3,4}$              |
| (12)(34) | (124)    | (234)              | (1, 2), (3, 4)                      | 3, (1, 2, 4)                           | (1, 2, 3, 4)                                | 2, (1, 3, 4)                                     | $1_{1,2,3,4}$              |
| (12)(34) | (142)    | (134)              | (1, 2), (3, 4)                      | 3, (1, 2, 4)                           | (1, 2, 3, 4)                                | 1, (2, 3, 4)                                     | $1_{1,2,3,4}$              |
| (12)(34) | (134)    | (142)              | (1, 2), (3, 4)                      | 2, (1, 3, 4)                           | (1, 2, 3, 4)                                | 4, (1, 2, 3)                                     | $1_{1,2,3,4}$              |
| (12)(34) | (143)    | (132)              | (1, 2), (3, 4)                      | 2, (1, 3, 4)                           | (1, 2, 3, 4)                                | 3, (1, 2, 4)                                     | $1_{1,2,3,4}$              |
| (12)(34) | (234)    | (124)              | (1, 2), (3, 4)                      | 1, (2, 3, 4)                           | (1, 2, 3, 4)                                | 4, (1, 2, 3)                                     | $1_{1,2,3,4}$              |
| (12)(34) | (243)    | (123)              | (1, 2), (3, 4)                      | 1, (2, 3, 4)                           | (1, 2, 3, 4)                                | 3, (1, 2, 4)                                     | $1_{1,2,3,4}$              |
| (12)(34) | (1234)   | (24)               | (1, 2), (3, 4)                      | (1, 2, 3, 4)                           | (1, 2, 3, 4)                                | 2, 4, (1, 3)                                     | $1_{1,2,3,4}$              |
| (12)(34) | (1243)   | (23)               | (1, 2), (3, 4)                      | (1, 2, 3, 4)                           | (1, 2, 3, 4)                                | 2, 3, (1, 4)                                     | $1_{1,2,3,4}$              |
| (12)(34) | (1324)   | (1423)             | (1, 2), (3, 4)                      | (1, 2, 3, 4)                           | (1, 2, 3, 4)                                | (1, 2, 3, 4)                                     | $-24[\text{pt}]_{1,2,3,4}$ |
| (12)(34) | (1342)   | (14)               | (1, 2), (3, 4)                      | (1, 2, 3, 4)                           | (1, 2, 3, 4)                                | 1, 4, (2, 3)                                     | $1_{1,2,3,4}$              |
| (12)(34) | (1423)   | (1324)             | (1, 2), (3, 4)                      | (1, 2, 3, 4)                           | (1, 2, 3, 4)                                | (1, 2, 3, 4)                                     | $-24[\text{pt}]_{1,2,3,4}$ |
| (12)(34) | (1432)   | (13)               | (1, 2), (3, 4)                      | (1, 2, 3, 4)                           | (1, 2, 3, 4)                                | 1, 3, (2, 4)                                     | $1_{1,2,3,4}$              |

| $\pi$ | $\sigma$ | $\pi \cdot \sigma$ | $\langle \pi \rangle \setminus [n]$ | $\langle \sigma \rangle \setminus [n]$ | $\langle \pi, \sigma \rangle \setminus [n]$ | $\langle \pi \cdot \sigma \rangle \setminus [n]$ | $g(\pi, \sigma)$                     |
|-------|----------|--------------------|-------------------------------------|--|---|--|--------------------------------------|
| (123) | (123)    | (132)              | 4, (1, 2, 3)                        | 4, (1, 2, 3)                           | 4, (1, 2, 3)                                | 4, (1, 2, 3)                                     | $-24[\text{pt}]_{1,2,3} \otimes 1_4$ |
| (123) | (132)    | id                 | 4, (1, 2, 3)                        | 4, (1, 2, 3)                           | 4, (1, 2, 3)                                | 1, 2, 3, 4                                       | $1_4 \otimes 1_{1,2,3}$              |
| (123) | (124)    | (13)(24)           | 4, (1, 2, 3)                        | 3, (1, 2, 4)                           | (1, 2, 3, 4)                                | (1, 4), (2, 3)                                   | $1_{1,2,3,4}$                        |
| (123) | (142)    | (143)              | 4, (1, 2, 3)                        | 3, (1, 2, 4)                           | (1, 2, 3, 4)                                | 1, (2, 3, 4)                                     | $1_{1,2,3,4}$                        |
| (123) | (134)    | (234)              | 4, (1, 2, 3)                        | 2, (1, 3, 4)                           | (1, 2, 3, 4)                                | 3, (1, 2, 4)                                     | $1_{1,2,3,4}$                        |
| (123) | (143)    | (14)(23)           | 4, (1, 2, 3)                        | 2, (1, 3, 4)                           | (1, 2, 3, 4)                                | (1, 2), (3, 4)                                   | $1_{1,2,3,4}$                        |
| (123) | (234)    | (12)(34)           | 4, (1, 2, 3)                        | 1, (2, 3, 4)                           | (1, 2, 3, 4)                                | (1, 3), (2, 4)                                   | $1_{1,2,3,4}$                        |
| (123) | (243)    | (124)              | 4, (1, 2, 3)                        | 1, (2, 3, 4)                           | (1, 2, 3, 4)                                | 2, (1, 3, 4)                                     | $1_{1,2,3,4}$                        |
| (123) | (1234)   | (1342)             | 4, (1, 2, 3)                        | (1, 2, 3, 4)                           | (1, 2, 3, 4)                                | (1, 2, 3, 4)                                     | $-24[\text{pt}]_{1,2,3,4}$           |
| (123) | (1243)   | (1324)             | 4, (1, 2, 3)                        | (1, 2, 3, 4)                           | (1, 2, 3, 4)                                | (1, 2, 3, 4)                                     | $-24[\text{pt}]_{1,2,3,4}$           |
| (123) | (1324)   | (24)               | 4, (1, 2, 3)                        | (1, 2, 3, 4)                           | (1, 2, 3, 4)                                | 2, 3, (1, 4)                                     | $1_{1,2,3,4}$                        |
| (123) | (1342)   | (34)               | 4, (1, 2, 3)                        | (1, 2, 3, 4)                           | (1, 2, 3, 4)                                | 1, 3, (2, 4)                                     | $1_{1,2,3,4}$                        |
| (123) | (1423)   | (1432)             | 4, (1, 2, 3)                        | (1, 2, 3, 4)                           | (1, 2, 3, 4)                                | (1, 2, 3, 4)                                     | $-24[\text{pt}]_{1,2,3,4}$           |
| (123) | (1432)   | (14)               | 4, (1, 2, 3)                        | (1, 2, 3, 4)                           | (1, 2, 3, 4)                                | 1, 2, (3, 4)                                     | $1_{1,2,3,4}$                        |

  

| $\pi$  | $\sigma$ | $\pi \cdot \sigma$ | $\langle \pi \rangle \setminus [n]$ | $\langle \sigma \rangle \setminus [n]$ | $\langle \pi, \sigma \rangle \setminus [n]$ | $\langle \pi \cdot \sigma \rangle \setminus [n]$ | $g(\pi, \sigma)$           |
|--------|----------|--------------------|-------------------------------------|--|---|--|----------------------------|
| (1234) | (1234)   | (13)(24)           | (1, 2, 3, 4)                        | (1, 2, 3, 4)                           | (1, 2, 3, 4)                                | (1, 3), (2, 4)                                   | $-24[\text{pt}]_{1,2,3,4}$ |
| (1234) | (1243)   | (132)              | (1, 2, 3, 4)                        | (1, 2, 3, 4)                           | (1, 2, 3, 4)                                | 3, (1, 2, 4)                                     | $-24[\text{pt}]_{1,2,3,4}$ |
| (1234) | (1324)   | (142)              | (1, 2, 3, 4)                        | (1, 2, 3, 4)                           | (1, 2, 3, 4)                                | 2, (1, 3, 4)                                     | $-24[\text{pt}]_{1,2,3,4}$ |
| (1234) | (1342)   | (143)              | (1, 2, 3, 4)                        | (1, 2, 3, 4)                           | (1, 2, 3, 4)                                | 1, (2, 3, 4)                                     | $-24[\text{pt}]_{1,2,3,4}$ |
| (1234) | (1423)   | (243)              | (1, 2, 3, 4)                        | (1, 2, 3, 4)                           | (1, 2, 3, 4)                                | 4, (1, 2, 3)                                     | $-24[\text{pt}]_{1,2,3,4}$ |
| (1234) | (1432)   | id                 | (1, 2, 3, 4)                        | (1, 2, 3, 4)                           | (1, 2, 3, 4)                                | 1, 2, 3, 4                                       | $1_{1,2,3,4}$              |

