

# LAGRANGIAN HYPERPLANES IN HOLOMORPHIC SYMPLECTIC VARIETIES

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ABSTRACT. We classify the cohomology classes of Lagrangian hyperplanes  $\mathbb{P}^4$  in a smooth manifold  $X$  deformation equivalent to a Hilbert scheme of 4 points on a K3 surface, up to the monodromy action. Classically, the cone of effective curves on a K3 surface  $S$  is generated by nonnegative classes  $C$ , for which  $(C, C) \geq 0$ , and nodal classes  $C$ , for which  $(C, C) = -2$ ; Hassett and Tschinkel conjecture that the cone of effective curves on a holomorphic symplectic variety  $X$  is similarly controlled by “nodal” classes  $C$  such that  $(C, C) = -\gamma$ , for  $(\cdot, \cdot)$  now the Beauville-Bogomolov form, where  $\gamma$  classifies the geometry of the extremal contraction associated to  $C$ . In particular, they conjecture that for  $X$  deformation equivalent to a Hilbert scheme of  $n$  points on a K3 surface, the class  $C = \ell$  of a line in a smooth Lagrangian  $n$ -plane  $\mathbb{P}^n$  must satisfy  $(\ell, \ell) = -\frac{n+3}{2}$ . We prove the conjecture for  $n = 4$  by computing the ring of monodromy invariants on  $X$ , and showing there is a unique monodromy orbit of Lagrangian hyperplanes.

## 1. INTRODUCTION

Let  $X$  be an irreducible holomorphic symplectic variety; thus,  $X$  is a smooth projective simply-connected variety whose space  $H^0(\Omega_X^2)$  of global two-forms is generated by a nowhere degenerate form  $\omega$ .  $H^2(X, \mathbb{Z})$  carries a deformation-invariant nondegenerate primitive integral form  $(\cdot, \cdot)$  called the Beauville-Bogomolov form [Bea83]. For  $X = S$  a K3 surface  $(\cdot, \cdot)$  is the intersection form, while for  $X = S^{[n]}$  a Hilbert scheme of  $n > 1$  points on  $S$  we have the orthogonal decomposition [Bea83, §8]

$$H^2(S^{[n]}, \mathbb{Z})_{(\cdot, \cdot)} \cong H^2(S, \mathbb{Z}) \oplus_{\perp} \mathbb{Z}\delta \tag{1.1}$$

where the form on  $H^2(S, \mathbb{Z})$  is the intersection form,  $2\delta$  is the divisor of non-reduced subschemes, and  $(\delta, \delta) = 2 - 2n$ . The embedding of  $H^2(S, \mathbb{Z})$  is achieved via the canonical isomorphism

$$H^2(S, \mathbb{Z}) \cong H^2(\mathrm{Sym}^n S, \mathbb{Z})$$

and pullback along the contraction  $\sigma : S^{[n]} \rightarrow \mathrm{Sym}^n S$ . The inverse of  $(\cdot, \cdot)$  defines a  $\mathbb{Q}$ -valued form on  $H_2(X, \mathbb{Z})$  which we will also denote  $(\cdot, \cdot)$ ; by Poincaré duality, we obtain a decomposition dual to (1.1). For example, the class  $\delta^{\vee} \in H_2(X, \mathbb{Z})$  Poincaré dual to the exceptional divisor  $\delta$  has square  $(\delta^{\vee}, \delta^{\vee}) = \frac{1}{2-2n}$ . The form induces an embedding  $H^2(X, \mathbb{Z}) \subset H_2(X, \mathbb{Z})$  under which the two forms match up, and since the determinant of  $(\cdot, \cdot)$  on  $H^2(X, \mathbb{Z})$  is  $2 - 2n$ , we can write any  $\ell \in H_2(X, \mathbb{Z})$  as  $\ell = \frac{\lambda}{2n-2}$  for some  $\lambda \in H^2(X, \mathbb{Z})$ . We will refer to smallest multiple of  $\ell$  that is in  $H^2(X, \mathbb{Z})$  as the Beauville-Bogomolov dual  $\rho$  of  $\ell$ .

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**Cones of effective curves.** Much of the geometry of a K3 surface  $S$  is encoded in its nodal classes, the indecomposable effective curve classes  $C$  for which  $(C, C) = -2$ . Suppose  $S$  has an ample divisor  $H$ ; let  $N_1(S, \mathbb{Z}) \subset H_2(S, \mathbb{Z})$  be the group of curve classes modulo homological equivalence, and  $\text{NE}_1(S) \subset N_1(S, \mathbb{R}) = N_1(S, \mathbb{Z}) \otimes \mathbb{R}$  the cone of effective curves. Then it is well-known that [LP80, Lemma 1.6]

$$\text{NE}_1(S) = \langle C \in N_1(S, \mathbb{Z}) \mid H \cdot C > 0 \text{ and } C \cdot C \geq -2 \rangle \quad (1.2)$$

By Kleiman’s criterion there is a dual statement for the ample cone; here by  $\langle \dots \rangle$  we mean “the cone generated by  $\dots$ ”.

Hassett and Tschinkel [HT10b] conjecture that the cone of effective curves in a holomorphic symplectic variety  $X$  is similarly determined intersection theoretically by the Beauville-Bogomolov form:

**Conjecture 1.** [HT10b, Thesis 1.1] *Let  $X$  be an irreducible holomorphic symplectic variety with polarization  $H$ . Then there is a (positive) rational constant  $c_X$  dependent only on the deformation class of  $X$  such that*

$$\text{NE}_1(X) = \langle C \in N_1(X, \mathbb{Z}) \mid H \cdot C > 0 \text{ and } (C, C) \geq -c_X \rangle$$

Further, if  $X$  contains a smoothly embedded Lagrangian hyperplane  $\mathbb{P}^n \subset X$ , and  $\ell \in \text{NE}_1(X)$  is the class of the line in  $\mathbb{P}^n$ , then the bound is realized:

$$(\ell, \ell) = -c_X$$

*Remark 1.* In fact, there is a more precise statement of (1.2): the cone of effective curves is generated by the closure  $\overline{C}_+$  of the positive cone and the nodal cone  $C_{-2}$ ,

$$\begin{aligned} \text{NE}_1(S) = & \langle C \in N_1(S, \mathbb{Z}) \mid H \cdot C > 0 \text{ and } C \cdot C \geq 0 \rangle \\ & + \langle C \in N_1(S, \mathbb{Z}) \mid H \cdot C > 0 \text{ and } C \cdot C = -2 \rangle \end{aligned}$$

A nodal class is necessarily represented by a smooth rational  $\mathbb{P}^1 \subset S$ , and generates an extremal ray in  $\text{NE}_1(S)$  whose associated extremal contraction is the contraction of  $\mathbb{P}^1$  to an isolated singularity.

Likewise, there is a finer version of the conjecture (see [HT01, §3] for the case of fourfolds), predicting that the cone of effective curves on  $X$  is the sum of the closure of the positive cone

$$\overline{C}_+ = \langle C \in N_1(X, \mathbb{Z}) \mid H \cdot C > 0 \text{ and } (C, C) \geq 0 \rangle$$

and cones of the form

$$C_{-\gamma} = \langle C \in N_1(X, \mathbb{Z}) \mid H \cdot C > 0 \text{ and } (C, C) = -\gamma \rangle$$

Further, we expect indecomposable curve classes in each  $C_{-\gamma}$ —“nodal classes of type  $-\gamma$ ”—to generate extremal rays corresponding to extremal contractions of the same geometry. For example, all curve classes that are lines in a Lagrangian hyperplane will belong to a single  $C_{-\gamma}$ . By Conjecture 1, Lagrangian hyperplanes play a particularly important role by picking out the largest value of  $\gamma$  that appears in the decomposition.

The sufficiency of the intersection theoretic criterion in Conjecture 1 has been worked out in full detail *à la* Remark 1 for  $X$  deformation equivalent to a Hilbert scheme of 2 points on a K3 surface [HT09, Theorem 1]. In this case, there are three “nodal” cones that appear,  $C_{-1/2}$ ,  $C_{-2}$ ,  $C_{-5/2}$ , and their extremal rays correspond to the 2 types of extremal contractions:

- (i) Divisorial extremal contractions. In this case, the exceptional divisor  $E$  is contracted to a  $K3$  surface  $T$ . The generic fiber over  $T$  is either an  $A_1$  or  $A_2$  configuration of rational curves [HT09, Theorem 21], and if  $C$  is the class of the generic fiber of the normalization, then either  $(C, C) = -2$  or  $-1/2$ , respectively.
- (ii) Small extremal contractions. In this case,  $f$  contracts a Lagrangian  $\mathbb{P}^2$  to an isolated singularity, and the class of a line  $\ell$  satisfies  $(\ell, \ell) = -5/2$ .

See [HT10b] for some speculations about the “nodal” cones that appear in higher dimensions.

**Lagrangian hyperplanes.** Generalizing slightly, let  $X$  be an irreducible holomorphic symplectic *manifold*—that is, a simply-connected Kähler manifold with  $H^0(\Omega_X^2) \cong \mathbb{C}$  generated by a nowhere degenerate 2-form. There are only two infinite families of deformation classes of irreducible holomorphic symplectic manifolds known: Hilbert schemes of points on  $K3$  surfaces and generalized Kummer varieties. We will be concerned with the former; following Markman, we define  $X$  to be of  $K3^{[n]}$ -type if it is deformation equivalent to a Hilbert scheme of  $n$  points on a  $K3$  surface. In this case, we expect:

**Conjecture 2.** [HT10b, Conjecture 1.2] *Let  $X$  be of  $K3^{[n]}$ -type,  $\mathbb{P}^n \subset X$  a smoothly embedded Lagrangian hyperplane, and  $\ell \in H_2(X, \mathbb{Z})$  the class of the line in  $\mathbb{P}^n$ . Then*

$$(\ell, \ell) = -\frac{n+3}{2}$$

The conjecture has been verified for  $n = 2$  in [HT09] and for  $n = 3$  in [HHT].

*Remark 2.* There is a similar conjecture for the class of a line  $\ell$  in a smoothly embedded Lagrangian hyperplane  $\mathbb{P}^n \subset X$  for  $X$  deformation equivalent to a  $2n$ -dimensional generalized Kummer variety  $K_n A$  of an abelian surface  $A$ . In this case, we expect

$$(\ell, \ell) = -\frac{n+1}{2}$$

This conjecture has been verified for  $n = 2$  in [HT10a].

**Monodromy.** The monodromy group of  $X$  will play a crucial role in our analysis. Recall that a monodromy operator is the parallel translation operator on  $H^*(X, \mathbb{Z})$  associated to a smooth family of deformations of  $X$ ; the monodromy group  $\text{Mon}(X)$  is the subgroup of  $\text{GL}(H^*(X, \mathbb{Z}))$  generated by all monodromy operators. Let  $\overline{\text{Mon}}^2(X) \subset \text{GL}(H^2(X, \mathbb{Z}))$  be the quotient acting nontrivially on degree 2 cohomology, and  $\overline{\text{Mon}}(X) \subset \text{GL}(H^*(X, \mathbb{C}))$  the Zariski closure of the full monodromy group  $\text{Mon}(X)$ . By the deformation invariance of the Beauville-Bogomolov form,  $\overline{\text{Mon}}^2(X)$  is actually contained in  $\text{O}(H^2(X, \mathbb{Z}))$ , the orthogonal group of  $H^2(X, \mathbb{Z})$  with respect to  $(\cdot, \cdot)$ . *A priori*, the full Lie group  $G_X = \text{SO}(H^2(X, \mathbb{C}))$  only acts on  $H^2(X, \mathbb{C})$ , but in fact

**Theorem 1.1.** [HHT, Proposition 4.1] *Let  $X$  be of  $K3^{[n]}$ -type. The full cohomology ring  $H^*(X, \mathbb{C})$  carries a representation of  $G_X = \text{SO}(H^2(X, \mathbb{C}))$  compatible with the Hodge structure and cup product.*

The basic reason for this is two-fold, both results of Markman:

- (a) the quotient  $\text{Mon}(X) \rightarrow \overline{\text{Mon}}^2(X)$  has finite kernel [Mar08, §4.3];
- (b)  $G_X$  is a connected component of  $\overline{\text{Mon}}(X)$  [Mar08, §1.8].

