Stochastic Processes and their Applications 42 (1992) 171-180
North-Holland


# Large deviations and the Strassen theorem in Hölder norm

P. Baldi
*Dipartimento di Matematica, Università di Tor Vergata, Rome, Italy, and Laboratoire de Probabilités, Paris, France*

G. Ben Arous
*Département de Mathématique, Université de Paris-Sud, Orsay, France*

G. Kerkyacharian
*U.E.R. de Mathématique, Université de Nancy I, Nancy, France*

Received 9 November 1990
Revised 4 April 1991

We prove that Schilder's theorem, giving large deviations estimates for the Brownian motion multiplied by a small parameter, still holds with the sup-norm replaced by any Hölder norm with exponent $\alpha < \frac{1}{2}$. We produce examples which show that this is effectively a stronger result and, as an application, we prove Strassen's Iterated Logarithm Law in these stronger topologies.

large deviations * iterated logarithm law

## 1. Introduction

Schilder's theorem giving the Large Deviations Principle for Wiener measure is usually stated with respect to the sup-norm topology. It becomes a stronger statement if one uses any Hölder norm (of exponent $\alpha < \frac{1}{2}$) instead.

We prove here that this stronger statement is true as a consequence of the general principle of large deviations for Gaussian measures on separable Banach spaces. Of course one has to handle the fact that the space of Hölder paths is not separable and that it is not wise to deal directly with an explicit description of its dual.

The second section of this paper contains a proof of this Hölder Large Deviations Principle. Many other proofs are possible as the one hinted at in Ben Arous and Léandre (1988), where the basic tool is Fernique-Landau-Schepp's integrability theorem.

*Correspondence to*: Dr. P. Baldi, Dipartimento di Matematica, Università di Tor Vergata, Via del Fontanile di Carcaricola, 00173 Rome, Italy.

0304-4149/92/$05.00 

A stronger topology gives of course a sharper Large Deviations Principle, the closure of a set being smaller and the interior larger, but it gives also more continuous (or semi-continuous) functions and so it gives a broader scope to the Varadhan-Laplace theorem. This was the starting point for the use of Hölder large deviations in Ben Arous and Léandre (1988) which enabled the second named author and R. Léandre to find a surprising example of an exponential decay for a degenerate heat kernel on the diagonal in small time, due to a drift term in the horizontal space.

The main point of this paper is the use of this Hölder Large Deviations Principle to improve Strassen's functional law of the Iterated Logarithm, namely to prove that in Strassen's theorem the convergence statement can be made in Hölder norm.

## 2. Large deviations for Hölder paths

Let us denote by $\mathscr{C}^\alpha$ the Banach Space of all $\alpha$-Hölder paths $\gamma:[0,1]\to\mathbb{R}^m$, such that $\gamma(0)=0$, endowed with the norm

$$\|\gamma\|_\alpha = \sup_{s,t\in[0,1]} \frac{|\gamma(t)-\gamma(s)|}{|t-s|^\alpha}.$$

For every $\delta>0$ let us set

$$\omega_\gamma(\delta) = \sup_{\substack{s,t\in[0,1]\\ |t-s|\le\delta}} \frac{|\gamma(t)-\gamma(s)|}{|t-s|^\alpha}$$

so that the modulus of continuity of $\gamma$ is $\delta^\alpha\omega_\gamma(\delta)$. We shall denote by $\mathscr{C}^{\alpha,0}$ the subspace of $\mathscr{C}^\alpha$ of all paths such that $\lim_{\delta\to 0}\omega_\gamma(\delta)=0$. It is wellknown (Ciesielski, 1960, for instance) that $\mathscr{C}^{\alpha,0}$ is a closed subspace of $\mathscr{C}^\alpha$, so that (endowed with the norm $\|\ \|_\alpha$) it is a Banach space, and that it is separable (whereas $\mathscr{C}^\alpha$ is not).

Let $B$ be a continuous Brownian motion. Since its sample paths are $\alpha$-Hölder continuous for every $\alpha<\frac{1}{2}$, we may consider $B$ as a r.v. taking values in $\mathscr{C}^{\alpha,0}$, $\alpha<\frac{1}{2}$.