**(2.2)** Deducing from the previous orbit computations the multiplication structure in the ring  $A\{S_4\}$  is easy. For example, we have the following:

$$\begin{aligned}
(\alpha_{1,2,3} \otimes \beta_4 \cdot (123)) \cdot (\alpha'_{1,2,3} \otimes \beta'_4 \cdot (123)) &= (\mathbf{e}\alpha\alpha')_{1,2,3} \otimes (\beta\beta')_4 \cdot (132) \\
(\alpha_{1,2,3} \otimes \beta_4 \cdot (123)) \cdot (\alpha'_{1,2,3} \otimes \beta'_4 \cdot (132)) &= \Delta(\alpha\alpha')_{1,2,3} \otimes (\beta\beta')_4 \cdot \text{id} \\
(\alpha_{1,2,3} \otimes \beta_4 \cdot (123)) \cdot (\alpha'_{1,2,4} \otimes \beta'_3 \cdot (124)) &= \Delta(\alpha\beta\alpha'\beta')_{13,24} \cdot (13)(24) \\
(\alpha_{1,2,3} \otimes \beta_4 \cdot (123)) \cdot (\alpha'_{1,2,4} \otimes \beta'_3 \cdot (142)) &= \Delta(\alpha\beta\alpha'\beta')_{2,134} \cdot (143) \\
(\alpha_{1,2,3} \otimes \beta_4 \cdot (123)) \cdot (\alpha'_{1,3,4} \otimes \beta'_2 \cdot (134)) &= \Delta(\alpha\beta\alpha'\beta')_{1,234} \cdot (234) \\
(\alpha_{1,2,3} \otimes \beta_4 \cdot (123)) \cdot (\alpha'_{1,3,4} \otimes \beta'_2 \cdot (143)) &= \Delta(\alpha\beta\alpha'\beta')_{14,23} \cdot (14)(23) \\
(\alpha_{1,2,3} \otimes \beta_4 \cdot (123)) \cdot (\alpha'_{2,3,4} \otimes \beta'_1 \cdot (234)) &= \Delta(\alpha\beta\alpha'\beta')_{12,34} \cdot (12)(34) \\
(\alpha_{1,2,3} \otimes \beta_4 \cdot (123)) \cdot (\alpha'_{2,3,4} \otimes \beta'_1 \cdot (243)) &= \Delta(\alpha\beta\alpha'\beta')_{3,124} \cdot (124)
\end{aligned}$$

where  $\mathbf{e} = -24[\text{pt}]$ .

Also

$$\begin{aligned}
(12)(12) &= \Delta(1)_{1,2} \otimes 1_3 \otimes 1_4(\text{id}) \\
(12)(13) &= 1_{123} \otimes 1_4(132) \\
(12)(23) &= 1_{123} \otimes 1_4(123)
\end{aligned}$$

(3.3) Let  $e_j^\vee = \sum_k t_{jk} e_k$  and  $e_j = \sum_k r_{jk} e_k^\vee$  such that the matrices  $(t_{jk})$  and  $(r_{jk})$  are inverses to each other. Thus

$$\sum_i t_{ij} r_{ik} = \delta_{j=k}$$

Also  $(e_j, e_k) = (\sum_i r_{ji} e_i^\vee, e_k) = r_{jk}$  and  $(e_j^\vee, e_k^\vee) = t_{jk}$ . Moreover,  $e_j e_k = (e_j, e_k) = r_{jk}[\text{pt}]$  and  $e_j^\vee e_k^\vee = t_{jk}[\text{pt}]$ .

### 3. MULTIPLICATION TABLE FOR INVARIANT CLASSES

(3.1) For computations using invariant classes one needs to construct a multiplication table which, because of symmetry, has  $18 + \binom{18}{2} = 171$  entries. All formulas were either checked by SAGE code, available on either author's webpage, or by hand; many were checked by both.

3.2.  $\delta$ . **Summary:**

|                                      |                |
|--------------------------------------|----------------|
| $\delta^2 = 2X - 3Y - Z + 3W$        | formula & sage |
| $\delta W = 4P - 4Q - 2R - 2S$       | formula & sage |
| $\delta X = 2P - R - T$              | sage           |
| $\delta Y = 2Q + R$                  | sage           |
| $\delta Z = 22Q + 2S + T$            | sage           |
| $\delta P = -3A - 3B - 3C - 4D - 4E$ | formula & sage |
| $\delta Q = 3C + D - F$              | sage           |
| $\delta R = 3B + 3C + 2D - 4F - G$   | sage           |
| $\delta S = 3A + 66C + 4E - 2G$      | sage           |
| $\delta T = 3A + 22D - G - 2H$       | formula        |

(3.2.1)

$$\begin{aligned}
\delta^2 &= \left( \sum_{(12)} 1_{12} \otimes 1_3 \otimes 1_4(12) \right)^2 \\
&= \sum_{(12)} 1_{12} \otimes 1_3 \otimes 1_4(12) ((12) + (13) + (14) + (23) + (24) + (34)) \\
&= \sum_{(12)} (\Delta(1)_{1,2} \otimes 1_3 \otimes 1_4(\text{id}) + 1_{1,2,3} \otimes 1_4(132) \\
&\quad + 1_{1,2,4} \otimes 1_3(142) + 1_{1,2,3} \otimes 1_4(123) + 1_{1,2,4} \otimes 1_3(124) + 1_{12} \otimes 1_{34}(12)(34)) \\
&= -3 \sum_1 [\text{pt}]_1 \otimes 1_2 \otimes 1_3 \otimes 1_4(\text{id}) - \sum_{(12)} \sum_j (e_j)_1 \otimes (e_j^\vee)_2 \otimes 1_3 \otimes 1_4(\text{id}) \\
&\quad + 3 \sum_{(123)} 1_{123} \otimes 1_4(123) + 2 \sum_{(12)(34)} 1_{12} \otimes 1_{34}(12)(34) \\
&= -3Y - Z + 3W + 2X
\end{aligned}$$

**(3.2.2)**

$$\begin{aligned}
\delta W &= \sum_{(12)} 1_{12} \otimes 1_3 \otimes 1_4(12) \sum 1_{abc} \otimes 1_d(abc) \\
&= \sum_{(12)} 1_{12} \otimes 1_3 \otimes 1_4(12) (1_{123} \otimes 1_4(123) + 1_{124} \otimes 1_3(124) + 1_{134} \otimes 1_2(134) + 1_{234} \otimes 1_1(234)) \\
&\quad + 1_{123} \otimes 1_4(132) + 1_{124} \otimes 1_3(142) + 1_{134} \otimes 1_2(143) + 1_{234} \otimes 1_1(243)) \\
&= \sum_{(12)} (\Delta(1)_{1,23} \otimes 1_4(23) + \Delta(1)_{1,24} \otimes 1_3(24) + 1_{1234}(1342) + 1_{1234}(1234)) \\
&\quad + \Delta(1)_{2,13} \otimes 1_4(13) + \Delta(1)_{2,14} \otimes 1_3(1,4) + 1_{1234}(1432) + 1_{1234}(1243)) \\
&= 2 \sum_{1,(23)} \Delta(1)_{1,23}(23) + 4 \sum_{(1234)} 1_{1234}(1234) \\
&= 4 \sum_{(12)} 1_{1234}(1234) - 2 \sum_{1,(23)} 1_1 \otimes [\text{pt}]_{23}(23) - 2 \sum_{1,(23)} [\text{pt}]_1 \otimes 1_{23}(23) - 2 \sum_{1,(23)} \sum_{j=1}^{22} (e_j)_1 \otimes (e_j^\vee)_{23}(23) \\
&= 4P - 4Q - 2R - 2S
\end{aligned}$$

**(3.2.3)**

$$\begin{aligned}
\delta X &= \sum_{(12)} (12) \sum_{(12)(34)} (12)(34) \\
&= \sum_{(12)} (\Delta(1)_{1,2} \otimes 1_{34}(34) + (1324) + (1423)) \\
&= \sum_{(12)} \left( -[\text{pt}]_1 \otimes 1_2 \otimes 1_{34}(34) - 1_1 \otimes [\text{pt}]_2 \otimes 1_{34}(34) - \sum_j (e_j)_1 \otimes (e_j^\vee)_2 \otimes 1_{34}(34) + (1324) + (1423) \right) \\
&= -R - T + 2P
\end{aligned}$$

**(3.2.4)**

$$\begin{aligned}
\delta Y &= \sum_{(12)} (12) ([\text{pt}]_1 + [\text{pt}]_2 + [\text{pt}]_3 + [\text{pt}]_4) (\text{id}) \\
&= \sum_{(12)} (2[\text{pt}]_{12} \otimes 1_3 \otimes 1_4(12) + 1_{12} \otimes [\text{pt}]_3 \otimes 1_4(12) + 1_{12} \otimes 1_3 \otimes [\text{pt}]_4(12)) \\
&= 2Q + R
\end{aligned}$$

**(3.2.5)**

$$\begin{aligned}
\delta Z &= \sum_{(12)} \sum_j (e_j)_1 \otimes (e_j^\vee)_2 \otimes 1_3 \otimes 1_4(\text{id}) ((12) + (13) + (14) + (23) + (24) + (34)) \\
&= \sum_{(12)} \sum_j ((e_j e_j^\vee)_{12} \otimes 1_3 \otimes 1_4(\text{id}) + (e_j)_{13} \otimes (e_j^\vee)_2 \otimes 1_4(13) + (e_j)_{14} \otimes (e_j^\vee)_2 \otimes 1_3(14) \\
&\quad + (e_j)_1 \otimes (e_j^\vee)_{23} \otimes 1_4(23) + (e_j)_1 \otimes (e_j^\vee)_{24} \otimes 1_3(24) + (e_j)_1 \otimes (e_j^\vee)_2 \otimes 1_{34}(34)) \\
&= 22Q + 2S + T
\end{aligned}$$

