

Flow Decomposition and Large Deviations

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We study large deviations properties related to the behavior, as ε goes to 0, of diffusion processes generated by $\varepsilon^2 L_1 + L_2$, where L_1 and L_2 are two second-order differential operators, extending recent results of Doss and Stroock and Rabehiramanana. The main tool is the decomposition theorem for flows of stochastic differential equations proved by Bismut and Kunita. We give another application of flow decomposition in a nonlinear filtering problem. © 1996 Academic Press, Inc.

1. INTRODUCTION

The purpose of this work is to show how the decomposition theorem for flows of stochastic differential equations proved by Bismut [4] and Kunita [9] can be used to obtain new large deviation principles for the diffusions generated by $\varepsilon^2 L_1 + L_2$, when L_1 and L_2 are two second-order differential operators, and when $\varepsilon \rightarrow 0$. This problem is now classical when L_2 is first order (see Freidlin and Wentzell [8], Azencott [1]). It has also been treated when L_2 is the Laplacian (Bezuidenhout [3]) and when L_1 can be written as a sum of squares of vector fields $L_1 = \frac{1}{2} \sum_i X_i^2$, where the Lie algebra generated by the X_i is abelian (Doss and Stroock [7]), or nilpotent (Rabehiramanana [11]). These authors give a large deviations principle for the law of the random variable R^ε , particular version of the conditional law of X^ε relative to (εB) , where X^ε is a solution to the Stratonovich stochastic differential equation

$$X_t^\varepsilon = x + \varepsilon \sum_{i=1}^r \int_0^t \sigma_i(X_s^\varepsilon) dB_s^i + \sum_{j=1}^l \tilde{\sigma}_j(X_s^\varepsilon) d\tilde{B}_s^j + \int_0^t \tilde{\sigma}_0(X_s^\varepsilon) ds; \quad (1.1)$$

- $x \in \mathbb{R}^n$;
- $t \in [0, 1]$;
- For all $i \in \{1, \dots, r\}$ and all $j \in \{0, \dots, l\}$, σ_i and $\tilde{\sigma}_j$ are sufficiently smooth vector fields on \mathbb{R}^n .

- B and \tilde{B} are two independent Brownian motions, with values in \mathbb{R}^r and \mathbb{R}^l , respectively, defined on the Wiener spaces $W = \mathcal{C}_0([0, 1], \mathbb{R}^r)$ and $\tilde{W} = \mathcal{C}_0([0, 1], \mathbb{R}^l)$. We will denote by P (respectively \tilde{P}) the Wiener measure on W (respectively \tilde{W}).

One could wonder whether a large deviations principle for the law of X^e is attainable. As a matter of fact, it is not, as Doss and Stroock have pointed out. Indeed, the support of the law of the nonperturbed diffusion X^0 is not compact, in general. Since the rate function λ associated to the large deviations of X^e vanishes on the support of P^0 , the level sets $\{\lambda \leq L\}$ cannot be compact. However, X^e can be considered as a random variable,

$$X_e: \begin{aligned} (W, P) &\rightarrow (L^p(\tilde{W}, \mathcal{C}_x([0, 1], \mathbb{R}^n))) \\ (\varepsilon B) &\rightarrow (\tilde{B} \rightarrow X_e(x)). \end{aligned}$$

In this case, there is no obvious contradiction to have a large deviations principle for the law P_e of X_e . The support of P_0 is now a point of $L^p(\tilde{W}, \mathcal{C}_x([0, 1], \mathbb{R}^n))$, which is obviously compact. And we do obtain a large deviations principle for the law of X_e .

Our result contains the results of [7, 11] and extends them to the general case with no hypothesis at all on the Lie algebra. The key observation is the fact that a contraction principle can be used if one has two ingredients

1. the decomposition principle: we recall this result in Section 2.
2. a large deviations principle for flows of stochastic differential equations.

Such a large deviations principle for flows has been obtained by Millet, Nualart, and Sanz-Solé [10] and Baldi and Sanz-Solé [2]. We need a slight extension of it to be able to control derivatives of the flow. We give the proof of this large deviations principle in Section 3, and we get in Section 4 to the main theorem that we now state.

THEOREM 7. *Let P_e be the law of the random variable X_e (P_e is a probability measure on $L^p(\tilde{W}, \mathcal{C}_x([0, 1], \mathbb{R}^n))$). Then P_e satisfies a large deviations principle with rate function A defined for all $z \in L^p(\tilde{W}, E_x)$ by*

$$A(z) = \inf \left\{ \frac{1}{2} \|h\|_{\mathcal{H}^r}^2, h \in \mathcal{H}^r, \text{ such that } \tilde{P} \text{ a.e.} \right.$$

$$\left. z_t = x + \sum_{i=1}^r \int_0^t \sigma_i(z_s) \dot{h}_s^i ds + \sum_{j=1}^l \tilde{\sigma}_j(z_s) d\tilde{B}_s^j + \int_0^t \tilde{\sigma}_0(z_s) ds \right\}.$$

This contains a large deviations principle for the conditional law R^ε of X^ε relative to (εB) (as in Doss and Stroock [7] and in Rabeherimanana [11]), as a trivial contraction principle shows.

Finally in Section 5, we show how this method can be used for a problem in nonlinear filtering, extending earlier results of Doss [6] and Rabeherimanana [11].

2. FLOW DECOMPOSITION FOR STOCHASTIC DIFFERENTIAL EQUATIONS

We mention here the results of [4, 9] for later use. For $i \in \{0, \dots, k\}$, let X_i and Y_i be \mathcal{C}_b^m vector fields on \mathbb{R}^n (that is differentiable up to order m , bounded with bounded derivatives). Let us consider the Stratonovich differential equation

$$dx_t = \sum_{i=1}^k X_i(x_t) dB_t^i + X_0(x_t) dt$$

$$x_0 = x.$$

Then there is a version of $(t, x) \mapsto x_t(x)$, which is a flow of C^m -diffeomorphisms in \mathbb{R}^n , that is an element of \mathcal{D}^n , where

$$\mathcal{D}^n \equiv \left\{ \begin{array}{l} \phi: [0, 1] \times \mathbb{R}^n \rightarrow \mathbb{R}^n, (t, x) \mapsto \phi_t(x) \text{ such that} \\ \forall t \in [0, 1], \phi_t \text{ is a } C^m \text{-diffeomorphism of } \mathbb{R}^n \\ \forall l \in \mathbb{N}^n, |l| \leq m, \frac{\partial^{|l|} \phi_t}{\partial x^l}(x), \frac{\partial^{|l|} (\phi_t)^{-1}}{\partial x^l}(x) \text{ are continuous in } (t, x). \end{array} \right\}$$

Let $\phi_t(x)$ denote this essentially unique version of $x_t(x)$. Almost surely, for all $t \in [0, 1]$, we can then define the stochastic vector fields

$$\phi_t^{-1} * Y_i(y) = \left(\frac{\partial \phi_t}{\partial x}(y) \right)^{-1} Y_i(\phi_t(y)).$$

Let us consider then the Stratonovich differential equation

$$dy_t = \sum_{i=1}^k \phi_t^{-1} * Y_i(y_t) dB_t^i + \phi_t^{-1} * Y_0(y_t) dt \quad (2.2)$$

$$y_0 = y.$$

THEOREM 1. 1. *There is a strong solution to (2.2) defined on $[0, 1]$.*

2. *Let $z_t \equiv \phi_t(y_t)$. Then z_t is solution to the stochastic differential equation*

$$\begin{aligned} dz_t &= \sum_{i=1}^k (X_i + Y_i)(z_t) dB_t^i + (X_0 + Y_0)(z_t) dt \\ z_0 &= y. \end{aligned} \tag{2.3}$$

Proof. Let ξ_t be the strong solution of Eq. (2.3). Let us consider the process defined for all $t \in [0, 1]$ by $\tilde{y}_t = \phi_t^{-1}(\xi_t)$. Then, by the generalized Itô formula (see Theorem 4.1 in [4]), \tilde{y}_t is solution to Eq. (2.2). Therefore 1 and 2 are proved. ■

Theorem 1 has its Itô counterpart.