The main goal of this section is to prove the following large deviations estimate.

**Theorem 2.1.** *For every Borel subset $A\subset\mathscr{C}^{\alpha,0}$,*

$$\limsup_{\varepsilon\to 0} \varepsilon^2\log P\{\varepsilon B\in A\}\le -\Lambda(\bar{A}),$$

$$\liminf_{\varepsilon\to 0} \varepsilon^2\log P\{\varepsilon B\in A\}\ge -\Lambda(\mathring{A}),$$

*where $\Lambda(A)=\inf_{\gamma\in A}\lambda(\gamma)$, $\lambda$ being defined by*

$$\lambda(\gamma)=\begin{cases}\frac{1}{2}\int_0^1|\gamma'(s)|^2\,\mathrm{d}s & \text{if } \gamma \text{ is a.c.,}\\ +\infty & \text{otherwise,}\end{cases} \tag{2.1}$$

*the operations of closure and interior part being taken in the topology of $\mathscr{C}^{\alpha,0}$.* □

Theorem 2.1 is clearly a refinement of the classical Schilder large deviations estimates (Schilder, 1966); we shall prove it as a consequence of the following wellknown large deviations result for Gaussian measures.

**Definition 2.2.** An abstract Wiener space is a quadrupole $(W, H, j, \mu)$ where
(a) $W$ is a separable real Banach space;
(b) $H$ is a separable real Hilbert space;
(c) $j$ is a continuous linear injection $H \overset{j}{\hookrightarrow} W$ such that $j(H)$ is dense in $W$;
(d) $\mu$ is a probability measure on $(W, \mathscr{B}(W))$ such that for every $w' \in W'$,

$$\int_W \exp \mathrm{i}\langle w', w\rangle \mu(\mathrm{d}w) = \exp -\tfrac{1}{2}\|j^* w'\|_H^2$$

where $\langle\, ,\rangle$ denotes the duality between $W$ and its dual $W'$ and $j^* : W' \to H' = H$ is the adjoint transformation of $j$.

**Theorem 2.3.** *Let $(W, H, j, \mu)$ be an abstract Wiener space. Then for every Borel subset $A \subset W$,*

$$\limsup_{\varepsilon\to 0} \varepsilon^2 \log \mu\left(\frac{1}{\varepsilon} A\right) \leqslant -\Lambda_\mu(\bar{A}),$$

$$\liminf_{\varepsilon\to 0} \varepsilon^2 \log \mu\left(\frac{1}{\varepsilon} A\right) \geqslant -\Lambda_\mu(\mathring{A}),$$

*where $\Lambda_\mu(A) = \inf_{w\in A} \lambda_\mu(w)$, $\lambda_\mu : W \to [0, +\infty]$ being defined by*

$$\lambda_\mu(w) = \begin{cases} \frac{1}{2}\|j^{-1} w\|_H^2 & \text{if } w \in j(H), \\ +\infty & \text{if } w \notin j(H). \end{cases} \qquad \square$$

A proof of Theorem 2.3 can be found in Deuschel and Stroock (1989, Theorem 3.4.12).

Let us denote by $\mathscr{C} = \mathscr{C}([0,1], \mathbb{R}^d)$ the set of all continuous paths $\gamma : [0,1] \to \mathbb{R}^d$ which are continuous and such that $\gamma(0) = 0$ and by $H_1$ the subspace of $\mathscr{C}$ of all paths $\gamma$ which are absolutely continuous and whose derivative is square integrable. $\mathscr{C}$ is a separable Banach space with respect to the uniform norm and $H_1$ is a Hilbert space with respect to the scalar product

$$\langle \gamma_1, \gamma_2\rangle_{H_1} = \int_0^1 (\gamma_1'(s), \gamma_2'(s))\, \mathrm{d}s.$$

If $j : H_1 \hookrightarrow \mathscr{C}$ and $\mu^{\mathrm{W}}$ is the Wiener measure, it is wellknown that $(\mathscr{C}, H_1, j, \mu^{\mathrm{W}})$ is an abstract Wiener space.