**(3.2.6)**

$$\begin{aligned}
\delta P &= \sum_{(12)} 1_{12} \otimes 1_3 \otimes 1_4(12) \sum 1_{1234}(abcd) \\
&= \sum_{(12)} 1_{12} \otimes 1_3 \otimes 1_4(12) ((1234) + (1243) + (1342) + (1324) + (1432) + (1423)) \\
&= \sum_{(12)} ((12)(1234) + (12)(1243) + (12)(1342) + (12)(1324) + (12)(1432) + (12)(1423)) \\
&= \sum_{(12)} (\Delta(1)_{1,234}(234) + \Delta(1)_{1,234}(243) + \Delta(1)_{2,134}(134) + \Delta(1)_{13,24}(13)(24) + \Delta(1)_{2,134}(143) + \Delta(1)_{14,23}(142)) \\
&= 3 \sum_{(234)} \Delta(1)_{1,234}(234) + 4 \sum_{(13)(24)} \Delta(1)_{13,24}(13)(24) \\
&= -3 \sum_{(234)} 1_1 \otimes [\text{pt}]_{234}(234) - 3 \sum_{(234)} [\text{pt}]_1 \otimes 1_{234}(234) - 3 \sum_{(234)} \sum_{j=1}^{22} (e_j)_1 \otimes (e_j^\vee)_{234}(234) \\
&\quad - 4 \sum_{(13)(24)} 1_{13} \otimes [\text{pt}]_{24}(13)(24) - 4 \sum_{(13)(24)} [\text{pt}]_{13} \otimes 1_{24}(13)(24) - 4 \sum_{(13)(24)} \sum_{j=1}^{22} (e_j)_{13} \otimes (e_j^\vee)_{24}(13)(24) \\
&= -3A - 3B - 3C - 4D - 4E
\end{aligned}$$

**(3.2.7)**

$$\begin{aligned}
\delta Q &= \sum_{(12)} [\text{pt}]_{12} \otimes 1_3 \otimes 1_4(12) ((12) + (13) + (14) + (23) + (24) + (34)) \\
&= \sum_{(12)} (\Delta([\text{pt}])_{1,2} \otimes 1_3 \otimes 1_4(\text{id}) + [\text{pt}]_{123} \otimes 1_4(132) + [\text{pt}]_{124} \otimes 1_3(142) + [\text{pt}]_{123} \otimes 1_4(123) + [\text{pt}]_{124} \otimes 1_3(143) \\
&\quad + [\text{pt}]_{12} \otimes 1_{34}(12)(34)) \\
&= - \sum_{(12)} [\text{pt}]_1 \otimes [\text{pt}]_2 \otimes 1_3 \otimes 1_4(\text{id}) + 3 \sum_{(123)} [\text{pt}]_{123} \otimes 1_4(123) + \sum_{(12)} [\text{pt}]_{12} \otimes 1_{34}(12)(34) \\
&= -F + 3C + D
\end{aligned}$$

**(3.2.8)**

$$\begin{aligned}
\delta R &= \sum_{(12),3} 1_{12} \otimes [\text{pt}]_3 \otimes 1_4(12)((12) + (13) + (14) + (23) + (24) + (34)) \\
&= \sum_{(12),3} (\Delta(1)_{1,2} \otimes [\text{pt}]_3 \otimes 1_4(\text{id}) + [\text{pt}]_{123} \otimes 1_4(132) + 1_{124} \otimes [\text{pt}]_3(142) + [\text{pt}]_{123} \otimes 1_4(123)) \\
&\quad + 1_{124} \otimes [\text{pt}]_3(124) + 1_{12} \otimes [\text{pt}]_{34}(12)(34) \\
&= -4 \sum_{(12)} [\text{pt}]_1 \otimes [\text{pt}]_2 \otimes 1_3 \otimes 1_4(\text{id}) - \sum_{(12),3} \sum_j (e_j)_1 \otimes (e_j^\vee)_2 \otimes [\text{pt}]_3 \otimes 1_3(\text{id}) \\
&\quad + 3 \sum_{(123)} [\text{pt}]_{123} \otimes 1_4(123) + 3 \sum_{(123)} 1_{123} \otimes [\text{pt}]_4(123) + 2 \sum_{(12)} [\text{pt}]_{12} \otimes 1_{34}(12)(34) \\
&= -4F - G + 3B + 3C + 2D
\end{aligned}$$

**(3.2.9)**

$$\begin{aligned}
\delta S &= \sum_{(12),3} \sum_j (e_j)_{12} \otimes (e_j^\vee)_3 \otimes 1_4(12)((12) + (13) + (14) + (23) + (24) + (34)) \\
&= \sum_{(12),3} \sum_j (\Delta(e_j)_{1,2} \otimes (e_j^\vee)_3 \otimes 1_4(\text{id}) + (e_j e_j^\vee)_{123} \otimes 1_4(132) + (e_j)_{124} \otimes (e_j^\vee)_3(142) + (e_j e_j^\vee)_{123} \otimes 1_4(123)) \\
&\quad + (e_j)_{124} \otimes (e_j^\vee)_3(124) + (e_j)_{12} \otimes (e_j^\vee)_{34}(12)(34) \\
&= - \sum_{(12),3} \sum_j ((e_j)_1 \otimes [\text{pt}]_2 \otimes (e_j^\vee)_3 \otimes 1_4 + [\text{pt}]_1 \otimes (e_j)_2 \otimes (e_j^\vee)_3 \otimes 1_4)(\text{id}) + 3 \sum_j \sum_{(123)} [\text{pt}]_{123} \otimes 1_4(123) \\
&\quad + 3 \sum_{(123)} \sum_j (e_j)_{123} \otimes (e_j^\vee)_4(123) + 4 \sum_{(12)(34)} \sum_j (e_j)_{12} (e_j^\vee)_{34}(12)(34) \\
&= -2G + 66C + 3A + 4E
\end{aligned}$$

(3.2.10)

$$\begin{aligned}\delta T &= \sum_{j,(12)} 1_{12} \otimes (e_j)_3 \otimes (e_j^\vee)_4(12) ((12) + (13) + (14) + (23) + (24) + (34)) \\ &= \sum_{j,(12)} (\Delta(1)_{1,2} \otimes (e_j)_3 \otimes (e_j^\vee)_3(\text{id}) + (e_j)_{123} \otimes (e_j^\vee)_4(132) + (e_j)_3 \otimes (e_j^\vee)_{124}(142) + (e_j)_{123} \otimes (e_j^\vee)_4(123) \\ &\quad + (e_j)_3 \otimes (e_j^\vee)_{124}(124) + 1_{12} \otimes [\text{pt}]_{34}(12)(34)) \\ &= \sum_{j,(34)} \left( -[\text{pt}]_1 \otimes 1_2 - 1_1 \otimes [\text{pt}]_2 - \sum_k (e_k)_1 \otimes (e_k^\vee)_2 \right) \otimes (e_j)_3 \otimes (e_j^\vee)_3(\text{id}) + \sum_j \sum_{(12)} 1_{12} \otimes [\text{pt}]_{34}(12)(34) \\ &\quad + 3 \sum_{j,(123)} (e_j)_{123} \otimes (e_j^\vee)_4(123) \\ &= - \sum_{j,1,(34)} [\text{pt}]_1 \otimes 1_2 \otimes (e_j)_3 \otimes (e_j^\vee)_3(\text{id}) - 2 \sum_{j,k,(12)(34)} (e_k)_1 \otimes (e_k^\vee)_2 \otimes (e_j)_3 \otimes (e_j^\vee)_3(\text{id}) \\ &\quad + 22 \sum_{(12)} 1_{12} \otimes [\text{pt}]_{34}(12)(34) + 3 \sum_{j,(123)} (e_j)_{123} \otimes (e_j^\vee)_4(123) \\ &= -G - 2H + 22D + 3A\end{aligned}$$

3.3. *W*. Summary:

|  |                |
|--|----------------|
| $W^2 = -3A - 3B - 27C - 8D - 8E + 4F + 2G$ | formula & sage |
| $WX = -3A - 3B - 3C$                       | sage           |
| $WY = B + 3C$                              | sage           |
| $WZ = 3A + 66C$                            | sage           |