THEOREM 2. *Let us define*

- $(X_0 + Y_0)^*(x) \equiv X_0(x) + Y_0(x) + \frac{1}{2} \sum_{i=1}^k (X_i + Y_i)'(x)(X_i + Y_i)(x)$.
- $Y_0^*(x) = Y_0(x) + \frac{1}{2} \sum_{i=1}^k Y_i'(x) Y_i(x)$.
- $\tilde{Y}_0(x) = Y_0^*(x) + \frac{1}{2} \sum_{i=1}^k \phi_t^{-1} * [X_i, Y_i](x)$
 $\quad - \frac{1}{2} \sum_{i=1}^k \frac{\partial^2 \phi_t}{\partial x^2} (\phi_t^{-1} * Y_i(x), \phi_t^{-1} * Y_i(x))$,

where $[X, Y]$ is the Lie bracket of the vector fields X and Y .

Let us consider the Itô stochastic differential equation

$$\begin{aligned} dy_t &= \sum_{i=1}^k \phi_t^{-1} * Y_i(y_t) \delta B_t^i + \phi_t^{-1} * \tilde{Y}_0(y_t) dt \\ y_0 &= y. \end{aligned} \tag{2.4}$$

Then

1. *There is a strong solution to (2.4) defined on $[0, 1]$.*

2. *Let $z_t = \phi_t(y_t)$. Then z_t is solution to the Itô stochastic differential equation*

$$\delta z_t = \sum_{i=1}^k (X_i + Y_i)(z_t) \delta B_t^i + (X_0 + Y_0)^*(z_t) dt.$$

Proof. The same as Theorem 1. ■

We will use here Theorem 2 in the following context:

- $k = l + r$;
- $B = (B^1, \dots, B^r, \tilde{B}^1, \dots, \tilde{B}^l)$;
- $\forall i \in \{1, \dots, r\}, Y_i = 0$;
- $\forall i \in \{r + 1, \dots, l + r\}, X_i = 0$,

so that Eq. (1.1) splits in two stochastic differential equations, one driven by the Brownian B , the other by the Brownian \tilde{B} .

3. LARGE DEVIATIONS FOR STOCHASTIC FLOWS

3.1. Notations and Result

In this section, we will consider the Stratonovich differential equation,

$$\begin{aligned} d\xi_t^\varepsilon &= \varepsilon \sum_{i=1}^r \sigma_i(\varepsilon, \xi_t^\varepsilon) dB_t^i \\ \xi_0^\varepsilon &= x, \end{aligned} \tag{3.5}$$

where

- $\sigma_i(\varepsilon, \cdot)$ are \mathcal{C}_b^{m+2} vector fields on \mathbb{R}^n , for some $m \geq n + 1$. We will assume that $\sigma_i(\varepsilon, \cdot)$ converges in \mathcal{C}_b^m uniformly on compact subsets of \mathbb{R}^n to some vector field σ_i , when ε goes to 0.

- B is a standard Brownian motion defined on the Wiener space (W, P) , where W is the space $\mathcal{C}_0([0, 1], \mathbb{R}^n)$, with the topology of uniform convergence, and P is the Wiener measure.

Let \mathcal{D}^n be defined as in Section 2. \mathcal{D}^n will be endowed with the $\mathcal{C}^{0,k}$, or $\tilde{\mathcal{C}}^{0,k}$ -topology, defined for all $k \leq m$ by

- $\phi^n \xrightarrow{\mathcal{C}^{0,k}} \phi$ iff $\forall K$ compact subset of \mathbb{R}^n ,

$$\sup_{\substack{x \in K; t \in [0, 1] \\ |\alpha| \leq k}} \left\| \frac{\partial^{|\alpha|} \phi_t}{\partial x^\alpha}(x) - \frac{\partial^{|\alpha|} \phi_t^n}{\partial x^\alpha}(x) \right\| \xrightarrow{n \rightarrow \infty} 0$$

- $\phi^n \xrightarrow{\tilde{\mathcal{C}}^{0,k}} \phi$ iff $\forall K$ compact subset of \mathbb{R}^n ,

$$\begin{aligned} &\sup_{\substack{x \in K; t \in [0, 1] \\ |\alpha| \leq k}} \left\| \frac{\partial^{|\alpha|} \phi_t}{\partial x^\alpha}(x) - \frac{\partial^{|\alpha|} \phi_t^n}{\partial x^\alpha}(x) \right\| \\ &+ \left\| \frac{\partial^{|\alpha|} (\phi_t)^{-1}}{\partial x^\alpha}(x) - \frac{\partial^{|\alpha|} (\phi_t^n)^{-1}}{\partial x^\alpha}(x) \right\| \xrightarrow{n \rightarrow \infty} 0. \end{aligned}$$

Let $\Phi^e(x)$ be the version of $\zeta^e_t(x)$ which is an element of \mathcal{D}^n . Our purpose is to show a large deviations principle for the law of Φ^e (probability on \mathcal{D}^n with the $\mathcal{C}^{0,k}$ -topology). Millet, Nualart, and Sanz-Solé [10], Baldi and Sanz-Solé [2] have already shown this result for the $\mathcal{C}^{0,0}$ topology. Before stating the result, we will introduce further notations.

Let \mathcal{H}^r be the Cameron–Martin space over \mathbb{R}^r

$$\mathcal{H}^r = \left\{ \begin{array}{l} h: [0, 1] \rightarrow \mathbb{R}^r, h(0) = 0, h \text{ absolutely continuous with respect to} \\ \text{Lebesgue measure, such that } \int_0^1 \|\dot{h}_t\|^2 dt < \infty. \end{array} \right\}$$

\mathcal{H}^r is a Hilbert space for the inner product $(h, g)_{\mathcal{H}^r} = \int_0^1 \dot{h}_t \dot{g}_t dt$. Given $h \in \mathcal{H}^r$, we associate to (3.5) the ordinary differential equation

$$\begin{aligned} dx_t(h) &= \sum_{i=1}^r \sigma_i(x_t(h)) \dot{h}_t^i dt \\ x_0(h) &= x. \end{aligned} \tag{3.6}$$

Under the assumptions made on the vector fields, $x(h)$ is an element of \mathcal{D}^n . Thus, we define a map

$$\begin{aligned} F: \mathcal{H}^r &\rightarrow \mathcal{D}^n \\ h &\mapsto (t, x \mapsto x_t(h)(x)). \end{aligned} \tag{3.7}$$

Using the results of Bismut [4], F can be extended in a measurable way to W : P -a.e. $F_t(\omega)(x)$ will be a solution to the stochastic differential equation

$$\begin{aligned} d\xi_t &= \sum_{i=1}^r \sigma_i(\xi_t) dB_t^i \\ \delta_0 &= x, \end{aligned}$$

This extension will still be denoted by F .