In order to prove Theorem 2.1 we shall check that $(\mathscr{C}^{\alpha,0}, H_1, j, \mu^{\mathrm{W}})$ is an abstract Wiener space. Even if this is certainly not a surprising fact we think that it deserves to be verified, which is done in the following statement. By the way it implies the wellknown fact that $\mathscr{C}^{\alpha,0}$ is the closure of $H_1$ in $\mathscr{C}^{\alpha}$.

It should be mentioned that even if we were unable to find the following Theorem 2.4 in the literature, many results similar to it are known (see Baxendale, 1976, for instance).

**Theorem 2.4.** *Let $W_1$, $W_2$ be separable Banach spaces, $H$ a separable Hilbert space such that $H \overset{j}{\hookrightarrow} W_1 \hookrightarrow W_2$, all the embeddings being continuous. Let $\mu$ be a probability measure on $W_2$ such that $(W_2, H, j, \mu)$ is an abstract Wiener space and $\mu^*(W_1) = 1$. Then if $j(H)$ is dense in $W_1$ and $\nu$ denotes the trace of $\mu$ on $W_1$, $(W_1, H, j, \nu)$ is an abstract Wiener space.*

**Proof.** The trace probability $\nu$ is defined on the trace $\sigma$-field $\mathscr{B}_1$ of $\mathscr{B}(W_2)$ on $W_1$. Let us prove first that $\mathscr{B}_1 = \mathscr{B}(W_1)$, or, which is equivalent by the Hahn–Banach theorem and separability, that every $w_1' \in W_1'$ is measurable with respect to $\mathscr{B}_1$ (the inclusion $\mathscr{B}_1 \subset \mathscr{B}(W_1)$ is obvious since $W_1 \hookrightarrow W_2$ continuously).

Let us remark first that $W_2'$ is dense in $W_1'$ in the weak* topology. Otherwise, by the Hahn–Banach theorem applied to the locally convex vector space $W_1$ endowed with its weak* topology, there would exist a $0 \neq w_1 \in W_1$ such that $\langle w_1, w_2' \rangle = 0$ for every $w_2' \in W_2'$; this is impossible since by assumption $W_1$, which contains $j(H)$, is dense in $W_2$ and thus $W_2'$ separates the points of $W_1$.

It could be seen that $W_1'$ is exactly the smallest space of functions on $W_1$ which contains $W_2'$ and is stable by pointwise limit of sequences of $W_1$. Clearly this space is contained in $W_1'$ by the Banach–Steinhaus theorem, and is closed in the weak* topology of $W_1'$, by the Krein–Smulian theorem and the metrizability of this topology when it is restricted to the balls of $W_1'$. So $W_1'$ is $\mathscr{B}_1$-measurable.

It is a wellknown fact that pointwise convergence of sequences of Gaussian r.v. implies that the limit is Gaussian and the convergence takes place in $L^2$. So, using the previous argument, it is easy to see that $W_1'$ is contained in the closure of $W_2'$ in $L^2(W_1, d\nu)$ and is a Gaussian space.

It remains to prove that for $w'$, $w''$ belonging to $W_1'$, we have

$$\int_{W_1} w' w'' \, d\nu = \langle j^*(w'), j^*(w'') \rangle_H. \tag{2.2}$$

For $w' \in W_1'$, let $M_{w'}$ be the set of $w'' \in W_1'$ for which (2.2) is true. If $w' \in W_2'$, $M_{w'}$ contains $W_2'$ and is stable by pointwise limit of sequence, because if $\{w_n\}_n$ converges weakly to $w$ in $W_1'$, $\{j^*(w_n)\}_n$ converges weakly to $j^*(w)$ in $H$. So $M_{w'} = W_1'$. But now, if $w' \in W_1'$, $M_{w'}$ contains $W_2'$ and is stable by pointwise limit of sequence. So $M_{w'} = W_1'$, and this ends the proof. □

**Remark.** Theorem 2.1 still holds if $B$ is considered as a r.v. taking values in $\mathscr{C}^{\alpha}$, $\alpha < \frac{1}{2}$. It suffices to remark that $B$ takes values in $\mathscr{C}^{\alpha,0}$ a.s., that $\lambda = +\infty$ outside $\mathscr{C}^{\alpha,0}$ and that $\mathscr{C}^{\alpha,0}$ is a closed convex subset of $\mathscr{C}^{\alpha}$ (see for instance Baldi, 1988, Theorem 1.2).