(3.3.1)

$$\begin{aligned}
W^2 &= \left( \sum_{(123)} 1_{123} \otimes 1_4(123) \right)^2 \\
&= \sum_{(123)} (123) ((123) + (132) + (124) + (142) + (134) + (143) + (234) + (243)) \\
&= \sum_{(123)} (-24[\text{pt}]_{123} \otimes 1_4(132) + \Delta(1)_{1,2,3} \otimes 1_4(\text{id}) + \Delta(1)_{13,24}(13)(24) + \Delta(1)_{2,134}(143) + \Delta(1)_{1,234}(234) \\
&\quad + \Delta(1)_{14,23}(14)(23) + \Delta(1)_{12,34}(12)(34) + \Delta(1)_{3,124}(124)) \\
&= \sum_{(123)} (-24[\text{pt}]_{123} \otimes 1_4(132) + \Delta(1)_{1,2,3} \otimes 1_4(\text{id})) + 8 \sum_{(12)(34)} \Delta(1)_{12,34}(12)(34) + 3 \sum_{(234)} \Delta(1)_{1,234}(234) \\
&= -24 \sum_{(123)} ([\text{pt}]_{123} \otimes 1_4(132)) + \sum_{(123)} \sum_{r=2}^4 ([\text{pt}]_1 \otimes [\text{pt}]_r \otimes 1 \otimes 1 + \sum_j [\text{pt}]_1 \otimes [\text{pt}]_r \otimes e_j \otimes e_j^\vee)(\text{id}) \\
&\quad - 8 \sum_{(12)(34)} ([\text{pt}]_{12} \otimes 1_{34} + 1_{12} \otimes [\text{pt}]_{34})(12)(34) - 8 \sum_{(12)(34)} \sum_j (e_j)_{12} \otimes (e_j^\vee)_{34}(12)(34) \\
&\quad - 3 \sum_{(234)} [\text{pt}]_{234} \otimes 1_1(234) - 3 \sum_{(234)} 1_{234} \otimes [\text{pt}]_1(234) - 3 \sum_{(234)} \sum_j (e_j)_{234} \otimes (e_j^\vee)_1(234) \\
&= -3A - 3B - 27C - 8D - 8E + 4F + 2G
\end{aligned}$$

3.4. X. Summary:

$$\begin{array}{ll}
X^2 = -2D - 2E + 2F + G + H & \text{formula \& sage} \\
XY = 2D & \text{sage} \\
XZ = 22D + 4E & \text{sage}
\end{array}$$

**(3.4.1)**

$$\begin{aligned}
X^2 &= \left( \sum_{(12)(34)} 1_{12} \otimes 1_{34} (12)(34) \right)^2 \\
&= \sum_{(12)(34)} (\Delta(1)_{1,2} \otimes \Delta(1)_{3,4}(\text{id}) + \Delta(1)_{14,23}(14)(23) + \Delta(1)_{13,24}(13)(24)) \\
&= \sum_{(12)(34)} \Delta(1)_{1,2} \otimes \Delta(1)_{3,4}(\text{id}) + 2 \sum_{(12)(34)} \Delta(1)_{12,34}(12)(34) \\
&= -2 \sum_{(12)(34)} ([\text{pt}]_{12} \otimes 1_{34} + 1_{12} \otimes [\text{pt}]_{34})(12)(34) - 2 \sum_{(12)(34)} \sum_j (e_j)_{12} \otimes (e_j^\vee)_{34}(12)(34) \\
&\quad + \sum_{(12)(34)} \left( -[\text{pt}]_1 \otimes 1_2 - 1_1 \otimes [\text{pt}]_2 - \sum_j (e_j)_1 \otimes (e_j^\vee)_2 \right) \otimes \\
&\quad \otimes \left( -[\text{pt}]_3 \otimes 1_4 - 1_3 \otimes [\text{pt}]_4 - \sum_k (e_k)_3 \otimes (e_k^\vee)_4 \right) (\text{id}) \\
&= -2D - 2E + 2 \sum_{(12)} [\text{pt}]_1 \otimes [\text{pt}]_2 \otimes 1_3 \otimes 1_4(\text{id}) + \sum_{(12)(34)} \sum_{j,k} (e_j)_1 \otimes (e_j^\vee)_2 \otimes (e_k)_3 \otimes (e_k^\vee)_4(\text{id}) \\
&\quad + \sum_{1,(23)} \sum_j [\text{pt}]_1 \otimes (e_j)_2 \otimes (e_j^\vee)_3 \otimes 1_4 \\
&= -2D - 2E + 2F + G + H
\end{aligned}$$

**3.5. Y. Summary:**

$$\begin{array}{ll}
Y^2 = 2F & \text{formula \& sage} \\
YZ = G & \text{sage}
\end{array}$$

**(3.5.1)**

$$\begin{aligned}
Y^2 &= \left( \sum_1 [\text{pt}]_1 \otimes 1_2 \otimes 1_3 \otimes 1_4(\text{id}) \right)^2 \\
&= \sum_1 ([\text{pt}] \otimes [\text{pt}] \otimes 1 \otimes 1 + [\text{pt}] \otimes 1 \otimes [\text{pt}] \otimes 1 + [\text{pt}] \otimes 1 \otimes 1 \otimes [\text{pt}]) \\
&= 2 \sum_{(12)} [\text{pt}]_1 \otimes [\text{pt}]_2 \otimes 1 \otimes 1 \\
&= 2F
\end{aligned}$$

**3.6. Z. Summary:**

$$Z^2 = 22F + 2G + 2H \quad \text{formula}$$

(3.6.1)

$$\begin{aligned}
Z^2 &= \left( \sum_j \sum_{(12)} (e_j)_1 \otimes (e_j^\vee)_2 \otimes 1_3 \otimes 1_4(\text{id}) \right)^2 \\
&= \sum_{j,k} \sum_{(12)} (e_j)_1 \otimes (e_j^\vee)_2 \otimes 1_3 \otimes 1_4(\text{id}) (e_k \otimes e_k^\vee \otimes 1 \otimes 1 + e_k \otimes 1 \otimes e_k^\vee \otimes 1 + e_k \otimes 1 \otimes 1 \otimes e_k^\vee \\
&\quad + 1 \otimes e_k \otimes e_k^\vee \otimes 1 + 1 \otimes e_k \otimes 1 \otimes e_k^\vee + 1 \otimes 1 \otimes e_k \otimes e_k^\vee) \\
&= \sum_{j,k,(12)} ((e_j e_k) \otimes (e_j^\vee e_k^\vee) \otimes 1 \otimes 1 + (e_j e_k) \otimes e_j^\vee \otimes e_k^\vee \otimes 1 + (e_j e_k) \otimes e_j^\vee \otimes 1 \otimes e_k^\vee \\
&\quad + e_j \otimes e_j^\vee e_k \otimes e_k^\vee \otimes 1 + e_j \otimes e_j^\vee e_k \otimes 1 \otimes e_k^\vee + e_j \otimes e_j^\vee \otimes e_k \otimes e_k^\vee) \\
&= \sum_{j,k,(12)} t_{jk} r_{jk} [\text{pt}] \otimes [\text{pt}] \otimes 1 \otimes 1 + \sum_{j,k,(12),3} r_{jk} [\text{pt}] \otimes \left( \sum_u t_{ju} e_u \right) \otimes e_k^\vee \otimes 1 \\
&\quad + \sum_{j,(12),3} e_j \otimes [\text{pt}] \otimes e_k^\vee \otimes 1 + \sum_{j,k,(12)} e_j \otimes e_j^\vee \otimes e_k \otimes e_k^\vee \\
&= 22F + \sum_{k,u,(12),3} [\text{pt}] \otimes e_u \otimes e_k^\vee \otimes 1 \sum_j r_{jk} t_{ju} + G + 2H \\
&= 22F + \sum_{k,u,(12),3} [\text{pt}] \otimes e_u \otimes e_k^\vee \otimes 1 \delta_{u=k} + G + 2H \\
&= 22F + \sum_{k,(12),3} [\text{pt}] \otimes e_k \otimes e_k^\vee \otimes 1 + G + 2H \\
&= 22F + 2G + 2H
\end{aligned}$$

since  $e_j e_k^\vee = \delta_{j=k} [\text{pt}]$ .

3.7. A. Summary:

|  |                |
|--|----------------|
| $A^2 = 8 \cdot 22[\text{pt}]_1 \otimes [\text{pt}]_2 \otimes [\text{pt}]_3 \otimes [\text{pt}]_4(\text{id})$ | formula        |
| $AB = 0$   | formula & sage |
| $AC = 0$   | formula & sage |
| $AD = 0$   | formula & sage |
| $AE = 0$   | formula & sage |
| $AF = 0$   | formula        |
| $AG = 0$   | formula        |
| $AH = 0$   | formula        |

**(3.7.1)**

$$\begin{aligned}
A^2 &= \left( \sum_{j=1}^{22} \sum_{(123)} (e_j)_{123} \otimes (e_j^\vee)_4(123) \right)^2 \\
&= \sum_{j=1}^{22} \sum_{k=1}^{22} \sum_{(123)} (e_j)_{123} \otimes (e_j^\vee)_4(123) \left( (e_k)_{123} \otimes (e_k^\vee)_4(123) + (e_k)_{123} \otimes (e_k^\vee)_4(132) \right. \\
&\quad + (e_k)_{124} \otimes (e_k^\vee)_3(124) + (e_k)_{124} \otimes (e_k^\vee)_3(142) + (e_k)_{134} \otimes (e_k^\vee)_2(134) \\
&\quad \left. + (e_k)_{134} \otimes (e_k^\vee)_2(143) + (e_k)_{234} \otimes (e_k^\vee)_1(234) + (e_k)_{234} \otimes (e_k^\vee)_1(243) \right) \\
&= \sum_{j=1}^{22} \sum_{k=1}^{22} \sum_{(123)} (e_j)_{123} \otimes (e_j^\vee)_4(123) (e_k)_{123} \otimes (e_k^\vee)_4(132) \\
&= \sum_{j=1}^{22} \sum_{k=1}^{22} \sum_{(123)} \Delta(e_j e_k)_{1,2,3} \otimes (e_j^\vee e_k^\vee)_4(\text{id}) \\
&= \sum_{(123)} \sum_{j=1}^{22} \sum_{k=1}^{22} (e_j, e_k) (e_j^\vee, e_k^\vee) [\text{pt}]_1 \otimes [\text{pt}]_2 \otimes [\text{pt}]_3 \otimes [\text{pt}]_4(\text{id}) \\
&= \sum_{(123)} 22 [\text{pt}]_1 \otimes [\text{pt}]_2 \otimes [\text{pt}]_3 \otimes [\text{pt}]_4(\text{id}) \\
&= 8 \cdot 22 [\text{pt}]_1 \otimes [\text{pt}]_2 \otimes [\text{pt}]_3 \otimes [\text{pt}]_4(\text{id})
\end{aligned}$$

as  $g((123), (123)) = -24[\text{pt}]_{123} \otimes 1_4$  while all the other defects are trivial.