We define now the rate function associated to the large deviations of the stochastic flow Φ^e . Let ϕ be in \mathcal{D}^n .

$$I(\phi) \equiv \inf \left\{ \frac{1}{2} \|h\|_{\mathcal{H}^r}^2, h \in \mathcal{H}^r, F(h) = \phi \right\}. \tag{3.8}$$

When A is a subset of \mathcal{D}^n , we will denote by $I(A) = \inf\{I(\phi), \phi \in A\}$. We have then the following result.

THEOREM 3. \mathcal{D}^n is provided with the $\mathcal{C}^{0,k}$ -topology, for some $k \leq m - 1 - [n/2]$. Then

1. I is lower semi-continuous, and for all $L > 0$, $\{I \leq L\}$ is a compact subset of \mathcal{D}^n .

2. For all $\phi \in \mathcal{D}^n$ such that $I(\phi) < \infty$, there exists a unique $h \in \mathcal{H}^r$ such that $\phi = F(h)$ and $I(\phi) = \frac{1}{2} \|h\|_{\mathcal{H}^r}^2$. Moreover, if $\sigma(x) = (\sigma_1(x) \cdots \sigma_r(x))$ and if $V \equiv \bigcap_{x \in \mathbb{R}^n} \text{Ker } \sigma(x)$, then dt -a.e., $\dot{h}_t \in V^\perp$.

3. $\forall A \subset \mathcal{D}^n$,

$$-I(\bar{A}) \leq \liminf_{\varepsilon \rightarrow 0} \varepsilon^2 \log P(\Phi^\varepsilon \in A) \leq \limsup_{\varepsilon \rightarrow 0} \varepsilon^2 \log P(\Phi^\varepsilon \in A) \leq -I(\bar{A}).$$

These results remain true when \mathcal{D}^n is provided with the $\tilde{\mathcal{C}}^{0,k}$ -topology.

3.2. Proof of Theorem 3

Large Deviations for the $\mathcal{C}^{0,k}$ -topology

First of all, let us note that the map F defined by (3.7) is continuous from $\mathcal{H}_a^r \equiv \{h \in \mathcal{H}^r, \|h\|_{\mathcal{H}^r} \leq a\}$ endowed with the uniform convergence, to $(\mathcal{D}^n, \mathcal{C}^{0,k})$. Let, indeed, f and g be two functions in \mathcal{H}_a^r and let $x = F(f)$ and $y = F(g)$. Then,

$$\begin{aligned} \|x_t(x) - y_t(x)\|^2 &\leq 2 \left\| \int_0^t (\sigma(x_s) - \sigma(y_s)) \dot{f}_s \, ds \right\|^2 + 2 \left\| \int_0^t \sigma(y_s) (\dot{f}_s - \dot{g}_s) \, ds \right\|^2 \\ &\leq 2Ka^2 \int_0^t \|x_s - y_s\|^2 \, ds + 2 \left\| \int_0^t \sigma(y_s) (\dot{f}_s - \dot{g}_s) \, ds \right\|^2. \end{aligned}$$

And an integration by part then yields

$$\begin{aligned} \|x_t - y_t\| &\leq K \|\sigma(y_t)(f_t - g_t)\| + \int_0^t \|\sigma'(y_s) \dot{y}_s(f_s - g_s)\| \, ds \\ &\quad + 2Ka^2 \int_0^t \|x_s - y_s\|^2 \, ds \\ &\leq K \|f - g\|_\infty \left(1 + \int_0^t \|\sigma'(y_s)\| \|\sigma(y_s) \dot{g}_s\| \, ds \right) \\ &\quad + 2Ka^2 \int_0^t \|x_s - y_s\|^2 \, ds. \end{aligned}$$

By Gronwall's lemma, we have then:

$$\|x_t - y_t\| \leq K \|f - g\|_\infty (1 + Ka).$$

The same arguments hold for the derivatives of $x - y$.

Therefore, Lemma 1.3 [1, p. 69], ensures that I is a “good” rate function. Moreover, when $I(\phi) < \infty$, the infimum in the definition of I is reached.

We are now going to prove that the point where the infimum is reached is unique. So, let $\phi \in \mathcal{D}^n$ be such that $I(\phi) < \infty$. Let $h_0 \in \mathcal{H}^r$ be such that $I(\phi) = \frac{1}{2} \|h_0\|_{\mathcal{H}^r}^2$ and $\phi = F(h_0)$. Let us define the map

$$\begin{aligned} \Pi: \mathcal{H}^r &\rightarrow \mathcal{H}^r \\ h &\mapsto \Pi h: \Pi h_t = \int_0^t P_{V^\perp} \dot{h}_s ds, \end{aligned} \tag{3.9}$$

where P_{V^\perp} is the orthogonal projection on V^\perp . Then it is easy to check that

- $\|\Pi h\|_{\mathcal{H}^r} \leq \|h\|_{\mathcal{H}^r}$;
- $\|\Pi h\|_{\mathcal{H}^r} = \|h\|_{\mathcal{H}^r} \Leftrightarrow dt\text{-a.e.}, \dot{h}_t \in V^\perp$;
- $F(h) = F(\Pi h)$.

Thus, $\phi = F(h_0) = F(\Pi h_0)$ and $\|h_0\|_{\mathcal{H}^r} = \|\Pi h_0\|_{\mathcal{H}^r}$. Therefore, dt -a.e., $\dot{h}_0(t) \in V^\perp$. Let us assume now that

$$\phi = F(h_0) = F(h_1), \quad I(\phi) = 1/2 \|h_0\|_{\mathcal{H}^r}^2 = 1/2 \|h_1\|_{\mathcal{H}^r}^2.$$

The equations satisfied by $F(h_1)$ and $F(h_0)$ yield

$$\forall t, \forall x, \quad \sigma(\phi_t(x)) \dot{h}_0(t) = \sigma(\phi_t(x)) \dot{h}_1(t).$$

Using the diffeomorphism property of ϕ_t , we derive that

$$\forall t, \forall x, \quad \sigma(x)(\dot{h}_0(t) - \dot{h}_1(t)) = 0.$$

Therefore, $\dot{h}_0(t) - \dot{h}_1(t) \in V$. But we already know that $\dot{h}_0(t) - \dot{h}_1(t) \in V^\perp$. Therefore, $\dot{h}_0 = \dot{h}_1$, and 2 is proved.

Following Azencott [1], we begin with the “quasicontinuity” of the map F , in order to obtain 3.