The representation of  $\text{Mon}(X)$  on  $H^*(X, \mathbb{C})$  extends to one of  $\overline{\text{Mon}}(X)$ , and since  $G_X$  and  $\text{Mon}(X)$  therefore only differ by finite groups, it lifts to the universal cover of  $G_X$ , and descends to  $G_X$  because of the vanishing of odd cohomology.

The action respects the Hodge structure, so we may consider  $I^*(X) = H^*(X, \mathbb{Q}) \cap H^*(X, \mathbb{C})^{G_X}$ , the ring of Hodge classes. Of course,  $I^*(X)$  contains the Chern classes of the tangent bundle of  $X$ , but there can be many other Hodge classes. The Beauville-Bogomolov form yields a class in  $\text{Sym}^2 H_2(X, \mathbb{Q})^* \cong \text{Sym}^2 H^2(X, \mathbb{Q})$ , and its image under cup product is a rational class  $\theta \in I^4(X)$  known as the Beauville-Bogomolov class. Markman [Mar11] constructs another series of Hodge classes  $k_i \in I^{2i}(X)$ ,  $i \geq 2$ , as characteristic classes of monodromy-invariant twisted sheaves.

Suppose given  $\lambda \in H^2(X, \mathbb{Q})$ , and let  $G_\lambda \subset G_X$  be the stabilizer of  $\lambda$ . Define  $I_\lambda^*(X) = H^*(X, \mathbb{Q}) \cap H^*(X, \mathbb{C})^{G_\lambda}$  to be the ring of cohomology classes invariant under the monodromy group preserving  $\lambda$ . For example, given a Lagrangian hyperplane  $\mathbb{P}^n \subset X$ , the deformations of  $X$  that deform  $\mathbb{P}^n$  are precisely those in  $H^{1,1}(X) \cap \rho^\perp$ , where  $\rho$  is the Beauville-Bogomolov dual of the class of the line in  $\mathbb{P}^n$ , and the orthogonal is taken with respect to the Beauville-Bogomolov form [Ran95, Voi92]. Thus, the class  $[\mathbb{P}^n] \in H^{2n}(X, \mathbb{Z})$  must lie in the subring  $I_\rho^*(X)$ .  $G_X$  will act on these cohomology classes, and up to this action we expect there is a unique Lagrangian hyperplane:

**Conjecture 3.** *For  $X$  of  $K3^{[n]}$ -type, there is a unique  $G_X$  orbit of smooth Lagrangian hyperplane classes  $[\mathbb{P}^n] \in H^*(X, \mathbb{Z})$ .*

In the  $n = 3$  case, the degree 6 part of  $I_\rho^*(X)$  is 3-dimensional, spanned by  $\rho c_2(X), \rho^3$  and the unique 6-dimensional Hodge class  $\eta \in I^6(X)$ . There is an affirmative answer to Conjectures 2 and 3:

**Theorem 1.2.** [HHT, Theorem 1.1] *Let  $X$  be of  $K3^{[3]}$ -type,  $\mathbb{P}^3 \subset X$  a smoothly embedded Lagrangian 3-plane,  $\ell \in H_2(X, \mathbb{Z})$  the class of the line in  $\mathbb{P}^3$ , and  $\rho = 2\ell \in H^2(X, \mathbb{Q})$ . Then  $\rho$  is integral, and*

$$[\mathbb{P}^3] = \frac{1}{48} (\rho^3 + \rho c_2(X))$$

Further, we must have  $(\ell, \ell) = -3$ .

Note that  $\rho$  is therefore the Beauville-Bogomolov dual to the line. We prove below an affirmative answer to both Conjectures 2 and 3 in the  $n = 4$  case:

**Theorem 1.3.** *Let  $X$  be of  $K3^{[4]}$ -type,  $\mathbb{P}^4 \subset X$  be a smoothly embedded Lagrangian 4-plane,  $\ell \in H_2(X, \mathbb{Z})$  the class of a line in  $\mathbb{P}^4$ , and  $\rho = 2\ell \in H^2(X, \mathbb{Q})$ . Then  $\rho$  is integral, and*

$$[\mathbb{P}^4] = \frac{1}{337920} (880\rho^4 + 1760\rho^2 c_2(X) - 3520\theta^2 + 4928\theta c_2(X) - 1408c_2(X)^2)$$

Further, we must have  $(\ell, \ell) = -\frac{7}{2}$ .

Since  $G_X$  acts transitively on  $\rho \in H^2(X, \mathbb{C})$  with  $(\rho, \rho) = -\frac{7}{2} \cdot 4 = -14$ , we have

**Corollary 1.4.** *For  $X$  of  $K3^{[4]}$ -type, there is a unique class of a Lagrangian hyperplane  $[\mathbb{P}^4] \in H^*(X, \mathbb{Z})$  up to the action of  $G_X$ .*

The structure of the paper is as follows. In Section 1 we compute the rings  $I^*(S^{[4]})$  and  $I_\delta^*(S^{[4]})$  using an explicit basis; the same rings  $I^*(X), I_\lambda^*(X)$  in the general case of  $X$  of  $K3^{[4]}$ -type and  $\lambda \in H^2(X, \mathbb{Z})$  will be isomorphic since  $G_X$  acts transitively on rays in  $H^2(X, \mathbb{Z})$ .

In Section 2 we compute a geometric basis for the middle cohomologies  $I^8(X), I_\lambda^8(X)$ , and the intersection form with respect to these bases. In Section 3 we take  $\lambda$  proportional to the Beauville-Bogomolov dual of the class of a line in a smooth Lagrangian hyperplane  $\mathbb{P}^4 \subset X$  and produce a diophantine equation in the coefficients of the class  $[\mathbb{P}^4]$  with respect to the basis from Section 2. In Section 4, we show the only solution to the diophantine equation is the conjectural one.

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## 2. STRUCTURE OF THE RING OF MONODROMY INVARIANTS

**2.1. The Lehn-Sorger formalism.** We briefly summarize the work of Lehn and Sorger in [LS03] on the cohomology ring of a Hilbert scheme of points on a K3 surface. Given a Frobenius algebra  $A$ , they construct a Frobenius algebra  $A^{[n]}$  such that when  $A = H^*(S, \mathbb{Q})$  for  $S$  a K3 surface,  $A^{[n]}$  is canonically  $H^*(S^{[n]}, \mathbb{Q})$ .

Let  $A$  be a Frobenius algebra (over  $\mathbb{Q}$ ). It comes equipped with a form  $T : A \rightarrow \mathbb{Q}$  and a multiplication  $m : A \otimes A \rightarrow A$  such that the pairing  $(x, y) = T(xy)$  is nondegenerate. There is also a comultiplication  $\Delta : A \rightarrow A \otimes A$  adjoint to  $m$  with respect to the form  $T \otimes T$  on  $A \otimes A$ .

For our purposes, we will eventually set  $A = H^*(S, \mathbb{Q})$ ,  $T = -\int_S$ , with  $m$  cup-product, and  $\Delta$  the push-forward along the diagonal. The euler class of  $A$  is defined to be  $\mathbf{e} = m(\Delta(1))$ . In this case, by a simple computation using adjointness, we have

**Lemma 2.2.** *Let  $S$  be a K3 surface,  $A = H^*(S, \mathbb{Z})$ . Let  $1 \in H^0(S, \mathbb{Z})$  be the unit,  $[\text{pt}] \in H^4(S, \mathbb{Z})$  the point class,  $e_i$  a basis for  $H^2(S, \mathbb{Z})$  and  $e_i^\vee$  the dual basis with respect to the intersection form. Then*

$$\begin{aligned} \Delta(1) &= -\sum_j e_j \otimes e_j^\vee - [\text{pt}] \otimes 1 - 1 \otimes [\text{pt}] \\ \Delta(e_j) &= -e_j \otimes [\text{pt}] - [\text{pt}] \otimes e_j \\ \Delta(e_j^\vee) &= -e_j^\vee \otimes [\text{pt}] - [\text{pt}] \otimes e_j^\vee \\ \Delta([\text{pt}]) &= -[\text{pt}] \otimes [\text{pt}] \end{aligned}$$

Thus  $\mathbf{e} = -24[\text{pt}]$ . Similarly, we have an  $n$ -fold multiplication

$$m[n] : A^{\otimes n} \rightarrow A$$

and its adjoint

$$\Delta[n] : A \rightarrow A^{\otimes n}$$

Note that  $m[1] = \Delta[1] = \text{id}$ ,  $m[2] = m$ , and  $\Delta[2] = \Delta$ .

**Lemma 2.3.** *In the setup of (2.2),*

$$\begin{aligned}\Delta[3](1) &= \sum_j \sum (e_j)_a \otimes (e_j^\vee)_b \otimes [\text{pt}]_c + \sum [\text{pt}]_a \otimes [\text{pt}]_b \otimes 1_c \\ \Delta[3](e_j) &= \sum [\text{pt}]_a \otimes [\text{pt}]_b \otimes (e_j)_c \\ \Delta[3](e_j^\vee) &= [\text{pt}]_a \otimes [\text{pt}]_b \otimes (e_j^\vee)_c \\ \Delta[3](\text{[pt]}) &= [\text{pt}] \otimes [\text{pt}] \otimes [\text{pt}]\end{aligned}$$

By  $[\text{pt}]_a \otimes [\text{pt}]_b \otimes 1_c \in A^{\otimes 3}$  we mean  $[\text{pt}]$  inserted in the  $a$ th and  $b$ th tensor factors, and 1 inserted in the  $c$ th factor. All unspecified sums in Lemma 2.3 are over bijections  $\{1, 2, 3\} \xrightarrow{\cong} \{a, b, c\}$ .

*Proof.* This follows from the relation

$$m[n] = m[2] \circ (m[n-1] \otimes \text{id})$$

for  $n \geq 2$  and the dual relation

$$\Delta[n] = (\Delta[n-1] \otimes \text{id}) \circ \Delta[2]$$

We have

$$\begin{aligned}\Delta[3](1) &= (\Delta \otimes \text{id}) \left( - \sum_j e_j \otimes e_j^\vee - [\text{pt}] \otimes 1 - 1 \otimes [\text{pt}] \right) \\ &= \sum_j (e_j \otimes [\text{pt}] + [\text{pt}] \otimes e_j) \otimes e_j^\vee + [\text{pt}] \otimes [\text{pt}] \otimes 1 + \left( \sum e_j \otimes e_j^\vee + [\text{pt}] \otimes 1 + 1 \otimes [\text{pt}] \right) \otimes [\text{pt}] \\ &= \sum_j \sum (e_j)_a \otimes (e_j^\vee)_b \otimes [\text{pt}]_c + \sum [\text{pt}]_a \otimes [\text{pt}]_b \otimes 1_c \\ \Delta[3](e_j) &= (\Delta \otimes \text{id})(-e_j \otimes [\text{pt}] - [\text{pt}] \otimes e_j) \\ &= \sum [\text{pt}]_a \otimes [\text{pt}]_b \otimes (e_j)_c \\ \Delta[3](e_j^\vee) &= (\Delta \otimes \text{id})(-e_j^\vee \otimes [\text{pt}] - [\text{pt}] \otimes e_j^\vee) \\ &= \sum [\text{pt}]_a \otimes [\text{pt}]_b \otimes (e_j^\vee)_c \\ \Delta[3](\text{[pt]}) &= (\Delta \otimes \text{id})(-[\text{pt}] \otimes [\text{pt}]) = [\text{pt}] \otimes [\text{pt}] \otimes [\text{pt}]\end{aligned}$$

□

Let  $[n] = \{k \in \mathbb{N} | k \leq n\}$ . Define the tensor product of  $A$  indexed by a finite set  $I$  of cardinality  $n$  as

$$A^I := \left( \bigoplus_{\varphi: [n] \xrightarrow{\cong} I} A_{\varphi(1)} \otimes \cdots \otimes A_{\varphi(n)} \right) / S_n$$

where  $S_n$  acts by permuting the tensor factors in each summand in the obvious way.  $A^I$  is a Frobenius algebra with multiplication  $m^I$  and form  $T^I$ .