**(3.7.2)** All other products are zero as the only terms with nonzero product in middle cohomology are of the form

$$A^\sigma \cdot \sigma \otimes A^{\sigma^{-1}} \cdot \sigma^{-1} \rightarrow A^{\text{id}} \cdot \text{id}$$

Similarly below.

**3.8. B. Summary:**

|  |                |
|--|----------------|
| $B^2 = 0$  | formula        |
| $BC = 8[\text{pt}]_1 \otimes [\text{pt}]_2 \otimes [\text{pt}]_3 \otimes [\text{pt}]_4(\text{id})$ | sage           |
| $BD = 0$   | formula & sage |
| $BE = 0$   | formula & sage |
| $BF = 0$   | formula & sage |
| $BG = 0$   | formula & sage |
| $BH = 0$   | formula        |

(3.8.1)

$$\begin{aligned} B^2 &= \left( \sum_{(123)} 1_{123} \otimes [\text{pt}]_4(123) \right)^2 \\ &= \sum_{(123)} 1_{123} \otimes [\text{pt}]_4(123) (1_{123} \otimes [\text{pt}]_4(123) + 1_{123} \otimes [\text{pt}]_4(132) + 1_{124} \otimes [\text{pt}]_3(124) \\ &\quad + 1_{124} \otimes [\text{pt}]_3(142) + 1_{134} \otimes [\text{pt}]_2(134) + 1_{134} \otimes [\text{pt}]_2(143) + 1_{234} \otimes [\text{pt}]_1(234) + 1_{234} \otimes [\text{pt}]_1(243)) \\ &= 0 \end{aligned}$$

3.9. *C*. Summary:

|           |                |
|-----------|----------------|
| $C^2 = 0$ | formula & sage |
| $CD = 0$  | formula & sage |
| $CE = 0$  | formula & sage |
| $CF = 0$  | formula & sage |
| $CG = 0$  | formula & sage |
| $CH = 0$  | formula        |

3.10. *D*. Summary:

|   |         |
|---|---------|
| $D^2 = 6[\text{pt}]_1 \otimes [\text{pt}]_2 \otimes [\text{pt}]_3 \otimes [\text{pt}]_4(\text{id})$ | sage    |
| $DE = 0$  | formula |
| $DF = 0$  | formula |
| $DG = 0$  | formula |
| $DH = 0$  | formula |

3.11. *E*. Summary:

|  |                |
|--|----------------|
| $E^2 = 66[\text{pt}]_1 \otimes [\text{pt}]_2 \otimes [\text{pt}]_3 \otimes [\text{pt}]_4(\text{id})$ | formula & sage |
| $EF = 0$   | formula        |
| $EG = 0$   | formula        |
| $EH = 0$   | formula        |

(3.11.1)

$$\begin{aligned} E^2 &= \left( \sum_{j=1}^{22} \sum_{(12)(34)} (e_j)_{12} \otimes (e_j^\vee)_{34}(12)(34) \right)^2 \\ &= \sum_{j=1}^{22} \sum_{k=1}^{22} \sum_{(12)(34)} (e_j)_{12} \otimes (e_j^\vee)_{34}(12)(34) \times \\ &\quad \times ((e_k)_{12} \otimes (e_k^\vee)_{34}(12)(34) + (e_k)_{13} \otimes (e_k^\vee)_{24}(13)(24) + (e_k)_{14} \otimes (e_k^\vee)_{23}(14)(23)) \\ &= \sum_{j=1}^{22} \sum_{k=1}^{22} \sum_{(12)(34)} (e_j)_{12} \otimes (e_j^\vee)_{34}(12)(34) (e_k)_{12} \otimes (e_k^\vee)_{34}(12)(34) \\ &= \sum_{j=1}^{22} \sum_{k=1}^{22} \sum_{(12)(34)} \Delta(e_j e_k)_{1,2} \otimes \Delta(e_j^\vee e_k^\vee)_{3,4}(\text{id}) \\ &= \sum_{j=1}^{22} \sum_{k=1}^{22} \sum_{(12)(34)} r_{jk} t_{jk} [\text{pt}]_1 \otimes [\text{pt}]_2 \otimes [\text{pt}]_3 \otimes [\text{pt}]_4(\text{id}) \\ &= 3 \sum_{j=1}^{22} \sum_{k=1}^{22} t_{jk} r_{jk} [\text{pt}]_1 \otimes [\text{pt}]_2 \otimes [\text{pt}]_3 \otimes [\text{pt}]_4(\text{id}) \\ &= 66 [\text{pt}]_1 \otimes [\text{pt}]_2 \otimes [\text{pt}]_3 \otimes [\text{pt}]_4(\text{id}) \end{aligned}$$

3.12. *F*. Summary:

$$\begin{aligned} F^2 &= 6[\text{pt}]_1 \otimes [\text{pt}]_2 \otimes [\text{pt}]_3 \otimes [\text{pt}]_4(\text{id}) && \text{sage} \\ FG &= 0 && \text{formula} \\ FH &= 0 && \text{formula} \end{aligned}$$

3.13. *G*. Summary:

$$\begin{aligned} G^2 &= 264[\text{pt}]_1 \otimes [\text{pt}]_2 \otimes [\text{pt}]_3 \otimes [\text{pt}]_4(\text{id}) && \text{formula} \\ GH &= 0 && \text{formula} \end{aligned}$$

(3.13.1)

$$\begin{aligned} G^2 &= \left( \sum_{j=1}^{22} \sum_{1,(23)} [pt]_1 \otimes (e_j)_2 \otimes (e_j^\vee)_3 \otimes 1_4(\text{id}) \right)^2 \\ &= \sum_{j=1}^{22} \sum_{k=1}^{22} \sum_{1,(23)} \sum_{a,(bc)} [pt]_1 \otimes (e_j)_2 \otimes (e_j^\vee)_3 \otimes 1_4(\text{id}) [pt]_a \otimes (e_k)_b \otimes (e_k^\vee)_c \otimes 1_a(\text{id}) \\ &= \sum_{j=1}^{22} \sum_{k=1}^{22} \sum_{1,(23)} [pt]_1 \otimes (e_j)_2 \otimes (e_j^\vee)_3 \otimes 1_4(\text{id}) 1_1 \otimes (e_k)_2 \otimes (e_k^\vee)_3 \otimes [pt]_4(\text{id}) \\ &= \sum_{j=1}^{22} \sum_{k=1}^{22} \sum_{1,(23)} [pt]_1 \otimes (e_j e_k)_2 \otimes (e_j^\vee e_k^\vee)_3 \otimes [pt]_4(\text{id}) \\ &= \sum_{1,(23)} 22 [pt]_1 \otimes [pt]_2 \otimes [pt]_3 \otimes [pt]_4(\text{id}) \\ &= 12 \cdot 22 [pt]_1 \otimes [pt]_2 \otimes [pt]_3 \otimes [pt]_4(\text{id}) \end{aligned}$$

since  $\alpha_1 \otimes \beta_1 \otimes \gamma_1 \otimes \delta_1(\text{id}) \alpha_2 \otimes \beta_2 \otimes \gamma_2 \otimes \delta_2(\text{id}) = \alpha_1 \alpha_2 \otimes \beta_1 \beta_2 \otimes \gamma_1 \gamma_2 \otimes \delta_1 \delta_2(\text{id})$ . The second to last line in the equation follows as before.