LEMMA 4. $\forall K$ compact subset of \mathbb{R}^n , $\forall a > 0$, $\forall L > 0$, $\forall R > 0$, there exists b_0 and $\varepsilon_0 > 0$ such that $\forall b \leq b_0$, $\forall \varepsilon \leq \varepsilon_0$, $\forall h \in \mathcal{H}^r$, $\|h\|_{\mathcal{H}^r} \leq a$,

$$P[\|\Phi^\varepsilon - F(h)\|_{\mathcal{C}^{0,k}([0,1] \times K)} \geq R; \|\varepsilon B - h\| \leq b] \leq \exp(-L/\varepsilon^2).$$

Proof of Lemma 4

Throughout the paper, C is a constant which can differ from one expression to the other. Since $m \geq k + 1 + [n/2]$, Sobolev’s embedding theorem gives $\|\cdot\|_{\mathcal{C}^k(K)} \leq C \|\cdot\|_{W^{m,2}(K)}$, where $W^{m,2}$ is the space of functions differentiable up to order m , whose derivatives are square integrable with respect to Lebesgue measure. $W^{m,2}(K)$ is a Hilbert space for the norm

$$\|f\|_{W^{m,2}(K)} = \int_K \sum_{|\alpha| \leq m} \left\| \frac{\partial^{|\alpha|} f}{\partial x^\alpha} \right\|^2 dx.$$

Therefore,

$$P[\|\Phi^\varepsilon - F(h)\|_{\mathcal{C}^{0,k}([0,1] \times K)} \geq R; \|\varepsilon B - h\| \leq b] \leq P\left[\sup_{t \in [0,1]} \|\Phi_t^\varepsilon - F_t(h)\|_{W^{m,2}(K)} \geq \frac{R}{C}; \|\varepsilon B - h\| \leq b\right].$$

So we have to show that $\forall K$ compact set of \mathbb{R}^n , $\forall a, R, L > 0$, there exist $b_0, \varepsilon_0 > 0$, such that $b \leq b_0$, $\varepsilon \leq \varepsilon_0$, $\|h\|_{\mathcal{H}^r} \leq a$ imply

$$\Pi_1 \equiv P\left[\sup_{t \in [0,1]} \|\Phi_t^\varepsilon - F_t(h)\|_{W^{m,2}(K)} \geq R; \|\varepsilon B - h\| \leq b\right] \leq \exp(-L/\varepsilon^2).$$

In the following, we will denote

$$\tau_R^\varepsilon = \inf\{t \text{ such that } \|\Phi_t^\varepsilon - F_t(h)\|_{W^{m,2}(K)} \geq R\}.$$

Then, $\Pi_1 = P[\sup_{t \in [0,1]} \|\Phi_{t \wedge \tau_R^\varepsilon}^\varepsilon - F_{t \wedge \tau_R^\varepsilon}(h)\|_{W^{m,2}(K)} \geq R; \|\varepsilon B - h\| \leq b]$.

Since the vector fields σ_i are \mathcal{C}_b^m , one can easily check that there is a constant M (depending on a and K), such that

$$\sup_{t \in [0,1]} \sup_{h; \|h\|_{\mathcal{H}^r} \leq a} \|F_t(h)\|_{W^{m,2}(K)} \leq M.$$

From the definition of τ_R^ε , it results then that $\sup_{t \leq \tau_R^\varepsilon} \|\Phi_t^\varepsilon\|_{W^{m,2}(K)} \leq R + M$. Therefore, since $m \geq n + 1$, it is proved in Appendix 1 that there is a constant C such that $\forall j \in \{1, \dots, r\}$, $\forall \varepsilon, \forall t \in [0, 1]$,

$$\begin{aligned} \|\sigma_j(\Phi_{t \wedge \tau_R^\varepsilon}^\varepsilon) - \sigma_j(F_{t \wedge \tau_R^\varepsilon}(h))\|_{W^{m,2}(K)} &\leq C \|\Phi_{t \wedge \tau_R^\varepsilon}^\varepsilon - F_{t \wedge \tau_R^\varepsilon}(h)\|_{W^{m,2}(K)} \\ \|\sigma_j(\varepsilon, \Phi_{t \wedge \tau_R^\varepsilon}^\varepsilon) - \sigma_j(\Phi_{t \wedge \tau_R^\varepsilon}^\varepsilon)\|_{W^{m,2}(K)} &\leq C \|\sigma_j(\varepsilon, \cdot) - \sigma_j(\cdot)\|_{\mathcal{C}^m(\mathcal{B}_{R+M})} \end{aligned}$$

(where \mathcal{B}_{R+M} is the ball of radius $R + M$ in \mathbb{R}^n). Thus, $\forall t \in [0, 1]$,

$$\begin{aligned} \|\Phi_{t \wedge \tau_R^\varepsilon}^\varepsilon - F_{t \wedge \tau_R^\varepsilon}(h)\|_{W^{m,2}(K)} &\leq \left\| \sum_{j=1}^r \int_0^{t \wedge \tau_R^\varepsilon} \sigma_j(\varepsilon, \Phi_s^\varepsilon) d(\varepsilon B_s^j - h_s^j) \right\|_{W^{m,2}(K)} \\ &\quad + \sum_{j=1}^r \int_0^{t \wedge \tau_R^\varepsilon} \|\sigma_j(\varepsilon, \Phi_s^\varepsilon) - \sigma_j(\Phi_s^\varepsilon)\|_{W^{m,2}(K)} |\dot{h}_s^j| ds \\ &\quad + \sum_{j=1}^r \int_0^{t \wedge \tau_R^\varepsilon} \|\sigma_j(\Phi_s^\varepsilon) - \sigma_j(F_s(h))\|_{W^{m,2}(K)} |\dot{h}_s^j| ds \end{aligned}$$

$$\begin{aligned}
&\leq \left\| \sum_{j=1}^r \int_0^{t \wedge \tau_R^\varepsilon} \sigma_j(\varepsilon, \Phi_s^\varepsilon) d(\varepsilon B_s^j - h_s^j) \right\|_{W^{m,2}(K)} \\
&\quad + C \left[a \sum_{j=1}^r \|\sigma_j(\varepsilon, \cdot) - \sigma_j(\cdot)\|_{\mathcal{C}^m(\mathcal{B}_{R+M})} \right. \\
&\quad \left. + \int_0^{t \wedge \tau_R^\varepsilon} \|\Phi_s^\varepsilon - F_s(h)\|_{W^{m,2}(K)} |\dot{h}_s^j| ds \right].
\end{aligned}$$

It follows from Gronwall's lemma that $\forall t \in [0, 1]$,

$$\begin{aligned}
&\sup_{t \in [0, 1]} \|\Phi_{t \wedge \tau_R^\varepsilon}^\varepsilon - F_{t \wedge \tau_R^\varepsilon}(h)\|_{W^{m,2}(K)} \\
&\leq C \left(\sup_{t \in [0, 1]} \left\| \sum_{j=1}^r \int_0^{t \wedge \tau_R^\varepsilon} \sigma_j(\varepsilon, \Phi_s^\varepsilon) d(\varepsilon B_s^j - h_s^j) \right\|_{W^{m,2}(K)} \right. \\
&\quad \left. + \sum_{j=1}^r \|\sigma_j(\varepsilon, \cdot) - \sigma_j(\cdot)\|_{\mathcal{C}^m(\mathcal{B}_{R+M})} \right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\Pi_1 &\leq P \left[\sup_t \left\| \sum_{j=1}^r \int_0^{t \wedge \tau_R^\varepsilon} \sigma_j(\varepsilon, \Phi_s^\varepsilon) d(\varepsilon B_s^j - h_s^j) \right\|_{W^{m,2}(K)} \geq \frac{R}{2C}; \|\varepsilon B - h\| \leq b \right] \\
&\quad + P \left[\sum_{j=1}^r \|\sigma_j(\varepsilon, \cdot) - \sigma_j(\cdot)\|_{\mathcal{C}^m(\mathcal{B}_{R+M})} \geq \frac{R}{2C}; \|\varepsilon B - h\| \leq b \right].
\end{aligned}$$