Note that for (finite) sets  $U, V$  and a bijection  $U \rightarrow V$  there is a canonical isomorphism  $A^U \rightarrow A^V$ , so we can always choose a bijection of  $I$  with some  $[k]$  to reduce to the usual

notion of finite self tensor products. In general, for any surjection  $\varphi : U \rightarrow V$ , there is an obvious ring homomorphism

$$\varphi^* : A^U \rightarrow A^V$$

using the ring structure to combine factors indexed by elements of  $U$  in the same fiber of  $\varphi$ . There is an adjoint map

$$\varphi_* : A^V \rightarrow A^U$$

with the important relation

$$\varphi_*(a \cdot \varphi^*(b)) = \varphi_*(a) \cdot b$$

which follows directly from the adjointness.

For any subgroup  $G \subset S_n$ , we can consider the left coset space  $G \setminus [n]$ , and form  $A^{G \setminus [n]}$ . In particular, for  $\sigma \in S_n$  and  $G = \langle \sigma \rangle$  the group generated by  $\sigma$ , we denote  $A^\sigma = A^{G \setminus [n]}$ . Let

$$A\{S_n\} = \bigoplus_{\sigma \in S_n} A^\sigma \cdot \sigma$$

A pure tensor element of  $A^\sigma$  is specified by attaching an element  $\alpha_i \in A$  to each orbit  $i \in I = \langle \sigma \rangle \setminus [n]$ . For example, for a function  $\nu : I \rightarrow \mathbb{Z}_{\geq 0}$ ,

$$\mathbf{e}^\nu = \otimes_{i \in I} \mathbf{e}^{\nu(i)} \in A^\sigma$$

There is a natural product structure on  $A\{S_n\}$ . For any inclusion of subgroups  $H \subset K$  of  $S_n$  there is a surjection  $H \setminus [n] \rightarrow K \setminus [n]$  and therefore maps

$$f^{H,K} : A^{H \setminus [n]} \rightarrow A^{K \setminus [n]}$$

$$f_{K,H} : A^{K \setminus [n]} \rightarrow A^{H \setminus [n]}$$

The product is then

$$\begin{aligned} A^\sigma \otimes A^\tau &\longrightarrow A^{\sigma\tau} \\ a \otimes b &\longrightarrow f_{\langle \sigma, \tau \rangle, \langle \sigma\tau \rangle} \left( f^{\langle \sigma \rangle, \langle \sigma, \tau \rangle}(a) \cdot f^{\langle \tau \rangle, \langle \sigma, \tau \rangle}(b) \cdot \mathbf{e}^{g(\sigma, \tau)} \right) \end{aligned}$$

where  $\langle \sigma, \tau \rangle$  is the subgroup of  $S_n$  generated by  $\sigma, \tau$ , and the graph defect  $g(\sigma, \tau) : \langle \sigma, \tau \rangle \setminus [n] \rightarrow \mathbb{Z}_{\geq 0}$  is

$$g(\sigma, \tau)(B) = \frac{1}{2} (|B| + 2 - |\langle \sigma \rangle \setminus B| - |\langle \tau \rangle \setminus B| - |\langle \sigma\tau \rangle \setminus B|)$$

$S_n$  acts naturally on  $A\{S_n\}$ . For any  $\tau \in A\{S_n\}$ , there is for any  $\sigma \in S_n$  a bijection  $\tau : \langle \sigma \rangle \setminus [n] \rightarrow \langle \tau\sigma\tau^{-1} \rangle \setminus [n]$ .  $\tau$  then acts on  $A\{S_n\}$  via  $\tau^* : A^\sigma \cdot \sigma \rightarrow A^{\tau\sigma\tau^{-1}} \cdot \tau\sigma\tau^{-1}$  on each factor. Define

$$A^{[n]} = A\{S_n\}^{S_n}$$

Note that for any partition  $\mu = (1^{\mu_1}, 2^{\mu_2}, \dots)$  of  $n$ , there is a piece

$$A_\mu^{[n]} = \left( \bigoplus_{\sigma \in C_\mu} A^\sigma \cdot \sigma \right)^{S_n} \cong \bigotimes_i \text{Sym}^{\mu_i} A \quad (2.1)$$

where  $C_\mu \subset S_n$  is the conjugacy class of permutations  $\sigma$  of cycle type  $\mu$ .

If  $A$  is a graded Frobenius algebra, then  $A^{[n]}$  is naturally graded.  $A^\sigma$  is graded as a tensor product of graded vector spaces, and we take

$$A^\sigma \cdot \sigma \cong A^\sigma[-2|\sigma|]$$

where if the cycle type of  $\sigma$  is  $\mu$ ,  $|\sigma| = \sum_i (i-1)\mu_i$ . In particular, the  $m$ th graded piece of (2.1) is

$$(A_\mu^{[n]})_m \cong \bigoplus_{\substack{(w,\mu) \\ |(w,\mu)|=m}} \bigotimes_i \text{Sym}^{\mu_i} A_{w_i} \quad (2.2)$$

where the sum is taken over weighted permutations  $(w, \mu)$ —*i.e.* a partition  $\mu$  and a weight  $w_i$  associated to each part—with

$$m = |(w, \mu)| = \sum_i (i-1)\mu_i + w_i$$

We then have

**Theorem 2.4.** [LS03, Theorem 1.1] *For  $S$  a K3 surface, there is a natural isomorphism of graded Frobenius algebras*

$$(H^*(S, \mathbb{Q})[2])^{[n]} \cong H^*(S^{[n]}, \mathbb{Q})[2n]$$

The grading shift on both sides is such that the 0th graded piece is middle cohomology.

*Remark 3.* It will be important in the next section to note that under the isomorphism of (2.4),

$$n![\text{pt}]_1 \otimes \cdots \otimes [\text{pt}]_n \cdot (\text{id}) \mapsto [\text{pt}]_{S^{[n]}}$$

**2.5. Monodromy invariants.** Let  $S$  be a K3 surface, and  $G_S = \text{SO}(H^2(S, \mathbb{C}))$  the special orthogonal group of the intersection form  $(\cdot, \cdot)$  on  $S$ .  $H^*(S, \mathbb{C})$  is naturally a representation of  $G_S$ , acting via the standard representation on  $H^2(S, \mathbb{C})$  and the trivial representations on  $H^0(S, \mathbb{C})$  and  $H^4(S, \mathbb{C})$ .

Recall (see for example [FH91]) that positive weights of the algebra  $\text{SO}_{\mathbb{C}}(k)$  of rank  $r$  ( $k = 2r$  or  $2k + 1$ ) are  $r$ -tuples  $\lambda = (\lambda_1, \dots, \lambda_r)$  with the  $\lambda_i$  either all integral or all half-integral, and either

$$\begin{aligned} \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{r-1} \geq |\lambda_r| \geq 0, & \quad k = 2r \\ \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{r-1} \geq \lambda_r \geq 0, & \quad k = 2r + 1 \end{aligned}$$

Let the representation of  $\text{SO}_{\mathbb{C}}$  of highest weight  $\lambda$  be denoted  $V(\lambda)$ . Thus,  $\mathbf{1} = V(0, \dots)$  is the trivial representation, and  $V = V(1, 0, \dots)$  the standard.  $\text{Sym}^k V$  is not irreducible, since the form yields an invariant  $\theta \in \text{Sym}^2 V$ , but  $V(k, 0, \dots) = \text{Sym}^k V / \text{Sym}^{k-2} V$ . In the sequel, we will only indicate the nonzero weights, *e.g.*  $V = V(1)$ .

If a Frobenius algebra  $A$  carries a representation of a group  $G$ ,  $A^{[n]}$  naturally carries a representation of  $G$  that can easily be read off of (2.2). Thus,

**Proposition 2.6.** *As a representation of  $G_S$ , we have*

$$\begin{aligned} H^2(S^{[4]}, \mathbb{C}) &\cong \mathbf{1}_S \oplus V_S(1) \\ H^4(S^{[4]}, \mathbb{C}) &\cong \mathbf{1}_S^4 \oplus V_S(1)^2 \oplus V_S(2) \\ H^6(S^{[4]}, \mathbb{C}) &\cong \mathbf{1}_S \oplus V_S(1)^5 \oplus V_S(1, 1) \oplus V_S(2)^2 \oplus V_S(3) \\ H^8(S^{[4]}, \mathbb{C}) &\cong \mathbf{1}_S^8 \oplus V_S(1)^6 \oplus V_S(1, 1) \oplus V_S(2)^4 \oplus V_S(2, 1) \oplus V_S(3) \oplus V_S(4) \end{aligned}$$

Poincaré duality is compatible with the  $G_S$  action, so the above determines all cohomology groups.

Note that the invariant class in  $H^2(S^{[n]}, \mathbb{C})$  is exactly  $\delta$ . The decomposition (1.1) identifies the action of  $G_S$  on  $H^*(S^{[n]}, \mathbb{C})$  with that of  $G_\delta \subset G_{S^{[n]}}$ , the stabilizer of  $\delta$ . In other words, deformations of  $S^{[n]}$  orthogonal to the exceptional divisor  $\delta$  remain Hilbert schemes of points of a  $K3$  surface, and therefore come from a deformation of  $S$ .

Recall that  $\mathrm{SO}_{\mathbb{C}}(k)$  has universal branching rules. For  $\mathrm{SO}_{\mathbb{C}}(k-1) \subset \mathrm{SO}_{\mathbb{C}}(k)$  the stabilizer of a nonisotropic vector  $v \in V$ ,  $(v, v) \neq 0$ , we have

$$\mathrm{Res}_{\mathrm{SO}_{\mathbb{C}}(k-1)}^{\mathrm{SO}_{\mathbb{C}}(k)} V(\lambda) = \bigoplus_{\lambda'} V(\lambda')$$

where the sum is taken over all weights  $\lambda'$  with

$$\lambda_1 \geq \lambda'_1 \geq \lambda_2 \geq \lambda'_2 \geq \dots \geq \lambda_r \geq |\lambda'_r| \geq 0$$

For  $X$  of  $K3^{[n]}$ -type, we can therefore deduce the structure of  $H^*(X, \mathbb{C})$  as a  $G_X$  representation from the structure of  $H^*(S^{[n]}, \mathbb{C})$  as a  $G_S$  representation:

**Corollary 2.7.** *For  $X$  of  $K3^{[4]}$ -type,*

$$\begin{aligned} H^2(X, \mathbb{C}) &\cong V_X(1) \\ H^4(X, \mathbb{C}) &\cong \mathbf{1}_X^2 \oplus V_X(1) \oplus V_X(2) \\ H^6(X, \mathbb{C}) &\cong \mathbf{1}_X \oplus V_X(1)^2 \oplus V_X(1, 1) \oplus V_X(2) \oplus V_X(3) \\ H^8(X, \mathbb{C}) &\cong \mathbf{1}_X^3 \oplus V_X(1)^2 \oplus V_X(2)^2 \oplus V_X(2, 1) \oplus V_X(4) \end{aligned}$$

Again, Poincaré duality determines the representations of the other cohomology groups.