3.14. *H*. Summary:

$$H^2 = 1584 [pt]_1 \otimes [pt]_2 \otimes [pt]_3 \otimes [pt]_4(\text{id}) \quad \text{formula}$$

(3.14.1)

$$\begin{aligned}
H^2 &= \left( \sum_{j=1}^{22} \sum_{k=1}^{22} \sum_{(12)(34)} (e_j)_1 \otimes (e_j^\vee)_2 \otimes (e_k)_3 \otimes (e_k^\vee)_4(\text{id}) \right)^2 \\
&= \sum_{j=1}^{22} \sum_{k=1}^{22} \sum_{u=1}^{22} \sum_{v=1}^{22} \sum_{(12)(34)} \sum_{(ab)(cd)} (e_j)_1 \otimes (e_j^\vee)_2 \otimes (e_k)_3 \otimes (e_k^\vee)_4(\text{id}) (e_u)_a \otimes (e_u^\vee)_b \otimes (e_v)_c \otimes (e_v^\vee)_d(\text{id}) \\
&= \sum_{j=1}^{22} \sum_{k=1}^{22} \sum_{u=1}^{22} \sum_{v=1}^{22} \sum_{(12)(34)} ((e_j e_u)_1 \otimes (e_j^\vee e_u^\vee)_2 \otimes (e_k e_v)_3 \otimes (e_k^\vee e_v^\vee)_4)(\text{id}) \\
&\quad + (e_j e_u)_1 \otimes (e_j^\vee e_v)_2 \otimes (e_k e_u^\vee)_3 \otimes (e_k^\vee e_v^\vee)_4(\text{id}) \\
&\quad + (e_j e_u)_1 \otimes (e_j^\vee e_v^\vee)_2 \otimes (e_k e_v)_3 \otimes (e_k^\vee e_u^\vee)_4(\text{id}) \\
&= \sum_{j=1}^{22} \sum_{k=1}^{22} \sum_{u=1}^{22} \sum_{v=1}^{22} \sum_{(12)(34)} (a_{ju} a_{ju}^\vee a_{kv} a_{kv}^\vee + a_{ju} a_{kv}^\vee \delta_{j=v} \delta_{k=u} + a_{ju} a_{ju}^\vee a_{kv} a_{ku}^\vee) \\
&= \sum_{j=1}^{22} \sum_{k=1}^{22} \sum_{(12)(34)} a_{jk} a_{kj}^\vee + \sum_{j=1}^{22} \sum_{k=1}^{22} \sum_{u=1}^{22} \sum_{v=1}^{22} \sum_{(12)(34)} (a_{ju} a_{ju}^\vee a_{kv} a_{kv}^\vee + a_{ju} a_{ju}^\vee a_{kv} a_{ku}^\vee) \\
&= \sum_{j=1}^{22} \sum_{k=1}^{22} \sum_{(12)(34)} a_{jk} a_{kj}^\vee + \sum_{(12)(34)} \left( \sum_{j=1}^{22} \sum_{u=1}^{22} a_{ju} a_{ju}^\vee \right)^2 + \sum_{j=1}^{22} \sum_{k=1}^{22} \sum_{u=1}^{22} \sum_{v=1}^{22} \sum_{(12)(34)} a_{ju} a_{ju}^\vee a_{kv} a_{ku}^\vee \\
&= \sum_{(12)(34)} (22 + 22^2 + 22) \\
&= 3 \cdot (2 \cdot 22 + 22^2) [\text{pt}]_1 \otimes [\text{pt}]_2 \otimes [\text{pt}]_3 \otimes [\text{pt}]_4(\text{id}) \\
&= 1584 [\text{pt}]_1 \otimes [\text{pt}]_2 \otimes [\text{pt}]_3 \otimes [\text{pt}]_4(\text{id})
\end{aligned}$$

since

$$\sum_{j=1}^{22} \sum_{k=1}^{22} a_{jk} a_{kj}^\vee = \text{Tr } I_{22} = 22$$

and

$$\begin{aligned}
\sum_{j=1}^{22} \sum_{k=1}^{22} \sum_{u=1}^{22} \sum_{v=1}^{22} a_{ju} a_{jv}^{\vee} a_{kv} a_{ku}^{\vee} &= \sum_{j=1}^{22} \sum_{k=1}^{22} \sum_{u=1}^{22} \sum_{v=1}^{22} (e_j, e_u) (e_j^{\vee}, e_v^{\vee}) (e_k, e_v) (e_k^{\vee}, e_u^{\vee}) \\
&= \sum_{j=1}^{22} \sum_{k=1}^{22} \sum_{u=1}^{22} \sum_{v=1}^{22} r_{ju} t_{jv} r_{kv} t_{ku} \\
&= \sum_{j=1}^{22} \sum_{k=1}^{22} \sum_{u=1}^{22} r_{ju} t_{ku} \left( \sum_{v=1}^{22} t_{jv} r_{vk} \right) \\
&= \sum_{j=1}^{22} \sum_{k=1}^{22} \sum_{u=1}^{22} r_{ju} t_{ku} \delta_{j=k} \\
&= \sum_{j=1}^{22} \sum_{u=1}^{22} r_{ju} t_{ju} \\
&= 22
\end{aligned}$$

#### 4. HODGE CLASSES OF $S^{[4]}$

We include here the details to the proofs of the statements in [BJ11, §2.7].

(4.1)  $c_2(S^{[4]}) = 3Z + 33Y - W$ , *cf.* [BJ11, Lemma 2.9].

We had

$$c_2(S^{[4]}) = \left( -\frac{4}{9}u - \frac{4}{27}v \right) W + \left( u - \frac{21}{4} \right) X + (v + 42)Y - \frac{v}{3}Z$$

This yields:

$$\begin{aligned}
945300 &= \theta^2 c_2(S^{[4]})^2 = \frac{1117}{2}(v+42)^2 - 16(v+42)(4u-21) + \frac{776}{27}(v+42)(3u+v) \\
&\quad - \frac{2365}{3}(v+42)v + \frac{611}{4}(4u-21)^2 - \frac{848}{27}(4u-21)(3u+v) \\
&\quad + \frac{220}{3}(4u-21)v + \frac{23488}{243}(3u+v)^2 - \frac{5720}{81}(3u+v)v + \frac{19525}{18}v^2 \\
&= \frac{79300}{27}u^2 + \frac{45200}{81}uv + \frac{221050}{243}v^2 - 22750u + \frac{43400}{3}v + \frac{4266675}{4} \\
-246600 &= \theta \delta^2 c_2(S^{[4]}) = -95(v+42)^2 + 52(v+42)(4u-21) - \frac{848}{9}(v+42)(3u+v) \\
&\quad + \frac{814}{3}(v+42)v - \frac{139}{2}(4u-21)^2 + \frac{992}{9}(4u-21)(3u+v) \\
&\quad - \frac{748}{3}(4u-21)v - \frac{7552}{81}(3u+v)^2 + \frac{6512}{27}(3u+v)v - \frac{2519}{9}v^2 \\
&= -\frac{5656}{9}u^2 - \frac{12608}{27}uv - \frac{4036}{81}v^2 + 1596u + 1288v - \frac{488187}{2} \\
177552 &= \delta^4 c_2(S^{[4]}) = 42(v+42)^2 - 48(v+42)(4u-21) + 96(v+42)(3u+v) \\
&\quad - 220(v+42)v + 87(4u-21)^2 - 192(4u-21)(3u+v) \\
&\quad + 352(4u-21)v + \frac{1280}{9}(3u+v)^2 - 352(3u+v)v + \frac{638}{3}v^2 \\
&= 368u^2 + \frac{1600}{3}uv - \frac{712}{9}v^2 + 1512u - 4032v + 154791
\end{aligned}$$

There are two solutions,  $(u, v) = (\frac{21}{4}, -9), (\frac{497}{116}, -\frac{285}{29})$ . Finally, we compute

$$\begin{aligned}
1992240 &= c_2(S^{[4]})^4 = \frac{113482}{729}u^4 + \frac{361912}{2187}u^3v + \frac{273944}{2187}u^2v^2 + \frac{67072}{19683}uv^3 - \frac{242858}{81}u^3 \\
&\quad + \frac{556552}{59049}v^4 - \frac{284438}{81}u^2v - \frac{356636}{243}uv^2 + \frac{146951}{4}u^2 + \frac{1193920}{2187}v^3 \\
&\quad + \frac{139013}{6}uv + \frac{476329}{18}v^2 - \frac{1416933}{8}u + \frac{2028159}{8}v + \frac{447500781}{128}
\end{aligned}$$

Which rules out  $(u, v) = (\frac{497}{116}, -\frac{285}{29})$ .

**(4.2)** The class  $\alpha = X - 3Y + Z$  intersects trivially with

$$\lambda^4\theta, \lambda^4c_2(X), \lambda^2\theta^2, \lambda^2\theta c_2(X), \lambda^2c_2(X)^2, \theta^3, \theta^2c_2(X), \theta c_2(X)^2, c_2(X)^3$$

and

$$\begin{aligned}
\alpha^2\theta^2 &= 9450 \\
\alpha^2\theta c_2(X) &= 14148 \\
\alpha^2c_2(X)^2 &= 21168
\end{aligned}$$

*cf.* [BJ11, Corollary 2.11]

We'll demonstrate the last 3 identities. Since

$$\theta = -\frac{1}{2}W - \frac{1}{3}X + \frac{45}{2}Y + \frac{13}{6}Z$$

cf. [BJ11, §1.10], we have

$$\begin{aligned}\alpha^2 &= -3G + 30D + 42F + 3H + 6E \\ \alpha\theta &= 88D + 8E - 27C + 3B - 88F + 20G + 4H \\ \alpha c_2(S^{[4]}) &= -54C + 132D + 6B + 12E + 30G + 6H - 132F \\ \theta^2 &= -8E + \frac{19}{2}H + \frac{215}{2}G - 64D - \frac{33}{4}A - \frac{97}{4}B + 1117F - \frac{873}{4}C \\ \theta c_2(S^{[4]}) &= -\frac{27}{2}A - \frac{747}{2}C - \frac{83}{2}B - 8E + 1630F + 153G + 13H - 48D \\ c_2(S^{[4]})^2 &= 18H - 8E - 69B - 8D + 218G - 21A - 621C + 2380F\end{aligned}$$

and the pairwise products are easily computed.