From the uniform convergence of $\sigma_i(\varepsilon, \cdot)$ to $\sigma_i(\cdot)$, we can choose ε_0 such that

$$\varepsilon \leq \varepsilon_0 \Rightarrow \|\sigma_j(\varepsilon, \cdot) - \sigma_j(\cdot)\|_{\mathcal{C}^m(\mathcal{B}_{R+M})} < R/2C.$$

Thus we are led to show that $\forall K$ compact of \mathbb{R}^n , $\forall R, R', L, a > 0$, there exists b_0 and ε_0 such that $b \leq b_0$, $\varepsilon \leq \varepsilon_0$, $\|h\|_{\mathcal{H}^r} \leq a$ imply

$$\begin{aligned}
\Pi_2 &\equiv P \left[\sup_t \left\| \sum_{j=1}^r \int_0^{t \wedge \tau_R^\varepsilon} \sigma_j(\varepsilon, \Phi_s^\varepsilon) d(\varepsilon B_s^j - h_s^j) \right\|_{W^{m,2}(K)} \geq R'; \|\varepsilon B - h\| \leq b \right] \\
&\leq \exp \left(-\frac{L}{\varepsilon^2} \right)
\end{aligned}$$

An integration by part yields $\Pi_2 \leq P_1 + P_2 + P_3 + P_4$, where

$$\begin{aligned}
P_1 &= P \left[\sup_{t \leq \tau_R^\varepsilon} \sum_{j=1}^r |\varepsilon B_t^j - h_t^j| \|\sigma_j(\varepsilon, \Phi_t^\varepsilon)\|_{W^{m,2}(K)} > \frac{R'}{4}; \|\varepsilon B - h\| \leq b \right] \\
P_2 &= P \left[\sup_t \left\| \sum_{j=1}^r \langle \sigma_j(\varepsilon, \Phi^\varepsilon), \varepsilon B^j \rangle_{t \wedge \tau_R^\varepsilon} \right\|_{W^{m,2}(K)} > \frac{R'}{4}; \|\varepsilon B - h\| \leq b \right]
\end{aligned}$$

$$P_3 = P \left[\sup_t \left\| \sum_{i,j=1}^r \int_0^{t \wedge \tau_R^\varepsilon} \sigma'_j(\varepsilon, \Phi_s^\varepsilon) \sigma_i(\varepsilon, \Phi_s^\varepsilon) (\varepsilon B_s^j - h_s^j) \varepsilon \delta B_s^i \right\|_{W^{m,2}(K)} \right. \\ \left. > \frac{R'}{4}; \|\varepsilon B - h\| \leq b \right],$$

where δ denotes the Itô differential,

$$P_4 = P \left[\varepsilon^2 \sup_t \left\| \sum_{j=1}^r \int_0^{t \wedge \tau_R^\varepsilon} (\varepsilon B_s^j - h_s^j) \text{Trace}(\sigma^* \sigma'_j \sigma)(\varepsilon, \Phi_s^\varepsilon) ds \right\|_{W^{m,2}(K)} \right. \\ \left. > \frac{2R'}{4}; \|\varepsilon B - h\| \leq b \right].$$

Treatment of P_1 . Since $\sup_t \|\Phi_{t \wedge \tau_R^\varepsilon}\|_{W^{m,2}(K)} \leq R + M$, it follows from Appendix 1 that there is a constant C such that $\forall \varepsilon \leq 1, \forall j \in \{1, \dots, r\}$, $\sup_t \|\sigma_j(\varepsilon, \Phi_t^\varepsilon)\|_{W^{m,2}(K)} \leq C$. Therefore, $P_1 \leq P[Cb > R'/4] = 0$ for sufficiently small b .

Treatment of P_2 .

$$\sum_{j=1}^r \langle \sigma_j(\varepsilon, \Phi^\varepsilon), \varepsilon B^j \rangle_{t \wedge \tau_R^\varepsilon} = \varepsilon^2 \sum_{j=1}^r \int_0^{t \wedge \tau_R^\varepsilon} \sigma'_j(\varepsilon, \Phi_s^\varepsilon) \sigma_j(\varepsilon, \Phi_s^\varepsilon) ds.$$

Appendix 1 yields then a constant C such that $P_2 \leq P[\varepsilon^2 C > R'/4]$, i.e., $P_2 = 0$ for $\varepsilon^2 \leq R'/4C$.

Treatment of P_4 . $P_4 \leq P[C\varepsilon^2 b > 2R'/4] = 0$ for ε and b sufficiently small.

Treatment of P_3 . The control of P_3 is given by an exponential inequality for martingales with value in some Hilbert space, proved in Appendix 2. Let $(e_n)_n$ be an orthonormal basis in $W^{m,2}(K)$. Let us denote

- $M_t^\varepsilon(x) = \sum_{i,j} \int_0^t \sigma'_j(\varepsilon, \Phi_s^\varepsilon) \sigma_i(\varepsilon, \Phi_s^\varepsilon) (\varepsilon B_s^j - h_s^j) \varepsilon \delta B_s^i$.
- $T_b^\varepsilon = \inf\{t \text{ such that } \|\varepsilon B_t - h_t\| \geq b\}$
- $S_{R'}^\varepsilon = \inf\{t \text{ such that } \|M_t^\varepsilon\|_{W^{m,2}(K)} \geq R'\}$
- $\tau = \tau_R^\varepsilon \wedge T_b^\varepsilon \wedge S_{R'}^\varepsilon$.

We have then to show that $\forall K$ compact of \mathbb{R}^n , $\forall L, R, R', a > 0$, there exists b_0 and ε_0 such that $\forall \varepsilon \leq \varepsilon_0, \forall b \leq b_0, \forall h, \|h\|_{\mathcal{H}^r} \leq a$,

$$\Pi_3 = P \left[\sup_{t \in [0,1]} \|M_{t \wedge \tau}^\varepsilon\|_{W^{m,2}(K)} \geq R' \right] \leq \exp(-L/\varepsilon^2).$$

For $t \leq \tau$, $M_t^\varepsilon \in W_K^{m,2}$, and if we denote $M_t^{\varepsilon,n} \equiv (M_t^\varepsilon, e_n)$, it can be checked that

$$M_t^{\varepsilon,n} = \sum_{i,j} \int_0^t (\sigma'_j(\varepsilon, \Phi_s^\varepsilon(\cdot)) \sigma_i(\varepsilon, \Phi_s^\varepsilon(\cdot)), e_n)(\varepsilon B_s^j - h_s^j) \varepsilon \delta B_s^i$$

by writing the stochastic integrals as L_2 -limits of Riemann sums. Therefore,

$$\begin{aligned} \langle M^{\varepsilon,n}, M^{\varepsilon,m} \rangle_t &= \sum_{i,j,l} \int_0^t \varepsilon^2 (\sigma'_j(\varepsilon, \Phi_s^\varepsilon) \sigma_i(\varepsilon, \Phi_s^\varepsilon), e_n)(\varepsilon B_s^j - h_s^j) \\ &\quad \times (\sigma'_l(\varepsilon, \Phi_s^\varepsilon) \sigma_i(\varepsilon, \Phi_s^\varepsilon), e_m)(\varepsilon B_s^l - h_s^l) ds. \end{aligned}$$