**2.8. A basis for  $I_\delta^*(S^{[4]})$ .** For a partition  $\mu = (1^{\mu_1}, 2^{\mu_2}, \dots)$  of  $n$ , the number of parts of  $\mu$  is  $\ell(\mu) = \sum \mu_i$ . By a labelled partition  $\mu$  we will mean a partition  $\mu$  and an ordered list of  $\ell(\mu)$  cohomology classes  $\alpha \in H^*(S, \mathbb{Q})$ . For example,  $(\{1\}_2, \{1, 1\}_1)$  is a labelled partition of 4, subordinate to the partition  $\mu = (1^2, 2)$ , and attaching the unit class to each part of  $\mu$ . Such a labelled partition  $\mu$  determines an element of the Lehn-Sorger algebra of  $H^*(S, \mathbb{Q})[2]$  by summing over all group elements  $\sigma \in S_n$  with cycle type  $\mu$ , for example

$$\begin{aligned} \delta = I(\{1\}_2, \{1, 1\}_1) &= \sum_{(12)} \mathbf{1}_{12} \otimes \mathbf{1}_3 \otimes \mathbf{1}_4(12) \\ &= \mathbf{1}_{12} \otimes \mathbf{1}_3 \otimes \mathbf{1}_4(12) + \mathbf{1}_{13} \otimes \mathbf{1}_2 \otimes \mathbf{1}_4(13) + \mathbf{1}_{14} \otimes \mathbf{1}_2 \otimes \mathbf{1}_3(14) \\ &\quad + \mathbf{1}_1 \otimes \mathbf{1}_{23} \otimes \mathbf{1}_4(23) + \mathbf{1}_1 \otimes \mathbf{1}_{24} \otimes \mathbf{1}_3(24) + \mathbf{1}_1 \otimes \mathbf{1}_2 \otimes \mathbf{1}_{34}(34) \end{aligned}$$

We can generate homogeneous classes of  $H^*(S^{[n]}, \mathbb{Q})$  invariant under  $G_S$  from partitions of  $n$  labelled by cohomology classes  $\{1, e, e^\vee, [\mathrm{pt}]\}$ , where every time we have a label  $e$ , there must be a paired  $e^\vee$  label, corresponding to inserting  $e_j$  and  $e_j^\vee$  in the corresponding tensor factors and summing over  $j$ .

$I_\delta^2(S^{[4]})$  is spanned by:

$$\delta = I(\{1\}_2, \{1, 1\}_1) = \sum_{(12)} \mathbf{1}_{12} \otimes \mathbf{1}_3 \otimes \mathbf{1}_4(12)$$

$I_\delta^4(S^4)$  is spanned by:

$$\begin{aligned}
W &= I(\{1\}_3, \{1\}_1) = \sum_{(123)} 1_{123} \otimes 1_4(123) \\
X &= I(\{1, 1\}_2) = \sum_{(12)(34)} 1_{12} \otimes 1_{34}(12)(34) \\
Y &= I(\{1, 1, 1, [\text{pt}]\}_1) = \sum_1 [\text{pt}]_1 \otimes 1_2 \otimes 1_3 \otimes 1_4(\text{id}) \\
Z &= I(\{1, 1, e, e^\vee\}_1) = \sum_{j, (12)} (e_j)_1 \otimes (e_j^\vee)_2 \otimes 1_3 \otimes 1_4(\text{id})
\end{aligned}$$

$I_\delta^6(S^4)$  is spanned by:

$$\begin{aligned}
P &= I(\{1\}_4) = \sum_{(1234)} 1_{1234}(1234) \\
Q &= I(\{[\text{pt}]\}_2, \{1, 1\}_1) = \sum_{(12)} [\text{pt}]_{12} \otimes 1_3 \otimes 1_4(12) \\
R &= I(\{1\}_2, \{1, [\text{pt}]\}_1) = \sum_{(12), 3} 1_{12} \otimes [\text{pt}]_3 \otimes 1_4(12) \\
S &= I(\{e^\vee\}_2, \{e, 1\}_1) = \sum_{j, 1, (23)} (e_j)_1 \otimes (e_j^\vee)_{23} \otimes 1_4(23) \\
T &= I(\{1\}_2, \{e, e^\vee\}_1) = \sum_{(12)} 1_{12} \otimes (e_j)_3 \otimes (e_j^\vee)_4(12)
\end{aligned}$$

$I_\delta^8(S^{[4]})$  is spanned by:

$$A = I(\{e\}_3, \{e^\vee\}_1) = \sum_{j, (123)} (e_j)_{123} \otimes (e_j^\vee)_4(123)$$

$$B = I(\{1\}_3, \{[\text{pt}]\}_1) = \sum_{(123)} 1_{123} \otimes [\text{pt}]_4(123)$$

$$C = I(\{[\text{pt}]\}_3, \{1\}_1) = \sum_{(123)} [\text{pt}]_{123} \otimes 1_4(123)$$

$$D = I(\{1, [\text{pt}]\}_2) = \sum_{(12)} [\text{pt}]_{12} \otimes 1_{34}(12)(34)$$

$$E = I(\{e, e^\vee\}_2) = \sum_{j, (12)(34)} (e_j)_{12} \otimes (e_j^\vee)_{34}(12)(34)$$

$$F = I(\{1, 1, [\text{pt}], [\text{pt}]\}_1) = \sum_{(12)} [\text{pt}]_1 \otimes [\text{pt}]_2 \otimes 1_3 \otimes 1_4(\text{id})$$

$$G = I(\{1, e, e^\vee, [\text{pt}]\}_1) = \sum_{j, 1, (23)} [\text{pt}]_1 \otimes (e_j)_2 \otimes (e_j^\vee)_3(\text{id})$$

$$H = I(\{e, e, e^\vee, e^\vee\}_1) = \sum_{j, k, (12)(34)} (e_j)_1 \otimes (e_j^\vee)_2 \otimes (e_k)_3 \otimes (e_k^\vee)_4 \cdot \text{id}$$

These classes are all clearly independent, and therefore by the computation of the dimensions of  $I_\delta^*(S^{[4]})$  in the previous section are a basis.

**2.9. Cup product on  $I_\delta^*(S^{[4]})$ .** We collect here without computation the product structure on  $I_\delta(S^{[4]})$ ; see [BJ11] for proofs.

**2.9.1. Multiplication by  $\delta$ .**

$$\begin{aligned} \delta^2 &= 2X - 3Y - Z + 3W \\ \delta W &= 4P - 4Q - 2R - 2S \\ \delta X &= 2P - R - T \\ \delta Y &= 2Q + R \\ \delta Z &= 22Q + 2S + T \\ \delta P &= -3A - 3B - 3C - 4D - 4E \\ \delta Q &= 3C + D - F \\ \delta R &= 3B + 3C + 2D - 4F - G \\ \delta S &= 3A + 66C + 4E - 2G \\ \delta T &= 3A + 22D - G - 2H \end{aligned}$$

At the very least we include:

$$\begin{aligned}
\delta^2 &= \left( \sum_{(12)} 1_{12} \otimes 1_3 \otimes 1_4(12) \right)^2 \\
&= \sum_{(12)} 1_{12} \otimes 1_3 \otimes 1_4(12) ((12) + (13) + (14) + (23) + (24) + (34)) \\
&= \sum_{(12)} (\Delta(1)_{1,2} \otimes 1_3 \otimes 1_4(\text{id}) + 1_{1,2,3} \otimes 1_4(132) \\
&\quad + 1_{1,2,4} \otimes 1_3(142) + 1_{1,2,3} \otimes 1_4(123) + 1_{1,2,4} \otimes 1_3(124) + 1_{12} \otimes 1_{34}(12)(34)) \\
&= -3 \sum_1 [\text{pt}]_1 \otimes 1_2 \otimes 1_3 \otimes 1_4(\text{id}) - \sum_{(12)} \sum_j (e_j)_1 \otimes (e_j^\vee)_2 \otimes 1_3 \otimes 1_4(\text{id}) \\
&\quad + 3 \sum_{(123)} 1_{123} \otimes 1_4(123) + 2 \sum_{(12)(34)} 1_{12} \otimes 1_{34}(12)(34) \\
&= -3Y - Z + 3W + 2X
\end{aligned}$$

Note that

$$\begin{aligned}
\delta^4 &= \delta(\delta \cdot \delta^2) \\
&= \delta(\delta(2X - 3Y - Z + 3W)) \\
&= \delta(16P - 40Q - 11R - 8S - 3T) \\
&= -81A - 81B - 729C - 192D - 96E + 84F + 30G + 6H
\end{aligned} \tag{2.3}$$

2.9.2. *Multiplication by  $W, X, Y, Z$ .*

$$\begin{aligned}
W^2 &= -3A - 3B - 27C - 8D - 8E + 4F + 2G \\
WX &= -3A - 3B - 3C \\
WY &= B + 3C \\
WZ &= 3A + 66C \\
X^2 &= -2D - 2E + 2F + G + H \\
XY &= 2D \\
XZ &= 22D + 4E \\
Y^2 &= 2F \\
YZ &= G \\
Z^2 &= 22F + 2G + 2H
\end{aligned}$$

Therefore,

$$\begin{aligned}
\delta^4 &= (\delta^2)^2 \\
&= (2X - 3Y - Z + 3W)^2 \\
&= -81A - 81B - 729C - 192D - 96E + 84F + 30G + 6H
\end{aligned}$$

which agrees with (2.3).

2.9.3. *Multiplication by  $A, B, C, D, E, F, G, H$ .* The only nonzero products are

$$\begin{aligned}
A^2 &= 8 \cdot 22[\text{pt}]_1 \otimes [\text{pt}]_2 \otimes [\text{pt}]_3 \otimes [\text{pt}]_4(\text{id}) \\
BC &= 8[\text{pt}]_1 \otimes [\text{pt}]_2 \otimes [\text{pt}]_3 \otimes [\text{pt}]_4(\text{id}) \\
D^2 &= 6[\text{pt}]_1 \otimes [\text{pt}]_2 \otimes [\text{pt}]_3 \otimes [\text{pt}]_4(\text{id}) \\
E^2 &= 66[\text{pt}]_1 \otimes [\text{pt}]_2 \otimes [\text{pt}]_3 \otimes [\text{pt}]_4(\text{id}) \\
F^2 &= 6[\text{pt}]_1 \otimes [\text{pt}]_2 \otimes [\text{pt}]_3 \otimes [\text{pt}]_4(\text{id}) \\
G^2 &= 264[\text{pt}]_1 \otimes [\text{pt}]_2 \otimes [\text{pt}]_3 \otimes [\text{pt}]_4(\text{id}) \\
H^2 &= 1584[\text{pt}]_1 \otimes [\text{pt}]_2 \otimes [\text{pt}]_3 \otimes [\text{pt}]_4(\text{id})
\end{aligned}$$

As a consistency check, from (3.3) we need

$$\begin{aligned}
\delta^8 &= 105(\delta, \delta)^4 \\
&= 105(-6)^8 \\
&= 136080
\end{aligned}$$

and indeed, from (2.3)

$$\begin{aligned}
\delta^8 &= (-81A - 81B - 729C - 192D - 96E + 84F + 30G + 6H)^2 \\
&= 136080 \cdot 24[\text{pt}]_1 \otimes [\text{pt}]_2 \otimes [\text{pt}]_3 \otimes [\text{pt}]_4(\text{id}) \\
&= 136080[\text{pt}]_{S^4}
\end{aligned}$$

Note that the remaining classes and products are determined by Poincaré duality.

2.10. **The Beauville-Bogomolov form.** From (1.1), we can explicitly write down  $\theta$  in the  $W, X, Y, Z$  basis:

$$\begin{aligned}
\theta &= \sum_j \left( \sum_1 (e_j)_1 \otimes 1_2 \otimes 1_3 \otimes 1_4(\text{id}) \right) \cdot \left( \sum_1 (e_j^\vee)_1 \otimes 1_2 \otimes 1_3 \otimes 1_4(\text{id}) \right) - \frac{1}{6}\delta^2 \\
&= 22Y + 2Z - \frac{1}{6}\delta^2 \\
&= -\frac{1}{2}W - \frac{1}{3}X + \frac{45}{2}Y + \frac{13}{6}Z
\end{aligned}$$

By direct computation, using the results of the previous section,

**Lemma 2.11.**

$$\begin{aligned}
\theta^4 &= 450225 \\
\delta^2\theta^3 &= -117450 = 19575(-6) \\
\delta^4\theta^2 &= 84564 = 2349 \cdot (-6)^2 \\
\delta^6\theta &= -93960 = 435 \cdot (-6)^3 \\
\delta^8 &= 136080 = 105 \cdot (-6)^4
\end{aligned}$$

### 3. HODGE CLASSES ON $X$

Let  $X$  be of  $K3^{[4]}$ -type and  $\lambda \in H^2(X, \mathbb{Q})$ . The rings  $I^*(X)$  and  $I_\lambda^*(X)$  are isomorphic to the rings  $I^*(S^{[4]})$  and  $I_\delta^*(S^{[4]})$  since the action of  $G_X$  is transitive on rays, but to construct an explicit isomorphism, we must find a geometric basis. To do this, we need to understand the products of Hodge classes.