## 5. FUJIKI CONSTANTS OF $X$

For the sake of completeness we describe the computation of the integrals

$$\int_{S^{[n]}} \delta^k c_\mu(S^{[n]})$$

for  $S = \mathbb{P}^2, \mathbb{P}^1 \times \mathbb{P}^1$  and  $\delta = \det \mathcal{O}^{[n]}$  by toric localization, cf. [BJ11, Section 2.1].

**(5.1)** First consider  $S = \mathbb{A}^2$ , which has an action by  $G = \mathbb{G}_m^2$  via  $(x, y) \mapsto (\lambda x, \mu y)$  where  $\lambda, \mu$  are the characters obtained by projecting to each factor. The only fixed point is the origin  $(0, 0)$ .  $G$  also acts on  $(\mathbb{A}^2)^{[n]}$ ; fixed points are length  $n$  subschemes  $Z$  fixed by  $G$ . Thus, they must be supported on a fixed point (i.e. the origin), and the ideal  $I_Z \subset A = \mathbb{C}[x, y]$  must be generated by monomials.  $I_Z$  is determined by the monomials  $x^a y^b$  left out of the ideal, which form a Young tableau with  $n$  boxes. Given such a Young tableau in the upper right quadrant, let  $(i, b_i - 1)$  for  $0 \leq i \leq n - 1$  be the extremal boxes, so  $b_i$  is the height of the  $i$ th column. A partition  $\mu$  of  $n$  uniquely determines a Young tableau by arranging  $\mu_i$  columns of height  $i$  in descending order.

**(5.2)** For a space  $X$  with an action by  $G$  with isolated fixed points, Bott localization implies

$$\int_X \varphi = \sum_{p \in X^G} \int \frac{i_p^* \varphi}{c_{\text{top}}(T_p X)}$$

where  $\varphi \in H_G^*(X)$ ,  $i_p^* : H_G^*(X) \rightarrow H_G^*(X^G) \cong H^*(X^G) \otimes H_G^*([\text{pt}])$  is the pull-back to a fixed point  $p \in X^G$ . The Chern class is the equivariant chern class of the  $G$  representation  $T_p X$ .

**(5.3)** For a partition  $\mu$  representing a fixed point  $p_\mu$  of  $X = (\mathbb{A}^2)^{[n]}$ , the Chern polynomial is [ES87, Lemma 3.2]

$$C(\mu; \alpha, \beta) := \sum_i t^i c_{2n-i}(T_{p_\mu} X) = \prod_{1 \leq i \leq j \leq n} \prod_{s=b_j}^{b_{j-1}-1} (t+(i-j-1)\alpha+(b_{i-1}-s-1)\beta)(t+(j-i)\alpha+(s-b_{i-1})\beta) \quad (5.1)$$

where  $\alpha = c_1(\lambda), \beta = c_1(\mu)$ .  $\mathcal{O}^{[n]}$  restricted to a point of  $\mathbb{A}^{[n]}$  corresponding to a subscheme  $Z$  is canonically  $\mathcal{O}_Z$ , so setting  $f = c_1(\mathcal{O}^{[n]})$ ,

$$Z(\underline{\mu}; \alpha, \beta) := i_{p_\mu}^* f = \sum_{i=0}^n \sum_{j=0}^{b_i-1} i\alpha + j\beta \quad (5.2)$$

(5.4) For  $S = \mathbb{P}^2$ , let  $\mathbb{G}_m^2$  act on  $[x, y, z]$  via  $[\lambda x, \mu y, z]$ . There are three fixed points  $p_0 = [0, 0, 1], p_1 = [0, 1, 0], p_2 = [1, 0, 0]$ , and a length  $n$  subscheme  $Z$  of  $\mathbb{P}^2$  will consist of a length  $n_i$  subscheme  $Z_i$  at  $p_i$  with  $\sum n_i = n$ . The tangent space at such a point is canonically

$$T_Z(\mathbb{P}^2)^{[n]} = \bigoplus_i T_{Z_i}(\mathbb{P}^2)^{[n_i]}$$

Note that at any point  $[Z] \in (\mathbb{P}^2)^{[n]}$  corresponding to a subscheme  $Z$  supported at  $p_i$ , there is a  $\mathbb{G}_m^2$ -stable Zariski neighborhood isomorphic to  $\mathbb{A}^{[n]}$  with torus action via  $(\lambda x, \mu y), (\lambda \mu^{-1} x, \mu^{-1} y), (\mu \lambda^{-1} x, \lambda^{-1} y)$  for  $i = 0, 1, 2$  respectively.

(5.5) A 3-vector partition  $\underline{\mu}$  of  $n$  will be three partitions  $(\mu_1, \mu_2, \mu_3)$  such that  $|\mu_1| + |\mu_2| + |\mu_3| = n$ ; 3-vector partitions of  $n$  classify fixed points  $p_{\underline{\mu}}$  of  $X = (\mathbb{P}^2)^{[n]}$ . By the above, the tangent space at  $p_{\underline{\mu}}$  has Chern polynomial

$$\sum t^i C_{2n-i}(\underline{\mu}; \alpha, \beta) = C(\mu_1; \alpha, \beta) C(\mu_2; \alpha - \beta, -\beta) C(\mu_3; \beta - \alpha, -\alpha) \quad (5.3)$$

Define  $C_i(\underline{\mu}; \alpha, \beta) = c_i(T_{p_{\underline{\mu}}} X)$ . Also,

$$Z(\underline{\mu}; \alpha, \beta) := i_{p_{\underline{\mu}}}^* f = Z(\mu_1; \alpha, \beta) + Z(\mu_2; \alpha - \beta, -\beta) + Z(\mu_3; \beta - \alpha, -\alpha) \quad (5.4)$$

The final answer is then, for  $X = (\mathbb{P}^2)^{[n]}$

$$\int_X f^{2n - \sum_i k_i} \prod_i c_{k_i}(TX) = \sum_{\underline{\mu}, |\underline{\mu}|=n} \frac{Z(\underline{\mu}; \alpha, \beta)^{2n - \sum_i k_i} \prod_i C_{k_i}(\underline{\mu}; \alpha, \beta)}{C_{2n}(\underline{\mu}; \alpha, \beta)} \quad (5.5)$$

(5.6) Let  $\mathbb{G}_m^2$  act on  $S = \mathbb{P}^1 \times \mathbb{P}^1$  via  $[\lambda x_1, y_1] \times [\mu x_2, y_2]$ . The fixed points are classified by 4-vector partitions  $\underline{\mu}$ . Now we have

$$\sum t^i C'_{2n-i}(\underline{\mu}; \alpha, \beta) = C(\mu_1; \alpha, \beta) C(\mu_2; -\alpha, \beta) C(\mu_3; \alpha, -\beta) C(\mu_4; -\alpha, -\beta) \quad (5.6)$$

Also,

$$Z'(\underline{\mu}; \alpha, \beta) := i_{p_{\underline{\mu}}}^* f = Z(\mu_1; \alpha, \beta) + Z(\mu_2; -\alpha, \beta) + Z(\mu_3; \alpha, -\beta) + Z(\mu_4; -\alpha, -\beta) \quad (5.7)$$

The final answer is then once again

$$\int_{(\mathbb{P}^1 \times \mathbb{P}^1)^{[n]}} f^{2n - \sum_i k_i} \prod_i c_{k_i}(TX) = \sum_{\underline{\mu}, |\underline{\mu}|=n} \frac{Z'(\underline{\mu}; \alpha, \beta)^{2n - \sum_i k_i} \prod_i C'_{k_i}(\underline{\mu}; \alpha, \beta)}{C'_{2n}(\underline{\mu}; \alpha, \beta)} \quad (5.8)$$

(5.7) Let  $\Phi$  be the universal genus, cf. [BJ11, §2.1]. We have

$$\sum_{n \geq 0} z^n \int_{S^{[n]}} \exp \det(\mathcal{O}^{[n]}) \Phi(S^{[n]}) = \mathbf{A}(z)^{c_1(S)^2} \mathbf{B}(z)^{c_2(S)}$$