This allows us to control the quantities appearing in Appendix 2.

$$\begin{aligned} &\sum_{n,m}^N \int_0^{t \wedge \tau} M_s^{\varepsilon,n} M_s^{\varepsilon,m} d\langle M^{\varepsilon,n}, M^{\varepsilon,m} \rangle_s \\ &= \sum_i \varepsilon^2 \int_0^{t \wedge \tau} \left[\sum_j (\sigma'_j(\varepsilon, \Phi_s^\varepsilon) \sigma_i(\varepsilon, \Phi_s^\varepsilon), P_N(M_s^\varepsilon))(\varepsilon B_s^j - h_s^j) \right]^2 ds \\ &\quad (\text{where } P_N \text{ is the orthogonal projection on } \text{Span}\{e_i, i \leq N\}) \\ &\leq C\varepsilon^2 b^2 \sum_i \int_0^{t \wedge \tau} \sum_j \|\sigma'_j(\varepsilon, \Phi_s^\varepsilon) \sigma_i(\varepsilon, \Phi_s^\varepsilon)\|_{W^{m,2}(K)}^2 \|P_N(M_s^\varepsilon)\|_{W^{m,2}(K)}^2 ds \\ &\leq C\varepsilon^2 b^2 R'^2 \quad \text{by Appendix 1.} \end{aligned}$$

Moreover,

$$\begin{aligned} \sum_{k=1}^N \langle M^{\varepsilon,k} \rangle_{t \wedge \tau} &= \sum_{i,k} \varepsilon^2 \int_0^{t \wedge \tau} \left[\sum_j (\sigma'_j(\varepsilon, \Phi_s^\varepsilon) \sigma_i(\varepsilon, \Phi_s^\varepsilon), e_k)(\varepsilon B_s^j - h_s^j) \right]^2 ds \\ &\leq C\varepsilon^2 b^2 \sum_i \int_0^{t \wedge \tau} \sum_j \|P_N(\sigma'_j(\varepsilon, \Phi_s^\varepsilon) \sigma_i(\varepsilon, \Phi_s^\varepsilon))\|_{W^{m,2}(K)}^2 ds \\ &\leq C\varepsilon^2 b^2 \quad \text{by Appendix 1.} \end{aligned}$$

Choosing $\varepsilon^2 \leq R'^2/Cb^2$, we obtain by Appendix 2

$$\Pi_3 \leq \exp \left[- \frac{(R'^2 - \varepsilon^2 b^2 C)^2}{8C\varepsilon^2 b^2 R'^2} \right] \leq \exp \left(- \frac{L}{\varepsilon^2} \right)$$

for b and ε sufficiently small. The proof of Lemma 4 is then complete. ■

From Lemma 4, and from the continuity of F from \mathcal{H}_a^r to $(\mathcal{D}^n, \mathcal{C}^{0,k})$, inequalities of large deviations are now classical. We refer the reader for instance to Azencott [1].

Large Deviations for the $\tilde{\mathcal{C}}^{0,k}$ -Topology. In the following, the \simeq index will concern the $\tilde{\mathcal{C}}^{0,k}$ -topology. Using the differential equations satisfied by the inverse flow, it is easy to see that the function F defined by (3.7) is continuous from \mathcal{H}_a^r to $(\mathcal{D}^n, \tilde{\mathcal{C}}^{0,k})$. Therefore, we derive as previously that I is a good rate function in $\tilde{\mathcal{C}}^{0,k}$ topology.

Now, $\forall A \subset \mathcal{D}^n$, $\tilde{A} \subset \bar{A}$ and $\mathring{A} \subset \tilde{\tilde{A}}$, so that $I(\tilde{A}) \geq I(\bar{A})$ and $I(\tilde{\tilde{A}}) \leq I(\mathring{A})$. But this does not allow us to conclude. The main point is that when ϕ is not a \mathcal{C}^m -diffeomorphism, $I(\phi) = \infty$. Therefore,

$$I(\bar{A}) = \inf\{I(\phi), \phi \in \bar{A}, \text{ and } \phi \text{ } \mathcal{C}^m\text{-diffeomorphism}\}.$$

Assume then that $I(\bar{A}) < \infty$ (the case $I(\bar{A}) = \infty$ is obvious). Let $\phi \in \bar{A}$, ϕ \mathcal{C}^m -diffeomorphism be such that $I(\phi) = I(\bar{A})$. Let ϕ_n be a sequence in A , such that $\phi_n \xrightarrow{\mathcal{C}^{0,k}} \phi$. Since ϕ is a \mathcal{C}^m -diffeomorphism, we deduce from the fact that $\phi \mapsto \phi^{-1}$ is an open mapping, that $\phi_n \xrightarrow{\tilde{\mathcal{C}}^{0,k}} \phi$. Therefore $\phi \in \tilde{\tilde{A}}$, and $I(\tilde{\tilde{A}}) \leq I(\phi) = I(\bar{A})$. A similar argument holds for the open sets. ■

4. LARGE DEVIATIONS FOR PERTURBED STOCHASTIC DIFFERENTIAL EQUATIONS

We will be interested in this section in the perturbed stochastic differential equation (1.1); (1.1) will be written in its Itô form

$$dX_t^\varepsilon = \varepsilon^2 \sigma_0^*(X_t^\varepsilon) dt + \varepsilon \sum_{i=1}^r \sigma_i(X_t^\varepsilon) \delta B_t^i + \tilde{\sigma}_0^*(X_t^\varepsilon) dt + \sum_{j=1}^l \tilde{\sigma}_j(X_t^\varepsilon) \delta \tilde{B}_t^j \quad (4.10)$$

$$X_0^\varepsilon = x,$$

where

$$\begin{aligned} \sigma_0^*(y) &= \frac{1}{2} \sum_{i=1}^r \left(\frac{\partial \sigma_i}{\partial x}(y), \sigma_i(y) \right) \\ \tilde{\sigma}_0^*(y) &= \tilde{\sigma}_0(y) + \frac{1}{2} \sum_{j=1}^l \left(\frac{\partial \tilde{\sigma}_j}{\partial x}(y), \tilde{\sigma}_j(y) \right): \end{aligned}$$

• B and \tilde{B} are two independent standard Brownian motions, respectively, defined on the Wiener spaces $W = \mathcal{C}_0([0, 1], \mathbb{R}^r)$ and $\tilde{W} = \mathcal{C}_0([0, 1], \mathbb{R}^l)$. W and \tilde{W} are endowed with the topology of uniform convergence, and their Borelian σ -fields. We will denote by P (respectively \tilde{P}) the Wiener measure on W (respectively \tilde{W}), and by \mathbb{P} the measure $P \otimes \tilde{P}$ on $W \times \tilde{W}$. So E (respectively \tilde{E} , \mathbb{E}) will be the expectation under P (respectively \tilde{P} , \mathbb{P}).