The arguments of this section also give a Large Deviation Principle in Hölder norm for the Brownian bridge.

**Example.** Let $\eta$, $p$, $m$, $a$, $b$ be positive numbers such that $p<m$ and

$$\frac{b}{a}\in\left]1,\frac{1+\frac{1}{2}m}{1+\frac{1}{2}p}\right[$$

so that

$$0<b-a<\tfrac{1}{2}(am-bp).$$

Let

$$A=\left\{\gamma\in\mathscr{C}([0,1],\mathbb{R}),\left(\int_0^1|\gamma_s|^p\,\mathrm{d}s\right)^b<\eta\left(\int_0^1|\gamma_s|^m\,\mathrm{d}s\right)^a\right\}.$$

It can be checked that the uniform closure of $A$ contains 0, so that the usual Schilder theorem gives no information about the decay of $\mu(A/\varepsilon)$ (for a similar line of reasoning see Lemma 1.5 of Ben Arous and Léandre 1988).

But it is also a simple fact that the closure of $A$ in $\mathscr{C}^{\alpha,0}$ for $(b-a)/(am-bp)<\alpha<\frac{1}{2}$ does not contain 0 so that by Theorem 2.1, $\limsup_{\varepsilon\to 0}\varepsilon^2\log\mu(A/\varepsilon)<0$.

In the same way if

$$F(\gamma)=\left(\int_0^1|\gamma_s|^p\,\mathrm{d}s\right)^b\Big/\left(\int_0^1|\gamma_s|^m\,\mathrm{d}s\right)^a$$

then the usual Schilder result can only give

$$\limsup_{\varepsilon\to 0}\varepsilon^2\log E(\mathrm{e}^{-F(\varepsilon\,\cdot\,)/\varepsilon^2})\leq 0$$

because the only l.s.c. definition of $F$ at 0 is $F(0)=0$. But in Hölder norm $F(0)=+\infty$ gives a l.s.c. function, so that

$$\limsup_{\varepsilon\to 0}\varepsilon^2\log E(\mathrm{e}^{-F(\varepsilon\,\cdot\,)/\varepsilon^2})<0.$$

## 3. Strassen's theorem in Hölder norm

Let $B$ be a continuous Brownian motion and let us consider for $t>0$,

$$\zeta_t(\omega)=\left(s\to\frac{B_{ts}(\omega)}{(t\log\log t)^{1/2}},0\leq s\leq 1\right). \tag{3.1}$$

For every $t>0$ $\zeta_t$ is a r.v. taking values in $\mathscr{C}$ and it is a classic result of Strassen (1964) that as $t\to\infty$, $(\zeta_t)_t$ is relatively compact and has

$$\mathscr{K}=\left\{\gamma;\ \gamma\text{ is absolutely continuous and }\tfrac{1}{2}\int_0^1|\gamma'(s)|^2\,\mathrm{d}s\leq 1\right\}$$

as limit set a.s. Our goal is to prove Strassen's theorem in Hölder norm.

**Theorem 3.1.** *Let $\zeta_t : \Omega \to \mathscr{C}^{\alpha,0}$, $\alpha < \frac{1}{2}$, be defined through* (3.1). *Then as $t \to \infty$ $(\zeta_t)_{t>0}$ is relatively compact and has $\mathscr{K}$ as limit set a.s. in the $\mathscr{C}^{\alpha,0}$ topology.* □

Theorem 3.1 will follow from Propositions 3.5 and 3.6 below. The idea of the proof is simple once one remembers the way Strassen's Law can be proved using Large Deviations estimates (see Baldi, 1986, for instance).