**3.1. Computation of the Fujiki constants for  $S^{[4]}$ .** Let  $X$  be smooth variety of dimension  $n$ , and  $\mu$  a partition of a nonnegative integer  $|\mu|$  (we allow the empty partition of 0). To each  $\mu$  we can associate a Chern monomial  $c_\mu(X) = \prod_{i=1}^k c_k^{\mu_k}(X)$ . Given a formal power series  $\varphi(x) \in \mathbb{Q}[[x]]$ , define the associated genus

$$\varphi(X) = \prod_i \varphi(x_i) \in H^*(X, \mathbb{Q})$$

where the  $x_i$  are the Chern roots of the tangent bundle  $TX$ . Taking the universal formal power series

$$\Phi(x) = 1 + a_1x + a_2x^2 + \cdots \in \mathbb{Q}[a_1, a_2, \dots][[x]]$$

we define the universal genus  $\Phi(X)$  of any smooth variety as an element of  $H^*(X, \mathbb{Q})[a_1, a_2, \dots]$ .  $\Phi(X)$  is a universal formal power series in the Chern classes  $c_1, c_2, \dots$  with coefficients polynomials in  $a_1, a_2, \dots$ . For example, up to degree 8 we have

$$\begin{aligned} \Phi &= 1 + a_1c_1 \\ &\quad + a_2c_1^2 + (a_1^2 - 2a_2)c_2 \\ &\quad + a_3c_1^3 + (a_1a_2 - 3a_3)c_1c_2 + (a_1^3 - 3a_1a_2 + 3a_3)c_3 \\ &\quad + a_4c_1^4 + (a_1^2a_2 - 2a_2^2 - a_1a_3 + 4a_4)c_1c_3 + (a_2^2 - 2a_1a_3 + 2a_4)c_2^2 + (a_1a_3 - 4a_4)c_1^2c_2 \\ &\quad + (a_1^4 - 4a_1^2a_2 + 2a_2^2 + 4a_1a_3 - 4a_4)c_4 + \cdots \end{aligned}$$

In particular, taking  $a_i = 1/i!$ , we get the Chern character.

Let  $S$  be a smooth surface,  $\varphi(x) \in \mathbb{Q}[[x]]$  a formal power series in  $x$ . Recall that  $\mathcal{O}^{[n]}$  is the push-forward of the structure sheaf of the universal subscheme  $Z \subset S \times S^{[n]}$  to  $S^{[n]}$ , and that  $\det \mathcal{O}^{[n]} = -\delta$ . A result of [EGL99, Theorem 4.2] implies that there are universal formal power series  $A(z), B(z)$  in  $z$  such that

$$\sum_{n \geq 0} z^n \int_{S^{[n]}} \exp(\det \mathcal{O}^{[n]}) \varphi(S^{[n]}) = A(z)^{c_1(S)^2} B(z)^{c_2(S)}$$

for any smooth surface  $S$ . Let

$$\mathbf{F}_S(z) = \sum_{n \geq 0} z^n \int_{S^{[n]}} \exp(\det \mathcal{O}^{[n]}) \Phi(S^{[n]}) \in \mathbb{Q}[a_1, a_2, \dots][[z]]$$

and let  $\mathbf{A}(z), \mathbf{B}(z) \in \mathbb{Q}[a_1, a_2, \dots][[z]]$  be the universal power series associated to  $\Phi$ .  $\mathbf{F}_{\mathbb{P}^2}(z) = \mathbf{A}(z)^9 \mathbf{B}(z)^3$  and  $\mathbf{F}_{\mathbb{P}^1 \times \mathbb{P}^1}(z) = \mathbf{A}(z)^8 \mathbf{B}(z)^4$  can be easily computed by equivariant localization

[BJ11], and therefore one can compute  $\mathbf{A}(z), \mathbf{B}(z)$ . For example, up to  $n = 2$  we have

$$\mathbf{A}(z) = 1 + a_2 z + \left( -a_1^3 + 3a_1^2 a_2 + \frac{1}{4} a_1^2 + a_1 a_2 - \frac{9}{2} a_2^2 + a_1 a_3 + \frac{1}{6} a_1 - \frac{3}{2} a_2 + 3a_3 - 10a_4 - \frac{1}{48} \right) z^2$$

$$\mathbf{B}(z) = 1 + (a_1^2 - 2a_2) z + \left( 2a_1^4 - 8a_1^2 a_2 - \frac{5}{4} a_1^2 + \frac{31}{2} a_2^2 - 15a_1 a_3 + \frac{5}{2} a_2 + 15a_4 + \frac{1}{48} \right) z^2$$

See [BJ11] for higher terms. Since  $\mathbb{P}^1 \times \mathbb{P}^1, \mathbb{P}^2$  generate the cobordism ring, this determines  $\mathbf{F}_S(z)$  for a  $K3$  surface  $S$ , and in particular we can compute all products

$$\int_{S^{[n]}} \delta^{2k} c_\mu(S^{[n]}) \quad (3.1)$$

By a result of Fujiki, (3.1) determines all products of the form

$$\int_X f^{2k} c_\mu(X)$$

for arbitrary  $f \in H^2(X, \mathbb{Q})$ :

**Theorem 3.2.** [Fuj87] *For  $X$  an irreducible holomorphic symplectic variety of dimension  $n$  and  $\mu$  a partition of an integer  $|\mu|$ , there are rational constants  $\gamma_X(k; \mu)$  such that, for any class  $f \in H^2(X, \mathbb{Z})$ ,*

$$\int_X f^{2k} c_\mu(X) = \gamma_X(k; \mu) \cdot (f, f)^k$$

Moreover, the constant  $\gamma_X(k; \mu)$  is a deformation invariant.

Of course,  $\gamma_X(k, \mu) = 0$  if  $\dim X \neq 2k + |\mu|$ . Also, because  $X$  is holomorphic symplectic, all odd Chern classes  $c_i(X)$  vanish, so we require  $\mu$  to be an even partition. We collect here the Fujiki constants  $\gamma(\mu)$  for  $n = 4$  for reference:

**Corollary 3.3.** *For  $X$  of  $K3^{[4]}$ -type, we have*

$$\begin{aligned} \gamma_X(0; 2^4) &= 1992240 & \gamma_X(1; 2^3) &= 59640 & \gamma_X(2; 2^2) &= 4932 & \gamma_X(3; 2^1) &= 630 & \gamma_X(4; \emptyset) &= 105 \\ \gamma_X(0; 2^2, 4^1) &= 813240 & \gamma_X(1; 2^1, 4^1) &= 24360 & \gamma_X(1; 4^1) &= 2016 & & & & \\ \gamma_X(0; 2^1, 6^1) &= 182340 & \gamma(1; 6^1) &= 5460 & & & & & & \\ \gamma_X(0; 8^1) &= 25650 & & & & & & & & \\ \gamma_X(0; 4^2) &= 332730 & & & & & & & & \end{aligned}$$

*Proof.* This follows from the deformation invariance and the degree 4 part of

$$\mathbf{F}_S(z) = \mathbf{B}(z)^{24}$$

for  $S$  a  $K3$  surface, see [BJ11]. Note that  $(\delta, \delta) = -6$ . □

**3.4. Generalized Fujiki constants.** Let  $X$  be of  $K3^{[n]}$ -type. In general, for  $\eta$  a Hodge class, an integral of the form  $\int_X f^{2k} \eta$  must be compatible with the  $G_X$  action, and therefore will be a rational multiple of  $(f, f)^k$ . For  $\eta$  a product of a power of  $\theta$  and a Chern monomial, these ratios are determined by the Fujiki constants of the previous section.

Define an augmented partition  $(k, \ell, \mu)$  to be a partition  $\mu$  of a nonnegative integer  $|\mu|$  and two nonnegative integers  $k, \ell$ . Set

$$|(k, \ell, \mu)| = 2k + 2\ell + |\mu|$$

**Proposition 3.5.** *For  $X$  of  $K3^{[n]}$ -type,  $n > 1$ , and  $(k, \ell, \mu)$  an augmented partition, there is a rational constant  $\gamma_X(k, \ell, \mu)$  such that for any  $f \in H^2(X, \mathbb{Z})$ ,*

$$\int_X f^{2k} \theta^\ell c_\mu(X) = \gamma_X(k, \ell, \mu) \cdot (f, f)^k$$

Furthermore, there are rational constants  $\alpha(k, \ell)$  independent of  $X$  such that

$$\gamma_X(k, \ell, \mu) = \alpha(k, \ell) \gamma_X(k + \ell; \mu)$$

Again,  $\gamma_X(k, \ell, \mu) = 0$  if either  $|(k, \ell, \mu)| \neq \dim X$ , or  $\mu$  is not an even partition.

*Proof.* As mentioned above, the interesting part is the existence of the  $\alpha$ . Let  $x_i$  be an orthonormal basis of  $H^2(X, \mathbb{C})$  with respect to the Beauville-Bogomolov form. Note that  $\theta = \sum_i x_i^2$ . It suffices to consider the case  $f = \sum_i x_i$ , which has  $(f, f) = 23$ . Let

$$p^k(a) = \left( \sum_i a_i x_i \right)^k$$

for  $a \in \mathbb{Q}^{23}$ . The  $p^k(a)$  span the space of degree  $k$  polynomials in  $x_i$ , so their symmetrizations

$$\bar{p}^k(a) = \frac{1}{23!} \sum_{\sigma \in S_{23}} \left( \sum_i a_i x_{\sigma(i)} \right)^k$$

span the space of degree  $k$  symmetric functions in  $x_i$ . We can therefore write

$$f^{2k} \theta^\ell = \sum_{a(k, \ell)} \lambda_{a(k, \ell)} \bar{p}^{2k+2\ell}(a(k, \ell))$$

where the sum is over finitely many  $a(k, \ell)$ . This expression has no dependence on the dimension of  $X$ . We have

$$\begin{aligned} \int_X f^{2k} \theta^\ell c_\mu(X) &= \frac{1}{23!} \sum_{a(k, \ell)} \lambda_{a(k, \ell)} \sum_{\sigma \in S_{23}} \int_X \left( \sum_i a(k, \ell)_i x_{\sigma(i)} \right)^{2k} c_\mu(X) \\ &= \frac{1}{23!} \sum_{a(k, \ell)} \lambda_{a(k, \ell)} \sum_{\sigma \in S_{23}} \left( \sum_i a(k, \ell)_i x_{\sigma(i)}, \sum_i a(k, \ell)_i x_{\sigma(i)} \right)^k \gamma_X(\mu) \\ &= \left( \sum_{a(k, \ell)} \lambda_{a(k, \ell)} \left( \sum_i a(k, \ell)_i^2 \right)^k \right) \gamma_X(\mu) \\ &= \alpha(k, \ell) \gamma_X(\mu) (f, f)^k \end{aligned}$$

where

$$\alpha(k, \ell) = \frac{1}{23^k} \sum_{a(k, \ell)} \lambda_{a(k, \ell)} \left( \sum_i a(k, \ell)_i^2 \right)^k$$

□

Explicitly,

$$\begin{aligned}
\int_X \theta c_\mu(X) &= \sum_i \int_X x_i^2 c_\mu(X) \\
&= \sum_i (x_i, x_i) \gamma_X(\mu) \\
&= 23 \cdot \gamma_X(\mu)
\end{aligned}$$

so  $\alpha(0, 1) = 23$ . Less trivially,

$$\begin{aligned}
\int_X \theta^2 c_\mu(X) &= \int_X \left( \sum_i x_i^2 \right)^2 c_\mu(X) \\
&= \int_X \left( \frac{1}{6} \sum_{i < j} (x_i + x_j)^4 + \frac{1}{6} \sum_{i < j} (x_i - x_j)^4 - \frac{19}{3} \sum_i x_i^4 \right) c_\mu(X) \\
&= \left( \frac{1}{6} \cdot \binom{23}{2} \cdot 2^2 + \frac{1}{6} \cdot \binom{23}{2} \cdot 2^2 - \frac{19}{3} \cdot 23 \cdot 1 \right) \gamma_X(\mu) \\
&= \frac{575}{3} \cdot \gamma_X(\mu)
\end{aligned}$$

The relevant values of the  $\alpha$  constants can be computed from (2.11) and by reducing to the  $K3^{[3]}$ -type case:

**Lemma 3.6.** *We have*

$$\begin{aligned}
\alpha(0, 1) &= 23 & \alpha(1, 1) &= \frac{25}{3} & \alpha(2, 1) &= \frac{27}{5} & \alpha(3, 1) &= \frac{29}{7} \\
\alpha(0, 2) &= \frac{575}{3} & \alpha(1, 2) &= 45 & \alpha(2, 2) &= \frac{783}{35} \\
\alpha(0, 3) &= 1035 & \alpha(1, 3) &= \frac{1305}{7} \\
\alpha(0, 4) &= \frac{30015}{7}
\end{aligned}$$

Of course,  $\alpha(k, 0) = 1$  for any  $k$ .