- E_x will be the space $\mathcal{C}_x([0, 1], \mathbb{R}^n)$ of all continuous paths starting from x with values in \mathbb{R}^n , endowed with the uniform convergence.
- σ_i and $\tilde{\sigma}_j$ are assumed to be in \mathcal{C}_b^k with $k \geq \max(n+1, 4 + [n/2])$.

4.1. “Pseudo” Large Deviations for X^ε

We derive from Theorem 3 some exponential lower and upper bounds for X^ε , extending the results of [7, 11]. Throughout, we will denote by \mathcal{H} the Cameron–Martin space associated to the Wiener space \tilde{W} . When $h \in \mathcal{H}^r$, and $\tilde{h} \in \mathcal{H}^l$, $G(h, \tilde{h})$ will be the solution to the ordinary differential equation

$$x_t = x + \int_0^t \sum_{i=1}^r \sigma_i(x_s) \dot{h}_s^i ds + \int_0^t \sum_{j=1}^l \tilde{\sigma}_j(x_s) \dot{\tilde{h}}_s^j ds + \int_0^t \tilde{\sigma}_0(x_s) ds.$$

PROPOSITION 5. 1. Let A be an open subset of E_x :

$$\begin{aligned} \liminf_{\varepsilon \rightarrow 0} \varepsilon^2 \log \mathbb{P}(X^\varepsilon \in A) \\ \geq -\inf \left\{ \frac{1}{2} \|h\|_{\mathcal{H}^r}^2, h \in \mathcal{H}^r \text{ such that } \exists \tilde{h} \in \mathcal{H}^l, G(h, \tilde{h}) \in A \right\} \end{aligned}$$

2. Let A be a closed subset of E_x :

$$\begin{aligned} \limsup_{\varepsilon \rightarrow 0} \varepsilon^2 \log \mathbb{P}(X^\varepsilon \in A) \\ \leq -\inf \left\{ \frac{1}{2} \|h\|_{\mathcal{H}^r}^2, h \in \bigcap_{\delta > 0} \overline{\{g \in \mathcal{H}^r, \exists \tilde{h} \in \mathcal{H}^l, G(g, \tilde{h}) \in A_\delta\}} \right\}. \end{aligned}$$

The closure is taken with respect to the uniform convergence, and

$$A_\delta = \{y \in E_x, \exists z \in A \ \|z - y\| < \delta\}.$$

The reader is referred to Section 4.3 for the proof of Proposition 5.

4.2. Large Deviations for a Perturbed Stochastic Differential Equation

Using the Bürkholder–Davies–Gundy inequality, it is easy to check that X^ε is in all $L^p(W \times \tilde{W}, E_x)$. But it has already been pointed out that one can not expect a large deviations principle for the law of X^ε . However, Fubini’s theorem allows us to consider the random variable X_ε defined for $p \geq 2$ by

$$\begin{aligned} X_\varepsilon: \quad W &\rightarrow L^p(\tilde{W}, E_x) \\ \omega &\mapsto X^\varepsilon(\omega, \cdot). \end{aligned} \tag{4.12}$$

Let P_ε be the law of X_ε (probability measure on $L^p(\tilde{W}, E_x)$). Our purpose is to show a large deviations principle for the family $(P_\varepsilon)_\varepsilon$. This will be done by writing the solution to Eq. (1.1) in terms of the stochastic flow defined by the stochastic differential equation (3.5) and by applying the contraction principle. Before stating the result, we define the rate function.

Let ϕ be in \mathcal{D}^n . We associate to ϕ the vector fields in \mathbb{R}^n

$$\begin{aligned}\tilde{s}_j^\phi(t, y) &= \phi_t^{-1} * \tilde{\sigma}_j(y) \quad \forall j \in \{1, \dots, l\} \\ \tilde{s}_0^\phi(t, y) &= \phi_t^{-1} * \tilde{\sigma}_0^*(y) - \frac{1}{2} \sum_{j=1}^l \left(\frac{\partial \phi_t}{\partial x}(y) \right)^{-1} \left[\frac{\partial^2 \phi_t}{\partial x^2}(y) (\tilde{s}_j^\phi(t, y), \tilde{s}_j^\phi(t, y)) \right].\end{aligned}\tag{4.13}$$

We consider then the Itô stochastic differential equation

$$\begin{aligned}d\tilde{z}_t^\phi &= \tilde{s}_0^\phi(t, \tilde{z}_t^\phi) dt + \sum_{j=1}^l \tilde{s}_j^\phi(t, \tilde{z}_t^\phi) \delta \tilde{B}_t^j \\ \tilde{z}_0^\phi &= x\end{aligned}\tag{4.14}$$

Without assumptions on ϕ , the existence of a strong solution to (4.14) is not ensured. So we will restrict ourselves to flows of diffeomorphisms in \mathcal{D}_b^n , where

$$\mathcal{D}_b^n = \left\{ \phi \in \mathcal{D}^n, \quad \begin{array}{l} \sup_{y \in \mathbb{R}^n, t \in [0, 1]} \left\| \frac{\partial^m (\phi_t)^{-1}}{\partial x^m}(y) \right\| < \infty, \quad m = 1, 2, \\ \sup_{y \in \mathbb{R}^n, t \in [0, 1]} \left\| \frac{\partial^m \phi_t}{\partial x^m}(y) \right\| < \infty, \quad m = 1, 2, 3. \end{array} \right\} \tag{4.15}$$

\mathcal{D}_b^n is an open set of \mathcal{D}^n with the $\tilde{C}^{0,3}$ topology. We define the topology on \mathcal{D}_b^n as the induced topology. When ϕ is in \mathcal{D}_b^n , the vector fields \tilde{s}_j^ϕ are bounded and Lipschitz in y , so that (4.14) has a strong solution. Moreover, it is easy to see that $\tilde{E}(\sup_{t \in [0, 1]} \|\phi_t(\tilde{z}_t^\phi)\|^p) < \infty$. For $p \geq 2$, we can then consider the map D

$$\begin{aligned}D: \quad \mathcal{D}_b^n &\rightarrow L^p(\tilde{W}, E_x) \\ \phi &\mapsto (\tilde{\omega} \mapsto \phi(\tilde{z}_\cdot^\phi(\tilde{\omega}))).\end{aligned}\tag{4.16}$$

D allows us to transfer on $L^p(\tilde{W}, E_x)$ the rate function I defined by (3.8).

$$\forall z \in L^p(\tilde{W}, E_x), \quad I(z) = \inf\{I(\phi), \phi \in \mathcal{D}_b^n, D(\phi) = z\} \tag{4.17}$$

(where $\inf(\emptyset) = +\infty$).

When A is a subset of $L^p(\tilde{W}, E_x)$, we will denote by $\Lambda(A) = \inf\{\Lambda(z), z \in A\}$. Using the expression of the rate function I , we have the following expression for Λ .