Let us note

$$\mathrm{LL}(t) = \log\log t, \qquad \phi(t) = \sqrt{t\,\mathrm{LL}(t)}.$$

Let be $c > 1$. In the following we shall make use repeatedly of the fact that

$$\exp(-M\,\mathrm{LL}(c^n)) = \mathrm{const}/n^M \tag{3.2}$$

is summable in $n$ if and only if $M > 1$. By scaling and Theorem 2.1 if $A$ is a closed subset of $\mathscr{C}^{\alpha,0}$, one has for every $\delta > 0$ and large $n$,

$$P\{\zeta_{c^n} \in A\} = P\left\{\frac{B}{\sqrt{\mathrm{LL}(c^n)}} \in A\right\} \leq \exp(-\mathrm{LL}(c^n)(\Lambda(A) - \delta)). \tag{3.3}$$

So that by (3.2), $P\{\zeta_{c^n} \in A\}$ is summable if $\Lambda(A) > 1$. The same argument gives that $P\{\zeta_{c^n} \in A\}$ for an open set $A \in \Omega$ is the general term of a divergent series if $\Lambda(A) < 1$.

Let us remind finally that $\mathscr{K}$ is compact (the embedding $H_1 \hookrightarrow \mathscr{C}^{\alpha,0}$ is compact) and that the functional $\lambda$ defined in (2.1) is l.s.c. We shall note

$$\mathscr{K}_\eta = \{v \in \mathscr{C}^{\alpha,0},\ \|v - \gamma\|_\alpha < \eta \text{ for some } \gamma \in \mathscr{K}\}.$$

**Lemma 3.2.** *For every $\eta > 0$ and $c > 1$ there exists $n_0 = n_0(\omega)$ such that $\zeta_{c^n} \in \mathscr{K}_\eta$ for every $n > n_0$.*

**Proof.** By the Borel–Cantelli lemma it is sufficient to prove that $P\{\zeta_{c^n} \in \mathscr{K}_\eta^c\}$ is summable. In view of our previous remarks ($\mathscr{K}_\eta^c$ is closed) we just need to check that $\Lambda(\mathscr{K}_\eta^c) > 1$. Indeed $\mathscr{K} = \{\lambda \leq 1\}$ and, since the level sets of $\lambda$ are compact, $\lambda$ has an absolute minimum $\gamma_0$ in $\mathscr{K}_\eta^c$. If $\Lambda(\mathscr{K}_\eta^c) \leq 1$ that would imply $\lambda(\gamma_0) \leq 1$ and thus $\gamma_0 \in \mathscr{K}$, which is impossible. □

Let us set

$$Y_n = \sup_{c^n \leq u \leq c^{n+1}} \frac{1}{\phi(c^n)} \|B_{u\cdot} - B_{c^n\cdot}\|_\alpha.$$

**Lemma 3.3.** *For every $\varepsilon > 0$ there exists $c_\varepsilon > 1$ such that for every $1 < c < c_\varepsilon$ there exists $n_0 = n_0(\omega)$ such that $Y_n(\omega) \leq \varepsilon$ for every $n \geq n_0$.*

**Proof.** We have to prove that $P\{Y_n > \varepsilon\}$ is summable.

$$
\begin{aligned}
P\{Y_n > \varepsilon\} &= P\left\{ \sup_{c^n \leq u \leq c^{n+1}} \frac{1}{\phi(c^n)} \|B_{u\cdot} - B_{c^n\cdot}\|_\alpha \geq \varepsilon \right\} \\
&= P\left\{ \sup_{c^n \leq u \leq c^{n+1}} \frac{\sqrt{u}}{\phi(c^n)} \|B_{\cdot} - B_{(c^n/u)\cdot}\|_\alpha \geq \varepsilon \right\} \\
&\leq P\left\{ \sup_{c^n \leq u \leq c^{n+1}} \frac{1}{\sqrt{\mathrm{LL}(c^n)}} \|B_{\cdot} - B_{(c^n/u)\cdot}\|_\alpha \geq \varepsilon/\sqrt{c} \right\} \\
&= P\left\{ \sup_{1 \leq v \leq c} \frac{1}{\sqrt{\mathrm{LL}(c^n)}} \|B_{\cdot} - B_{\cdot/v}\|_\alpha \geq \varepsilon/\sqrt{c} \right\} \\
&= P\left\{ \frac{1}{\sqrt{\mathrm{LL}(c^n)}} B \in A \right\}
\end{aligned}
$$