*Proof.*  $\alpha(3, 1), \alpha(2, 2), \alpha(1, 3), \alpha(0, 4)$  are all determined by (2.11), using  $\gamma_{S^{[4]}(\emptyset)} = 105$ . From the computations of [HHT] for the  $K3^{[3]}$ -type cases, we have

$$\begin{aligned}
(\theta_{S^{[3]}})^3 &= 15525 = 1035 \cdot \gamma_{S^{[3]}(\emptyset)} \\
(\delta_{S^{[3]}})^2 (\theta_{S^{[3]}})^2 &= -2700 = 45 \cdot (\delta_{S^{[3]}}, \delta_{S^{[3]}}) \cdot \gamma_{S^{[3]}(\emptyset)} \\
(\delta_{S^{[3]}})^4 (\theta_{S^{[3]}}) &= 1296 = \frac{27}{5} \cdot (\delta_{S^{[3]}}, \delta_{S^{[3]}})^2 \cdot \gamma_{S^{[3]}(\emptyset)} \\
(\theta_{S^{[3]}})^2 c_2(S^{[3]}) &= 20700 = \frac{575}{3} \cdot \gamma_{S^{[3]}(2^1)} \\
(\delta_{S^{[3]}})^2 (\theta_{S^{[3]}}) c_2(S^{[3]}) &= -3600 = \frac{25}{3} \cdot (\delta_{S^{[3]}}, \delta_{S^{[3]}}) \cdot \gamma_{S^{[3]}(2^1)}
\end{aligned}$$

since  $\gamma_{S^{[3]}(\emptyset)} = 15$ ,  $\gamma_{S^{[3]}(2^1)} = 108$  and  $c_2(S^{[3]}) = \frac{4}{3} \theta_{S^{[3]}}$ .

□

**3.7. A geometric basis.**  $I^8(X)$  is 3-dimensional, so we expect there to be a relation among  $\theta^2, \theta c_2(X), c_2(X)^2, c_4(X)$ :

**Lemma 3.8.** *For  $X$  of  $K3^{[4]}$ -type,*

$$\theta^2 = \frac{7}{5}\theta c_2 - \frac{31}{60}c_2^2 + \frac{1}{15}c_4 \quad (3.2)$$

*Proof.* Using the results of the previous section, we know the intersection form restricted to  $I^8(X)$  in terms of the basis  $\theta^2, \theta c_2(X), c_2(X)^2, c_4(X)$ :

$$\begin{pmatrix} 450225 & 1035 \cdot 630 & \frac{575}{3} \cdot 4932 & \frac{575}{3} \cdot 2016 \\ 1035 \cdot 630 & \frac{575}{3} \cdot 4932 & 23 \cdot 59640 & 23 \cdot 24360 \\ \frac{575}{3} \cdot 4932 & \frac{575}{3} \cdot 4932 & 1992240 & 813240 \\ \frac{575}{3} \cdot 2016 & 23 \cdot 24360 & 813240 & 332730 \end{pmatrix} \quad (3.3)$$

As expected, the matrix is rank 3. By Poincaré duality, a generator of the kernel gives the relation.  $\square$

**Corollary 3.9.**  $c_2(S^{[4]}) = 3Z + 33Y - W$

*Proof.* Suppose

$$c_2(S^{[4]}) = wW + xX + yY + zZ$$

for  $w, x, y, z \in \mathbb{Q}$ . Taking the product with  $\theta^3, \delta^2\theta^2, \delta^4\theta, \delta^4$  yields the equation

$$\begin{pmatrix} -6075 & -2700 & \frac{30375}{2} & \frac{96525}{2} \\ 15066 & 6696 & -3213 & -16335 \\ -19116 & -8496 & 1854 & 14058 \\ 29160 & 12960 & -1620 & -17820 \end{pmatrix} \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 652050 \\ -170100 \\ 122472 \\ -136080 \end{pmatrix}$$

The matrix has rank 2. Computing generators of the kernel, we can write

$$c_2(S^{[4]}) = \left(-\frac{4}{9}u - \frac{4}{27}v\right)W + \left(u - \frac{21}{4}\right)X + (v + 42)Y - \frac{v}{3}Z$$

Similarly, computing  $c_2(S^{[4]})^2$  and intersecting with  $\theta^2, \delta^2\theta, \delta^4$  yields 3 equations, whose solutions are  $(u, v) = (\frac{21}{4}, -9), (\frac{497}{116}, -\frac{285}{29})$ . Finally, setting  $1992240 = c_2(X)^4$  from (3.3), we obtain  $(u, v) = (\frac{21}{4}, -9)$ . See [BJ11] for details.  $\square$

Recall that  $I^{12}(X)$  is 2-dimensional. Given the class  $\lambda \in H^2(X, \mathbb{Q})$ , define  $\alpha \in I_\lambda^4(X)$  by Poincaré duality to be the unique class (up to a multiple) that intersects trivially with  $\lambda^6$  and  $I^{12}(X)$ .

**Lemma 3.10.** *For  $X = S^{[4]}$  and  $\lambda = \delta$ ,  $\alpha = X - 3Y + Z$ .*

*Proof.* By intersecting with  $\theta$  and  $c_2(S^{[4]})$  using (3.3) and (3.11),  $\theta^3$  and  $\theta^2 c_2(S^{[4]})$  are independent in  $I^{12}(S^{[4]})$ . From (3.9), we find that  $(X - 3Y + Z)\delta^6, (X - 3Y + Z)\theta^3$ , and  $(X - 3Y + Z)\theta^2 c_2(S^{[4]})$  are all zero. See [BJ11] for details.  $\square$

Because the cup-product structure on  $H^*(S^{[4]}, \mathbb{Z})$  is preserved under deformation, we have

**Corollary 3.11.** *For  $\alpha$  chosen as above with respect to  $\lambda \in H^2(X, \mathbb{Z})$ ,  $\alpha$  intersects trivially with*

$$\lambda^4\theta, \lambda^4 c_2(X), \lambda^2\theta^2, \lambda^2\theta c_2(X), \lambda^2 c_2(X)^2, \theta^3, \theta^2 c_2(X), \theta c_2(X)^2, c_2(X)^3$$

Further, up to a rational square,

$$\begin{aligned}\alpha^2\theta^2 &= 9450 \\ \alpha^2\theta c_2(X) &= 14148 \\ \alpha^2c_2(X)^2 &= 21168\end{aligned}$$

*Proof.* By direct computation in  $I_\delta^*(S^{[4]})$ . See [BJ11].  $\square$

**3.12. Middle cohomology.** Putting (3.3), (3.6), and (3.11) together, we now know the complete intersection form on middle cohomology  $I_\lambda^8(X)$  with respect to the basis:

$$\lambda^4, \lambda^2\theta, \lambda^2c_2(X), \theta^2, \theta c_2(X), c_2(X)^2, \alpha\theta, \alpha c_2(X) \quad (3.4)$$

Denoting it by  $M(\lambda)$ , it is:

$$\begin{pmatrix} 105(\lambda, \lambda)^4 & 435(\lambda, \lambda)^3 & 630(\lambda, \lambda)^3 & 2349(\lambda, \lambda)^2 & 3402(\lambda, \lambda)^2 & 4932(\lambda, \lambda)^2 & 0 & 0 \\ 435(\lambda, \lambda)^3 & 2349(\lambda, \lambda)^2 & 3402(\lambda, \lambda)^2 & 19575(\lambda, \lambda) & 28350(\lambda, \lambda) & 44110(\lambda, \lambda) & 0 & 0 \\ 630(\lambda, \lambda)^3 & 3402(\lambda, \lambda)^2 & 4932(\lambda, \lambda)^2 & 28350(\lambda, \lambda) & 44110(\lambda, \lambda) & 59640(\lambda, \lambda) & 0 & 0 \\ 2349(\lambda, \lambda)^2 & 19575(\lambda, \lambda) & 28350(\lambda, \lambda) & 450225 & 652050 & 945300 & 0 & 0 \\ 3402(\lambda, \lambda)^2 & 28350(\lambda, \lambda) & 44110(\lambda, \lambda) & 652050 & 945300 & 1371720 & 0 & 0 \\ 4932(\lambda, \lambda)^2 & 44110(\lambda, \lambda) & 59640(\lambda, \lambda) & 945300 & 1371720 & 1992240 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 9450 & 14148 \\ 0 & 0 & 0 & 0 & 0 & 0 & 14148 & 21168 \end{pmatrix}$$

Note that this matrix is nonsingular if  $(\lambda, \lambda) \neq 0$  (which must be the case for geometric reasons), and therefore (3.4) is in fact a basis.

#### 4. LAGRANGIAN HYPERPLANES IN $X$

Let  $X$  be a  $2n$  dimensional holomorphic symplectic variety, and suppose that  $\mathbb{P}^n \subset X$  is a smoothly embedded Lagrangian hyperplane. By a simple calculation,

**Lemma 4.1.** [HHT] *Denote by  $h$  the hyperplane class on  $\mathbb{P}^n$ . Then in the above setup,*

$$c_{2j}(T_X|_{\mathbb{P}^n}) = (-1)^j h^{2j} \binom{n+1}{j}$$

*Proof.* We have

$$0 \rightarrow T_{\mathbb{P}^n} \rightarrow T_X|_{\mathbb{P}^n} \rightarrow N_{\mathbb{P}^n/X} \rightarrow 0$$

and since  $\mathbb{P}^n$  is Lagrangian,  $N_{\mathbb{P}^n/X} \cong T_{\mathbb{P}^n}^*$ , so

$$c(T_X|_{\mathbb{P}^n}) = (1+h)^{n+1}(1-h)^{n+1} = (1-h^2)^{n+1}$$

$\square$

Let  $\theta$  be the Beauville-Bogomolov class. Then for  $n = 4$ ,

**Lemma 4.2.**  $\theta|_{\mathbb{P}^4} = -\frac{7}{2}h^2$ .

*Proof.* Let  $\theta|_{\mathbb{P}^n} = nh^2$ . By (3.2),

$$60n^2 = 7 \cdot 12n(-5) - 31(-5)^2 + 4(10)$$

$$4n^2 = -28n - 49$$

$$n = -\frac{7}{2}$$

□

Finally, the last intersection theoretic piece of data we need is

$$[\mathbb{P}^n]^2 = c_4(N_{\mathbb{P}^4/X}) = c_4(T_{\mathbb{P}^4}^*) = 5 \quad (4.1)$$

since  $\mathbb{P}^4$  is Lagrangian.