(5.8) The universal genus for vanishing odd Chern classes up to degree 8 is:

$$\begin{aligned}
\Phi = & 1 + (a_1^2 - 2a_2)c_2 \\
& + (a_2^2 - 2a_1a_3 + 2a_4)c_2^2 \\
& + (a_1^4 - 4a_1^2a_2 + 2a_2^2 + 4a_1a_3 - 4a_4)c_4 \\
& + (a_3^2 - 2a_2a_4 + 2a_1a_5 - 2a_6)c_2^3 \\
& + (a_1^2a_2^2 - 2a_1^3a_3 - 2a_2^3 + 4a_1a_2a_3 + 2a_1^2a_4 - 3a_3^2 + 2a_2a_4 - 6a_1a_5 + 6a_6)c_2c_4 \\
& + (a_1^6 - 6a_1^4a_2 + 9a_1^2a_2^3 + 6a_1^3a_3 - 2a_2^3 - 12a_1a_2a_3 - 6a_1^2a_4 + 3a_3^2 + 6a_2a_4 + 6a_1a_5 - 6a_6)c_6 \\
& + (a_4^2 - 2a_3a_5 + 2a_2a_6 - 2a_1a_7 + 2a_8)c_2^4 \\
& + (a_1^2a_3^2 - 2a_1^2a_2a_4 + 2a_1^3a_5 - 2a_2a_3^2 + 4a_2^2a_4 - 4a_1a_2a_5 \\
& - 2a_1^2a_6 - 4a_4^2 + 8a_3a_5 - 4a_2a_6 + 8a_1a_7 - 8a_8)c_2^2c_4 \\
& + (a_1^4a_2^2 - 2a_1^5a_3 - 4a_1^2a_2^3 + 8a_1^3a_2a_3 + 2a_1^4a_4 + 2a_2^4 - 9a_1^2a_3^2 \\
& - 6a_1^2a_2a_4 - 2a_1^3a_5 + 2a_2a_3^2 - 4a_2^2a_4 + 16a_1a_3a_4 + 4a_1a_2a_5 \\
& + 2a_1^2a_6 - 4a_4^2 - 8a_3a_5 + 4a_2a_6 - 8a_1a_7 + 8a_8)c_2c_6 \\
& + (a_2^4 - 4a_1a_2^2a_3 + 2a_1^2a_3^2 + 4a_1^2a_2a_4 - 4a_1^3a_5 + 4a_2a_3^2 - 4a_2^2a_4 - 8a_1a_3a_4 \\
& + 8a_1a_2a_5 + 4a_1^2a_6 + 6a_4^2 - 4a_3a_5 - 4a_2a_6 - 4a_1a_7 + 4a_8)c_4^2 \\
& + (a_1^8 - 8a_1^6a_2 + 20a_1^4a_2^2 + 8a_1^5a_3 - 16a_1^2a_2^3 - 32a_1^3a_2a_3 - 8a_1^4a_4 + 2a_2^4 \\
& + 24a_1a_2^2a_3 + 12a_1^2a_3^2 + 24a_1^2a_2a_4 + 8a_1^3a_5 - 8a_2a_3^2 - 8a_2^2a_4 - 16a_1a_3a_4 \\
& - 16a_1a_2a_5 - 8a_1^2a_6 + 4a_4^2 + 8a_3a_5 + 8a_2a_6 + 8a_1a_7 - 8a_8)c_8
\end{aligned}$$

(5.9) By localization, we compute:

$$\begin{aligned}
\mathbf{A}(z) = & 1 + a_2 z + \left( -a_1^3 + 3a_1^2 a_2 + \frac{1}{4}a_1^2 + a_1 a_2 - \frac{9}{2}a_2^2 + a_1 a_3 + \frac{1}{6}a_1 - \frac{3}{2}a_2 + 3a_3 - 10a_4 - \frac{1}{48} \right) z^2 \\
& + \left( -3a_1^5 + 4a_1^4 a_2 + \frac{5}{2}a_1^4 + 5a_1^3 a_2 - \frac{23}{3}a_1^2 a_2^2 - \frac{8}{3}a_1^3 a_3 + 5a_1^3 - \frac{87}{4}a_1^2 a_2 - 21a_1 a_2^2 + \frac{179}{6}a_2^3 \right. \\
& + 62a_1^2 a_3 - \frac{125}{3}a_1 a_2 a_3 - \frac{160}{3}a_1^2 a_4 - \frac{19}{12}a_1^2 - \frac{22}{3}a_1 a_2 + 39a_2^2 - 19a_1 a_3 - 44a_2 a_3 - \frac{56}{3}a_3^2 \\
& \left. - 11a_1 a_4 + 138a_2 a_4 - \frac{64}{3}a_1 a_5 - \frac{17}{40}a_1 + \frac{223}{48}a_2 - \frac{15}{2}a_3 + 37a_4 - 55a_5 + \frac{364}{3}a_6 + \frac{19}{240} \right) z^3 \\
& + \left( -7a_1^7 + 7a_1^6 a_2 + 12a_1^6 + 15a_1^5 a_2 - \frac{37}{2}a_1^4 a_2^2 - 9a_1^5 a_3 + \frac{159}{4}a_1^5 - \frac{507}{4}a_1^4 a_2 - \frac{281}{2}a_1^3 a_2^2 \right. \\
& + \frac{827}{6}a_1^2 a_2^3 + 370a_1^4 a_3 - \frac{623}{3}a_1^3 a_2 a_3 - 150a_1^4 a_4 - \frac{2461}{96}a_1^4 - \frac{451}{4}a_1^3 a_2 + \frac{3563}{8}a_1^2 a_2^2 + \frac{353}{2}a_1 a_2^3 \\
& - \frac{4663}{24}a_2^4 - \frac{1007}{4}a_1^3 a_3 - 719a_1^2 a_2 a_3 + \frac{2147}{6}a_1 a_2^2 a_3 - \frac{21}{2}a_1^2 a_3^2 - 67a_1^3 a_4 + \frac{1799}{3}a_1^2 a_2 a_4 \\
& + 2a_1^3 a_5 - \frac{1403}{80}a_1^3 + \frac{6535}{48}a_1^2 a_2 + \frac{717}{4}a_1 a_2^2 - \frac{2495}{4}a_2^3 - \frac{4045}{12}a_1^2 a_3 + \frac{1501}{2}a_1 a_2 a_3 + \frac{1153}{2}a_2^2 a_3 \\
& - 840a_1 a_3^2 + \frac{1027}{3}a_2 a_3^2 + \frac{1033}{2}a_1^2 a_4 + 741a_1 a_2 a_4 - 1611a_2^2 a_4 + 686a_1 a_3 a_4 - 1147a_1^2 a_5 \\
& + \frac{1424}{3}a_1 a_2 a_5 + 973a_1^2 a_6 + \frac{18001}{2880}a_1^2 + \frac{2291}{80}a_1 a_2 - \frac{6689}{32}a_2^2 + \frac{1509}{16}a_1 a_3 + \frac{793}{3}a_2 a_3 \\
& + 235a_3^2 + \frac{322}{3}a_1 a_4 - 1597a_2 a_4 + 305a_3 a_4 - 782a_4^2 + 431a_1 a_5 + 1096a_2 a_5 + 633a_3 a_5 \\
& + 127a_1 a_6 - \frac{7148}{3}a_2 a_6 + 383a_1 a_7 + \frac{10061}{10080}a_1 - \frac{7219}{480}a_2 + \frac{1403}{80}a_3 - \frac{1037}{8}a_4 + \frac{545}{3}a_5 \\
& \left. - 731a_6 + 889a_7 - 1608a_8 - \frac{37861}{161280} \right) z^4
\end{aligned}$$

(5.10)

$$\begin{aligned}
\mathbf{B}(z) = & 1 + (a_1^2 - 2a_2)z + \left( 2a_1^4 - 8a_1^2a_2 - \frac{5}{4}a_1^2 + \frac{31}{2}a_2^2 - 15a_1a_3 + \frac{5}{2}a_2 + 15a_4 + \frac{1}{48} \right) z^2 \\
& + \left( 3a_1^6 - 18a_1^4a_2 - \frac{45}{4}a_1^4 + \frac{461}{6}a_1^2a_2^2 - \frac{245}{3}a_1^3a_3 + 45a_1^2a_2 - \frac{317}{3}a_2^3 + \frac{490}{3}a_1a_2a_3 \right. \\
& + \frac{245}{3}a_1^2a_4 + \frac{121}{48}a_1^2 - 65a_2^2 + 40a_1a_3 + \frac{280}{3}a_3^2 - 350a_2a_4 + \frac{560}{3}a_1a_5 - \frac{121}{24}a_2 - 40a_4 \\
& \left. - \frac{560}{3}a_6 - \frac{1}{18} \right) z^3 \\
& + \left( 5a_1^8 - 40a_1^6a_2 - 50a_1^6 + \frac{1565}{6}a_1^4a_2^2 - \frac{845}{3}a_1^5a_3 + 300a_1^4a_2 - \frac{2170}{3}a_1^2a_2^3 + \frac{3380}{3}a_1^3a_2a_3 \right. \\
& + \frac{845}{3}a_1^4a_4 + \frac{4207}{96}a_1^4 - \frac{7375}{8}a_1^2a_2^2 + \frac{19265}{24}a_2^4 + \frac{2575}{4}a_1^3a_3 - \frac{10585}{6}a_1a_2^2a_3 + \frac{7955}{6}a_1^2a_3^2 \\
& - \frac{7510}{3}a_1^2a_2a_4 + \frac{4130}{3}a_1^3a_5 - \frac{4207}{24}a_1^2a_2 + \frac{4175}{4}a_2^3 - \frac{2575}{2}a_1a_2a_3 - \frac{4130}{3}a_2a_3^2 - \frac{2575}{4}a_1^2a_4 \\
& + \frac{9035}{2}a_2^2a_4 - 1275a_1a_3a_4 - \frac{8260}{3}a_1a_2a_5 - \frac{4130}{3}a_1^2a_6 - \frac{3455}{576}a_1^2 + \frac{21883}{96}a_2^2 - \frac{1685}{16}a_1a_3 \\
& - \frac{735}{2}a_3^2 + \frac{4045}{2}a_2a_4 + 1950a_4^2 - 735a_1a_5 - 2625a_3a_5 + \frac{16135}{3}a_2a_6 - 2625a_1a_7 + \frac{3455}{288}a_2 \\
& \left. + \frac{1685}{16}a_4 + 735a_6 + 2625a_8 + \frac{625}{4608} \right) z^4
\end{aligned}$$

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