where $A \in \mathscr{B}(\mathscr{C}^{\alpha,0})$ is the set of paths

$$
A = \left\{ \gamma \in \mathscr{C}^{\alpha,0},\ \sup_{1 \leq v \leq c} \|\gamma_{\cdot} - \gamma_{\cdot/v}\|_\alpha \geq \varepsilon/\sqrt{c} \right\}.
$$

Since $A$ is closed in $\mathscr{C}^{\alpha,0}$ we have just to prove that if $c > 1$ small enough then $\Lambda(A) > 1$. Indeed

$$
\sup_{1 \leq v \leq c} \|\gamma_{\cdot} - \gamma_{\cdot/v}\|_\alpha = \sup_{1 \leq v \leq c}\ \sup_{0 \leq s < t \leq 1} \frac{1}{|t-s|^\alpha} |(\gamma_t - \gamma_{t/v}) - (\gamma_s - \gamma_{s/v})|.
$$

Thus if $\gamma \in A$ one has for some values of $s < t$, $1 \leq v \leq c$,

$$
\begin{aligned}
\frac{\varepsilon}{2\sqrt{c}} |t-s|^\alpha &\leq |(\gamma_t - \gamma_{t/v}) - (\gamma_s - \gamma_{s/v})| \\
&= \left| \int_{t/v}^{t} \gamma'(r)\,\mathrm{d}r - \int_{s/v}^{s} \gamma'(r)\,\mathrm{d}r \right| \\
&= \left| \int_{s \vee (t/v)}^{t} \gamma'(r)\,\mathrm{d}r - \int_{s/v}^{s \wedge (t/v)} \gamma'(r)\,\mathrm{d}r \right| \\
&\leq (|t - s \vee (t/v)|^{1/2} + |s \wedge (t/v) - s/v|^{1/2}) \|\gamma'\|_{L^2}.
\end{aligned}
$$

Thus if $\gamma \in A$,

$$
\lambda(\gamma) \geq \frac{\varepsilon^2}{8c} \left( \frac{|t-s|^\alpha}{|t - s \vee (t/v)|^{1/2} + |s \wedge (t/v) - s/v|^{1/2}} \right)^2
$$

so that from the following Lemma 3.4,

$$
\lambda(\gamma) \geq (\varepsilon^2/32c)(c-1)^{2\alpha-1}
$$

which implies $\Lambda(A) \geq (\varepsilon^2/32c)(c-1)^{2\alpha-1}$, and since $2\alpha - 1 < 0$, for $c > 1$ small enough one has $\Lambda(A) > 1$. $\square$

**Lemma 3.4.** *If*

$$F(v,s,t)=\left(\frac{|t-s|^{\alpha}}{|t-s\vee(t/v)|^{1/2}+|s\wedge(t/v)-s/v|^{1/2}}\right)$$

*then*

$$F(s,t,v)\geq\tfrac{1}{2}|c-1|^{\alpha-1/2}$$

*for every* $1\leq v\leq c$, $0\leq s<t\leq 1$.