Assume now that  $X$  is deformation equivalent to a Hilbert scheme of 4 points on a K3 surface. Then

$$[\mathbb{P}^4] = a\lambda^4 + b\lambda^2\theta + c\lambda^2c_2(X) + d\theta^2 + e\theta c_2(X) + fc_2(X)^2 + g\theta\alpha + hc_2(X)\alpha$$

Assume that  $\alpha|_{\mathbb{P}^4} = yh^2$ , for  $y \in \mathbb{Q}$ . Intersecting this class with each of (3.4),

$$\lambda^4, \lambda^2\theta, \lambda^2c_2(X), \theta^2, \theta c_2(X), c_2(X)^2, \alpha\theta, \alpha c_2(X)$$

yields by (4.1) and (4.2) the equation

$$M(\lambda)[\mathbb{P}^4] = \begin{pmatrix} \left(\frac{(\lambda,\lambda)}{6}\right)^4 \\ -\frac{7}{2}\left(\frac{(\lambda,\lambda)}{6}\right)^2 \\ -5\left(\frac{(\lambda,\lambda)}{6}\right)^2 \\ \frac{49}{4} \\ \frac{35}{2} \\ 25 \\ -\frac{7}{2}y \\ -5y \end{pmatrix} \quad (4.2)$$

from which it follows that

$$[\mathbb{P}^4] = \begin{pmatrix} \frac{1}{608256} \left(25 + \frac{700}{(\lambda,\lambda)} + \frac{1764}{(\lambda,\lambda)^2}\right) \\ -\frac{1}{2737152} \left(25(\lambda,\lambda) + 3276 + \frac{15876}{(\lambda,\lambda)}\right) \\ \frac{1}{38016} \left(23 + \frac{126}{(\lambda,\lambda)}\right) \\ \frac{1}{5474304} \left((\lambda,\lambda)^2 + 252(\lambda,\lambda) - 41148\right) \\ -\frac{1}{190080} \left(5(\lambda,\lambda) - 2142\right) \\ -\frac{1}{240} \\ \frac{31y}{1188} \\ -\frac{7y}{396} \end{pmatrix} \quad (4.3)$$

Finally, (4.1) yields:

$$5 = \frac{25}{788299776}x^4 + \frac{175}{98537472}x^3 + \frac{403}{10948608}x^2 - \frac{7}{2376}y^2 + \frac{7}{33792}x + \frac{65}{67584}$$

where  $x = (\lambda, \lambda)$ . This we may be rewritten as

$$y^2 = \frac{5^2}{2^{12} \cdot 3^4 \cdot 7} x^4 + \frac{5^2}{2^9 \cdot 3^4} x^3 + \frac{13 \cdot 31}{2^9 \cdot 3^2 \cdot 7} x^2 + \frac{3^2}{2^7} x - \frac{3^2 \cdot 5 \cdot 7^2 \cdot 197}{2^8} \quad (4.4)$$

Note that while we may have  $y \in \mathbb{Q}$ ,  $x$  must be integral. Also note that there is a solution compatible with Conjecture 2, namely  $(x, y) = (-126, 0)$ . By the analysis of the next section,

**Proposition 4.3.** *The only solution of (4.4) with  $x \in \mathbb{Z}$  and  $y \in \mathbb{Q}$  is  $(x, y) = (-126, 0)$ .*

It then follows that

**Theorem 4.4.** *Let  $X$  be of  $K3^{[4]}$ -type,  $\mathbb{P}^4 \subset X$  be a smoothly embedded Lagrangian 4-plane,  $\ell \in H_2(X, \mathbb{Z})$  the class of a line in  $\mathbb{P}^4$ , and  $\rho = 2\ell \in H^2(X, \mathbb{Q})$ . Then  $\rho$  is integral, and*

$$[\mathbb{P}^4] = \frac{1}{337920} (880\rho^4 + 1760\rho^2 c_2(X) - 3520\theta^2 + 4928\theta c_2(X) - 1408c_2(X)^2) \quad (4.5)$$

Further, we must have  $(\ell, \ell) = -\frac{7}{2}$ .

*Proof.* (4.5) is obtained from (4.3) by substituting  $(\lambda, \lambda) = -126$  and  $y = 0$ , after setting  $\rho = \frac{1}{3}\lambda$ . It remains to show that  $\rho$  is integral. Following [HHT], after deforming to a Hilbert scheme of points on a K3 surface  $S$ , we can write

$$\ell = D + m\delta^\vee$$

using the decomposition dual to (1.1), for  $D \in H_2(S, \mathbb{Z})$ . Since

$$(\ell, \ell) = D^2 - \frac{m^2}{6} = -\frac{7}{2}$$

and  $D^2 \in 2\mathbb{Z}$ ,  $3|m$ . For  $2\ell$  to be an integral class in  $H^2(X, \mathbb{Z})$ , by Poincaré duality it is sufficient for the form  $(2\ell, \cdot)$  on  $H_2(X, \mathbb{Z})$  to be integral, which it obviously is, since  $(\delta^\vee, \delta^\vee) = -\frac{1}{6}$ .  $\square$

## 5. SOLVING THE DIOPHANTINE EQUATION

The diophantine equation (4.4) to solve is

$$y^2 = \frac{5^2}{2^{12} \cdot 3^4 \cdot 7} x^4 + \frac{5^2}{2^9 \cdot 3^4} x^3 + \frac{13 \cdot 31}{2^9 \cdot 3^2 \cdot 7} x^2 + \frac{3^2}{2^7} x - \frac{3^2 \cdot 5 \cdot 7^2 \cdot 197}{2^8}$$

with  $x \in \mathbb{Z}$  and  $y \in \mathbb{Q}$ . Let  $\mathcal{C}$  be the affine curve described by the equation.

**Lemma 5.1.** *For every point  $(x, y) \in \mathcal{C}$  with  $x \in \mathbb{Z}$  and  $y \in \mathbb{Q}$  the point  $(x_1, y_1) = (x + 126, 2^6 \cdot 3^2 \cdot 7y)$  is an integral point on the curve  $\mathcal{C}_1$ :*

$$y_1^2 = 5^2 \cdot 7 \cdot x_1^4 - 2^6 \cdot 5^2 \cdot 7^2 \cdot x_1^3 + 2^7 \cdot 3^2 \cdot 7 \cdot 23 \cdot 71 \cdot x_1^2 - 2^{11} \cdot 3^4 \cdot 7^2 \cdot 11^2 \cdot x_1$$

*Proof.* The equation for the curve  $\mathcal{C}$  is equivalent to

$$y_1^2 = 5^2 \cdot 7 \cdot x^4 + 2^3 \cdot 5^2 \cdot 7^2 \cdot x^3 + 2^3 \cdot 3^2 \cdot 7 \cdot 13 \cdot 31 \cdot x^2 + 2^5 \cdot 3^6 \cdot 7^2 \cdot x - 2^4 \cdot 3^6 \cdot 5 \cdot 7^4 \cdot 197$$

having made the change of variables  $y_1 = 2^6 \cdot 3^2 \cdot 7y$ .

We will use the fact that this equation has  $(-126, 0)$  as a solution, which we expect to be the only solution with  $x_1 \in \mathbb{Z}$  and  $y_1 \in \mathbb{Q}$ . Making the change of variables  $x_1 = x + 126$  we get the equation

$$y_1^2 = 5^2 \cdot 7 \cdot x_1^4 - 2^6 \cdot 5^2 \cdot 7^2 \cdot x_1^3 + 2^7 \cdot 3^2 \cdot 7 \cdot 23 \cdot 71 \cdot x_1^2 - 2^{11} \cdot 3^4 \cdot 7^2 \cdot 11^2 \cdot x_1$$

$\square$

**Lemma 5.2.** For an integer  $v$  consider the elliptic curve  $\mathcal{E}_v$  given by the Weierstrass equation

$$y_2^2 = x_2^3 - 2^6 \cdot 5^2 \cdot 7^2 \cdot v \cdot x_2^2 + 2^7 \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot 23 \cdot 71 \cdot v^2 x_2 - 2^{11} \cdot 3^4 \cdot 5^4 \cdot 7^4 \cdot 11^2 \cdot v^3$$

Then every integral point  $(x_1, y_1) \neq (0, 0)$  on the curve  $\mathcal{C}_1$  corresponds to an integral point  $(x_2, y_2)$  on one of the curves  $\mathcal{E}_v$  where

$$\begin{aligned} x_1 &= u^2 v & x_2 &= 5^2 \cdot 7 \cdot v^2 u^2 \\ y_1 &= uvw & y_2 &= 5^2 \cdot 7 \cdot v^2 w \end{aligned}$$

for some integers  $u, v, w$  where  $v$  is a square-free divisor of  $2^{11} \cdot 3^4 \cdot 7^2 \cdot 11^2$ .

*Proof.* Certainly if  $x_1 = 0$  then  $y_1 = 0$  and it can be checked that if  $y_1 = 0$  then  $x_1 = 0$  is the only rational solution. So let us assume for the remaining that  $x_1, y_1 \neq 0$ . Note that since  $x_1 \in \mathbb{Z}$  it follows that  $y_1 \in \mathbb{Z}$  and  $x_1 \mid y_1^2$ . Since  $x_1, y_1 \neq 0$  we may write  $x_1 = u^2 v$  and  $y_1 = uvw$  for  $u, v, w \in \mathbb{Z}$  with  $v$  square-free. Rewriting the equation we get

$$vw^2 = 5^2 \cdot 7 \cdot u^6 v^3 - 2^6 \cdot 5^2 \cdot 7^2 \cdot u^4 v^2 + 2^7 \cdot 3^2 \cdot 7 \cdot 23 \cdot 71 \cdot u^2 v - 2^{11} \cdot 3^4 \cdot 7^2 \cdot 11^2$$

and we conclude that  $v$  is a square-free divisor of  $2^{11} \cdot 3^4 \cdot 7^2 \cdot 11^2$ .

Multiplying by  $5^4 \cdot 7^2 \cdot v^3$  and making the change of variables  $y_2 = 5^2 \cdot 7 \cdot v^2 \cdot w$  and  $x_2 = 5^2 \cdot 7 \cdot v^2 \cdot u^2$  we get the equation

$$\begin{aligned} (5^2 \cdot 7 \cdot v^2 \cdot w)^2 &= (5^2 \cdot 7 \cdot v^2 \cdot u^2)^3 - 2^6 \cdot 5^2 \cdot 7^2 \cdot v \cdot (5^2 \cdot 7 \cdot v^2 \cdot u^2)^2 + 2^7 \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot 23 \cdot 71 \cdot v^2 (5^2 \cdot 7 \cdot v^2 \cdot u^2) \\ &\quad - 2^{11} \cdot 3^4 \cdot 5^4 \cdot 7^4 \cdot 11^2 \cdot v^3 \\ y_2^2 &= x_2^3 - 2^6 \cdot 5^2 \cdot 7^2 \cdot v \cdot x_2^2 + 2^7 \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot 23 \cdot 71 \cdot v^2 x_2 - 2^{11} \cdot 3^4 \cdot 5^4 \cdot 7^4 \cdot 11^2 \cdot v^3 \end{aligned}$$

which gives a point  $(x_2, y_2) \in \mathcal{E}_v(\mathbb{Z})$ . □

Thus to find the required points on  $\mathcal{C}$  we need to find the integral solutions of the elliptic curve  $E_v$  above whenever  $v$  is a square-free divisor of  $2^{11} \cdot 3^4 \cdot 7^2 \cdot 11^2$ , of which there are 32 (positive and negative).

**Lemma 5.3.** If the curve  $\mathcal{E}_v$  where  $v$  is a square-free divisor of  $2^{11} \cdot 3^4 \cdot 7^2 \cdot 11^2$  such that  $7 \nmid v$  has an integral solution  $(5^2 \cdot 7 \cdot u^2 v^2, 5^2 \cdot 7 \cdot v^2 w)$  then  $v \in \{-1, -2, -11, -22\}$ .