**Proof.** We shall consider separately the two cases $t/v\leq s$ and $t/v\geq s$. If $t/v\leq s$ then $(t/v)\vee s=s$, $(t/v)\wedge s=t/v$ so that the denominator equals

$$|t-s|^{1/2}+|(1/v)(t-s)|^{1/2}=(1+1/v^{1/2})|t-s|^{1/2}.$$

Thus

$$F(v,s,t)=\frac{v^{1/2}}{1+v^{1/2}}|t-s|^{\alpha-1/2}=s^{\alpha-1/2}\frac{v^{1/2}}{1+v^{1/2}}|t/s-1|^{\alpha-1/2}$$

$$\geq\tfrac{1}{2}|v-1|^{\alpha-1/2}\geq\tfrac{1}{2}|c-1|^{\alpha-1/2}.$$

If conversely $t/v\geq s$ then $(t/v)\vee s=t/v$, $(t/v)\wedge s=s$ and the denominator equals $(t^{1/2}+s^{1/2})(1-1/v)^{1/2}$ so that

$$F(v,s,t)=\frac{|t-s|^{\alpha}}{|1-1/v|^{1/2}(t^{1/2}+s^{1/2})}=\frac{|1-s/t|^{\alpha}}{|1-1/v|^{1/2}(t^{-\alpha+1/2}+s^{1/2}t^{-\alpha})}$$

and since $s/t\leq 1/v<1$, $s^{1/2}t^{-\alpha}\leq t^{-\alpha+1/2}\leq 1$ and

$$F(v,s,t)\geq\frac{(1-1/v)^{\alpha}}{2(1-1/v)^{1/2}}\geq\tfrac{1}{2}(v-1)^{\alpha-1/2}v^{-\alpha+1/2}$$

$$\geq\tfrac{1}{2}(v-1)^{\alpha-1/2}\geq\tfrac{1}{2}(c-1)^{\alpha-1/2}$$

which ends the proof. □

Let us remark now that

$$\sup_{c^n\leq u\leq c^{n+1}}\|\zeta_u-\zeta_{c^n}\|_{\alpha}$$

$$\leq\sup_{c^n\leq u\leq c^{n+1}}\frac{1}{\phi(u)}\|B_{u\cdot}-B_{c^n\cdot}\|_{\alpha}+\sup_{c^n\leq u\leq c^{n+1}}\left|\frac{1}{\phi(c^n)}-\frac{1}{\phi(u)}\right|\|B_{c^n\cdot}\|_{\alpha}$$

$$\leq\sup_{c^n\leq u\leq c^{n+1}}\frac{1}{\phi(c^n)}\|B_{u\cdot}-B_{c^n\cdot}\|_{\alpha}+\left|\frac{1}{\phi(c^n)}-\frac{1}{\phi(c^{n+1})}\right|\|B_{c^n\cdot}\|_{\alpha}.$$

Now the first term in the right hand side is what we called $Y_n$, which is $\leq\frac{1}{2}\varepsilon$ for $n\geq n_0$ by Lemma 3.3 if $c>1$ is small enough. As for the second term $\|B_{c^n}\|_{\alpha}\leq\phi(c^n)(1+\delta)$ for $n\geq n_0$ by Lemma 3.2, whereas

$$\lim_{n\to\infty}\phi(c^n)\left|\frac{1}{\phi(c^n)}-\frac{1}{\phi(c^{n+1})}\right|=1-c^{-1/2}.$$

Thus, if $c$ is chosen close enough to 1, for $n \geq n_0$,

$$\sup_{c^n \leq u \leq c^{n+1}} \|\zeta_u - \zeta_{c^n}\|_\alpha \leq \varepsilon. \tag{3.4}$$

Combining (3.4) with Lemma 3.2 we get easily:

**Proposition 3.5.** *For every $\varepsilon > 0$ there exists $t_\varepsilon = t_\varepsilon(\omega)$ such that for $t \geq t_\varepsilon$ $\zeta_t$ is in a neighborhood of radius $\varepsilon$ of $\mathcal{K}$.* □

Proposition 3.5 implies that $(\zeta_t)_{t>0}$ is relatively compact and that all its limit points lie in $\mathcal{K}$. The proof that every $f \in \mathcal{K}$ is a limit point is not different from the case of the sup-norm. We give a sketch of it here only for sake of completeness. We shall use in the following the notation:

$$\|\gamma\|_{\alpha,u,v} = \sup_{u \leq t,s \leq v} \frac{|\gamma(t) - \gamma(s)|}{|t-s|^\alpha}.$$

It is easy to check that for $0 \leq r \leq u \leq v \leq 1$, $\|f\|_{\alpha,r,u} + \|f\|_{\alpha,u,v} \geq \|f\|_{\alpha,r,v}$. Let $f \in \mathscr{C}^{\alpha,0}$ be such that $\lambda(f) < 1$ and let us define

$$Z_n(t) = \begin{cases} 0, & t \leq c^{-1}, \\ (B_{c^n t} - B_{c^{n-1}})/\phi(c^n), & c^{-1} \leq t \leq 1. \end{cases}$$

$\{Z_n\}_n$ is a sequence of independent $\mathscr{C}^{\alpha,0}$-valued r.v., because of the independence of the increments of the Brownian motion. Moreover, for $c^{-1} \leq t \leq 1$, $Z_n(t) = \zeta_{c^n}(t) - \zeta_{c^n}(c^{-1})$, thus by (3.3) with

$$A = \{\gamma; \|\gamma - \gamma(c^{-1}) - (f - f(c^{-1}))\|_{\alpha,c^{-1},1} < \tfrac{1}{2}\eta\}$$

one has

$$P\{\|Z_n - (f(t) - f(c^{-1}))\|_{\alpha,c^{-1},1} \leq \tfrac{1}{2}\eta\} \geq \exp(-\mathrm{LL}(c^n)(\lambda(f) + \delta)).$$

Thus by the Borel-Cantelli Lemma and (3.2),

$$\|\zeta_{c^n} - \zeta_{c^n}(c^{-1}) - (f - f(c^{-1}))\|_{\alpha,c^{-1},1} \leq \tfrac{1}{2}\eta \tag{3.5}$$

for infinitely many values of $n$. Since

$$\lim_{n\to\infty} \frac{\phi(c^{n-1})}{\phi(c^n)} = \frac{1}{\sqrt{c}}$$

by Lemma 3.2,

$$\|\zeta_{c^n}\|_{\alpha,0,c^{-1}} = \|\zeta_{c^{n-1}}\|_\alpha \frac{\phi(c^{n-1})}{\phi(c^n)} c^\alpha \leq 2c^{\alpha-1/2} \tag{3.6}$$

for $n \geq n_1 = n_1(\omega)$. Also easily

$$\|f\|_{\alpha,0,c^{-1}} \leq \sqrt{2}c^{\alpha-1/2}. \tag{3.7}$$

Putting together (3.5), (3.6) and (3.7) one has

$$\|\zeta_{c^n} - f\|_\alpha \leq \tfrac{1}{2}\eta(2+\sqrt{2})c^{\alpha-1/2}$$

for infinitely many values of $n$. Choosing $c>1$ large enough, $\eta$ being arbitrary, this yields that $\zeta_t$ is in any fixed neighborhood of $f$ for a set of values $t$ which is unbounded. We have thus proved the following:

**Proposition 3.6.** *Every $f \in \mathcal{K}$ is a limit point of $(\zeta_t)_t$ as $t \to +\infty$.* □

## References

P. Baldi, Large deviations and functional iterated logarithm law for diffusion processes, Probab. Theory Rel. Fields 71 (1986) 435-453.

P. Baldi, Large deviations and stochastic homogenization, Ann. Mat. Pura Appl. 151 (1988) 161-178.

P. Baxendale, Gaussian measures on function spaces, Amer. J. Math. 98 (1976) 891-952.

G. Ben Arous and R. Léandre, Décroissance exponentielle du noyau de la chaleur sur la diagonale II, to appear in: Probab. Theory Rel. Fields (1988).

Z. Ciesielski, On the isomorphism of the spaces $H_\alpha$ and $m$, Bull. Acad. Pol. Sci. 7 (1960) 217-222.

J.-M. Deuschel and D. Strook, Large Deviations (Academic Press, Boston, MA, 1989).

M. Schilder, Asymptotic formulas for Wiener integrals, Trans. Amer. Math. Soc. 125 (1966) 63-85.

V. Strassen, An invariance principle for the law of the iterated logarithm, Z. Wahrsch. Verw. Gebiete 3 (1964) 211-226.