*Proof.* Note from the equation

$$vw^2 = 5^2 \cdot 7 \cdot u^6 v^3 - 2^6 \cdot 5^2 \cdot 7^2 \cdot u^4 v^2 + 2^7 \cdot 3^2 \cdot 7 \cdot 23 \cdot 71 \cdot u^2 v - 2^{11} \cdot 3^4 \cdot 7^2 \cdot 11^2$$

we deduce that  $7 \mid vw^2$ . Since  $7 \nmid v$  it follows that  $7 \mid w$  so it must be that  $5^2 u^6 v^3 + 2^7 \cdot 3^2 \cdot 23 \cdot 71 u^2 v \equiv 0 \pmod{7}$  in other words  $u^2 v \equiv 3u^6 v^3 \pmod{7}$ . Since  $v$  is invertible we get  $5u^2 \equiv u^6 v^2$ . If  $7 \nmid u$  then we would have that 5 is a quadratic residue mod 7, which is not true. So  $7 \mid u$ . Rewriting the equation for  $w = 7w_1$  and  $u = 7u_1$  we get

$$vw_1^2 = 5^2 \cdot 7^5 \cdot u_1^6 v^3 - 2^6 \cdot 5^2 \cdot 7^4 \cdot u_1^4 v^2 + 2^7 \cdot 3^2 \cdot 7 \cdot 23 \cdot 71 \cdot u_1^2 v - 2^{11} \cdot 3^4 \cdot 11^2$$

so necessarily  $vw_1^2 \equiv 3 \pmod{7}$ . But the only square-free divisors  $v$  of  $2 \cdot 3 \cdot 11$  for which such  $w_1$  exist are 3, 6, 33, 66, -1, -2, -11, -22.

If  $3 \mid v$  then we could write  $v = 3v_1$  so we would get

$$v_1 w_1^2 = 5^2 \cdot 3^2 \cdot 7^5 \cdot u_1^6 v_1^3 - 2^6 \cdot 3 \cdot 5^2 \cdot 7^4 \cdot u_1^4 v_1^2 + 2^7 \cdot 3^2 \cdot 7 \cdot 23 \cdot 71 \cdot u_1^2 v_1 - 2^{11} \cdot 3^3 \cdot 11^2$$

which would imply that  $3 \mid v_1 w_1^2$ . Since  $3 \nmid v_1$  (as  $v$  is square-free) it follows that  $3^2 \mid v_1 w_1^2$  but then  $3^2$  divides the right hand side so we deduce that  $3 \mid u_1$ . Writing  $w_1 = 3w_2$  and  $u_1 = 3u_2$  we get

$$v_1 w_2^2 = 5^2 \cdot 3^6 \cdot 7^5 \cdot u_2^6 v_1^3 - 2^6 \cdot 3^3 \cdot 5^2 \cdot 7^4 \cdot u_2^4 v_1^2 + 2^7 \cdot 3^2 \cdot 7 \cdot 23 \cdot 71 \cdot u_2^2 v_1 - 2^{11} \cdot 3 \cdot 11^2$$

As before, we get that  $3^2 \mid v w^2$  but now  $3^2$  cannot divide the right hand side.

The remaining possibilities for  $v$  are  $-1, -2, -11, -22$ .  $\square$

**Lemma 5.4.** *If the curve  $\mathcal{E}_v$  where  $v$  is a square-free divisor of  $2^{11} \cdot 3^4 \cdot 7^2 \cdot 11^2$  such that  $7 \mid v$  has an integral solution  $(5^2 \cdot 7 \cdot u^2 v^2, 5^2 \cdot 7 \cdot v^2 w)$  then  $v \in \{7, 14, 77, 154\}$ .*

*Proof.* Writing  $v = 7v_1$  we get

$$v_1 w^2 = 5^2 \cdot 7^3 \cdot u^6 v_1^3 - 2^6 \cdot 5^2 \cdot 7^3 \cdot u^4 v_1^2 + 2^7 \cdot 3^2 \cdot 7 \cdot 23 \cdot 71 \cdot u^2 v_1 - 2^{11} \cdot 3^4 \cdot 7 \cdot 11^2$$

Since  $v$  is square-free  $7 \nmid v_1$  so we deduce that  $7 \mid w$ . Writing  $w = 7w_1$  we get

$$7v_1 w_1^2 = 5^2 \cdot 7^2 \cdot u^6 v_1^3 - 2^6 \cdot 5^2 \cdot 7^2 \cdot u^4 v_1^2 + 2^7 \cdot 3^2 \cdot 23 \cdot 71 \cdot u^2 v_1 - 2^{11} \cdot 3^4 \cdot 11^2$$

which implies that  $u^2 v_1 \equiv 4 \pmod{7}$ . The only  $v_1$  among the square-free divisors of  $2 \cdot 3 \cdot 11$  for which such  $u$  exist are  $1, 2, 11, 22, -3, -6, -33, -66$  giving  $v \in \{7, 14, 77, 154, -21, -42, -231, -462\}$ .

As in the previous lemma, under the assumption that  $3 \mid v$  we get a contradiction. The remaining possibilities are  $v \in \{7, 14, 77, 154\}$ .  $\square$

**Lemma 5.5.** *If  $v \in \{-1, -2, 7, 14, 77, 154\}$  the curve  $\mathcal{E}_v$  has no integral points of the form  $(5^2 \cdot 7 \cdot u^2 v^2, 5^2 \cdot 7 \cdot v^2 w)$ .*

*Proof.* (1) The elliptic curve  $\mathcal{E}_7$  has rank 1 and a computation in SAGE shows that it has no integral point.

(2) The elliptic curve  $\mathcal{E}_{14}$  has rank 4. A computation in SAGE shows that there are 17 integral points (with an ambiguity on sign for the  $y$ -coordinate) on  $\mathcal{E}_{14}$ :  
 $(564480, 49392000), (604905, 101433675), (632100, 129859800), (683844, 180931128),$   
 $(755825, 251976375), (940800, 451113600), (1063680, 599510400), (1317120, 945033600),$   
 $(1361220, 1010272200), (2257920, 2617776000), (3066624, 4451914368), (3327780, 5110549800),$   
 $(11863929, 38995732083), (12603780, 42818542200), (13848576, 49513570176),$   
 $(72195620, 608777597400), (1964277504, 87032792472192),$  but none of the  $x$ -coordinates are of the form  $2^2 \cdot 5^2 \cdot 7^3 \cdot u^2$ .

(3) A computation in SAGE (7 hours) shows that  $\mathcal{E}_{77}$  has no integral points.

(4) A computation in SAGE (7 hours) shows that  $\mathcal{E}_{154}$  has no integral points.

(5) A computation in SAGE (2 hours) shows that  $\mathcal{E}_{-1}$  has 3 integral points (with an ambiguity on sign for the  $y$ -coordinate):  $(-39196, 156792), (-27900, 2266200), (166980, 85186200),$  but none of the  $x$ -coordinates are of the required form  $x = 5^2 \cdot 7 \cdot (-1)^2 \cdot u^2$ .

(6) A computation in SAGE (2 hours) show that the only integral point (with an ambiguity on sign for the  $y$ -coordinate) is  $(0, 15523200)$ . However, the  $x$ -coordinate was assumed to be nonzero.  $\square$

**Lemma 5.6.** *If  $v \in \{-11, -22\}$  the curve  $\mathcal{E}_v$  has no integral points of the form  $(5^2 \cdot 7 \cdot u^2 v^2, 5^2 \cdot 7 \cdot v^2 w)$ .*

*Proof.* These two curves are not amenable to computations in SAGE. The authors are grateful to Michael Stoll for explaining how to do these computations in Magma. The idea is that the computation of integral points on elliptic curves in SAGE uses the mwrank library for the computation of a basis of the Mordell-Weil group, and this does not terminate in acceptable time for the two curves in question. Instead, we use Magma to compute two descents on the two curves, giving rational points on the elliptic curves, and then the Cremona-Pricket-Siksek method to find generators for the two rank one curves.

(1) We start with the curve  $E = \mathcal{E}_{-11}$ . The elliptic curve  $E$  is

$$y^2 = x^3 + 2^6 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot x^2 + 2^7 \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot 11^2 \cdot 23 \cdot 71 \cdot x + 2^{11} \cdot 3^4 \cdot 5^4 \cdot 7^4 \cdot 11^5$$

Via the change of variables  $x = 4x_1 - 287468, y = 8y_1$  we get the minimal Weierstrass equation  $E'$

$$y_1^2 = x_1^3 - x_1^2 + 1933249267x_1 + 116312127942837$$

Two descent in Magma gives a double cover by

$$y^2 = -1386x^4 + 5698x^3 + 52976x^2 + 33572x + 482944$$

which has a rational point  $(x, y) = (-3, -770)$  giving the rational point of infinite order

$$P = \left( \frac{195693}{4}, \frac{144883425}{8} \right)$$

on  $E'$  which will then have rank 1 (note that  $E'(\mathbb{Q})$  has trivial torsion).

Writing  $h$  for the logarithmic height on  $E'$  and  $\widehat{h}$  for the canonical height, the Cremona-Pricket-Siksek bound is a real number  $B$  such that  $h(Q) \leq \widehat{h}(Q) + B$  for all  $Q \in E'(\mathbb{Q})$ . This constant can be computed in Magma to be  $B = 11.424\dots$ . Also in Magma one computes  $\widehat{h}(P) = 11.289\dots$

Suppose  $P$  is not a generator of  $E'(\mathbb{Q})$ . Then if  $P_0$  is a generator, it follows that  $P = nP_0$  for some integer  $n$  such that  $|n| \geq 2$ . Then  $\widehat{h}(P_0) \leq \frac{\widehat{h}(P)}{n^2} \leq \frac{\widehat{h}(P)}{4}$  so  $h(P_0) = \widehat{P}_0 + h(P_0) - \widehat{h}(P_0) \leq \frac{\widehat{h}(P)}{4} + B$ . Thus, to find  $P_0$  one only needs to search in SAGE rational points of height at most  $\frac{1}{4}\widehat{h}(P) + B$ . This computation shows that in fact  $P = P_0$  is a generator.

Transferring back to  $E(\mathbb{Q})$  one obtains the generator  $(x, y) = (-91775, 144883425)$  of  $E(\mathbb{Q})$ . Using SAGE to compute the integral points, inputting manually the basis for  $E(\mathbb{Q})$ , one obtains that  $E(\mathbb{Q})$  has only one integral point (up to ambiguity in the sign of  $y$ ), but  $x = -91775$  is not of the required form.

(2) The elliptic curve  $E = \mathcal{E}_{-22}$  is

$$y^2 = x^3 + 2^7 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot x^2 + 2^9 \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot 11^2 \cdot 23 \cdot 71 \cdot x + 2^{14} \cdot 3^4 \cdot 5^4 \cdot 7^4 \cdot 11^5$$

via the change of variables  $x = 16x_1 - 574928, y = 64y_1$  gives the minimal model  $E'$

$$y_1^2 = x_1^3 + x_1^2 + 483312317x_1 + 14539257649013$$

Again, two descent in Magma gives the curve  $C$

$$y^2 = -41715x^4 + 152820x^3 - 54260x^2 - 2920x + 54900$$

which shows that  $E$  has rank 1. The rational point  $\left(\frac{2}{3}, \frac{2310}{3}\right)$  on  $C$  gives the point  $P = (-17428, -907137)$  in  $E'(\mathbb{Q})$  of infinite order. Its height is  $\widehat{h}(P) = 5.106\dots$  while  $B = 10.774\dots$ . As before this allows one to show that  $P$  is a generator of  $E'(\mathbb{Q})$ . The point  $P$  corresponds to the point  $(-853776, 58056768)$ , a generator of  $E(\mathbb{Q})$ . Again, feeding this into SAGE gives that the only possible integral value for  $x$  is  $-853776$ , which is negative, and hence not of the desired form. □

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