PROPOSITION 6. *For all $z \in L^p(\tilde{W}, E_x)$,*

$$\begin{aligned} \Lambda(z) = \inf & \left\{ \frac{1}{2} \|h\|_{\mathcal{H}^r}^2, h \in \mathcal{H}^r \text{ such that } \tilde{P} \text{ a.e. } \forall t, \right. \\ & z_t = x + \sum_{i=1}^r \int_0^t \sigma_i(z_s) \dot{h}_s^i ds \\ & \left. + \sum_{j=1}^l \int_0^t \tilde{\sigma}_j(z_s) d\tilde{B}_s^j + \int_0^t \tilde{\sigma}_0(z_s) ds \right\}. \end{aligned} \quad (4.18)$$

Proof. It is easily seen from the expression (3.8) of I that

$$\Lambda(z) = \inf \left\{ \frac{1}{2} \|h\|_{\mathcal{H}^r}^2, h \in \mathcal{H}^r, F(h) \in \mathcal{D}_b^n, D \circ F(h) = z \right\},$$

where F is the map defined by (3.7). Now using the ordinary differential equations satisfied by the derivatives of $F(h)$ and $F(h)^{-1}$, one can deduce from Gronwall's lemma that $F(\mathcal{H}^r)$ is included in \mathcal{D}_b^n . Furthermore, Theorem 2 shows that $D \circ F(h)$ is a solution to the stochastic differential equation

$$z_t = x + \sum_{i=1}^r \int_0^t \sigma_i(z_s) \dot{h}_s^i ds + \sum_{j=1}^l \int_0^t \tilde{\sigma}_j(z_s) d\tilde{B}_s^j + \int_0^t \tilde{\sigma}_0(z_s) ds. \quad \blacksquare$$

Once the rate function Λ is defined, we can state the large deviations principle for the family (P_ε) .

THEOREM 7. • *Λ is a good rate function.*

• $\forall A \subset L^p(\tilde{W}, E_x)$,

$$-\Lambda(\dot{A}) \leq \liminf_{\varepsilon \rightarrow 0} \varepsilon^2 \log P_\varepsilon(A) \leq \limsup_{\varepsilon \rightarrow 0} \varepsilon^2 \log P_\varepsilon(A) \leq -\Lambda(\bar{A}). \quad (4.19)$$

Before proving Theorem 7, we would like to underline that it extends to the nonnilpotent case the results of Doss and Stroock [7] when the vector fields σ_i commute and those of Rabehimanana [11] when the Lie algebra generated by the σ_i is nilpotent. In these two papers, a particular version $R^{\varepsilon, \cdot}$ of the conditional law of the process X^ε relative to (εB) is considered, and a large deviations principle is obtained for the law Q_ε of the random variable $\omega \in W \mapsto R^{\varepsilon, \varepsilon B(\omega)} \in \mathcal{M}_1(E_x)$ (Q_ε is an element of $\mathcal{M}_1(\mathcal{M}_1(E_x))$).

Applying the contraction principle, we deduce this large deviations principle from Theorem 7 *without nilpotence assumptions*.

COROLLARY 8. 1. *The map D can be extended to $F(W)$, where F is defined by (3.7).*

2. *When $\phi \in F(W)$, let N^ϕ be the law of the process $D(\phi)$. Let us define $R^{\varepsilon B}$ by $N^{F(\varepsilon B)}$. Then $R^{\varepsilon B}$ is a version of the conditional law of X^ε relative to (εB) .*

3. *When μ is an element of $\mathcal{M}_1(E_x)$, let us define*

$$\tilde{\Lambda}(\mu) = \inf \left\{ \frac{1}{2} \|h\|_{\mathcal{H}^r}^2, h \in \mathcal{H}^r, \text{ such that } \mu \text{ is the law of the process } z_t \text{ solution to (4.18)} \right\}.$$

If $\sigma = (\sigma_1 \cdots \sigma_r)$ is a constant matrix such that $a = \sigma\sigma^*$ is invertible, then

$$\tilde{\Lambda}(\mu) = \begin{cases} \frac{1}{2} \int_0^1 \|\dot{E}_\mu(\omega_s) - E_\mu(\tilde{\sigma}_0^*(\omega_s))\|_{a^{-1}}^2 ds, & \text{if } \mu \in \{R^h, h \in \mathcal{H}^r\} \\ +\infty, & \text{otherwise.} \end{cases}$$

4. *Let Q_ε be the law of the random variable $\omega \mapsto R^{\varepsilon B(\omega)}$, and let A be a subset of $\mathcal{M}_1(E_x)$. Then*

$$-\tilde{\Lambda}(\dot{A}) \leq \liminf_{\varepsilon \rightarrow 0} \varepsilon^2 \log Q_\varepsilon(A) \leq \limsup_{\varepsilon \rightarrow 0} \varepsilon^2 \log Q_\varepsilon(A) \leq -\tilde{\Lambda}(\bar{A}). \quad (4.20)$$

Proof of Corollary 8. 1 and 2 are consequences of Theorem 2, where it is proved that, when ϕ is in $F(W)$, Eq. (4.14) has a strong solution defined on $[0, 1]$ and that \mathbb{P} -a.e. $\forall t \in [0, 1]$, $X_t^\varepsilon(\omega, \tilde{\omega}) = D(F(\varepsilon B(\omega)))_t(\tilde{\omega})$.

Statement 3 is quite obvious, since if $\mu = R^h$, then taking the expectation in (4.18) yields

$$E_\mu(\omega_t) = x + \int_0^t \sigma \dot{h}_s ds + \int_0^t E_\mu \tilde{\sigma}_0^*(\omega_s) ds.$$

Since $\|\dot{E}_\mu(\omega_t) - E_\mu(\tilde{\sigma}_0^*(\omega_t))\|_{a^{-1}}^2 = \inf \{ \|x\|^2, \sigma x = \dot{E}_\mu(\omega_t) - E_\mu(\tilde{\sigma}_0^*(\omega_t)) \}$, $\frac{1}{2} \|h\|_{\mathcal{H}^r}^2 \geq \frac{1}{2} \int_0^1 \|\dot{E}_\mu(\omega_s) - E_\mu(\tilde{\sigma}_0^*(\omega_s))\|_{a^{-1}}^2 ds$. This inequality being true for all h such that $\mu = R^h$, it also holds for $\tilde{\Lambda}(\mu)$.

Moreover, let P be the orthogonal projection on $G \equiv (\text{Ker}(\sigma))^\perp$, and define Πh by $\Pi h_t = \int_0^t P \dot{h}_s ds$; then $\mu = R^{\Pi h}$, $\sigma \dot{\Pi h}_t = \dot{E}_\mu(\omega_t) - E_\mu(\tilde{\sigma}_0^*(\omega_t))$. But, since $\sigma\sigma^*$ is invertible, $\sigma_G: G \rightarrow \mathbb{R}^n$ is invertible, and

$$\dot{\Pi h}_t = \sigma_G^{-1}(\dot{E}_\mu(\omega_t) - E_\mu(\tilde{\sigma}_0^*(\omega_t))).$$

Thus,

$$\begin{aligned}\|\Pi h\|_{\mathcal{H}^r}^2 &= \int_0^1 \|\widehat{E_\mu(\omega_s)} - E_\mu(\tilde{\sigma}_0^*(\omega_s))\|_{(\sigma_G \sigma_G^*)^{-1}}^2 ds \\ &= \int_0^1 \|\widehat{E_\mu(\omega_s)} - E_\mu(\tilde{\sigma}_0^*(\omega_s))\|_{a^{-1}}^2 ds\end{aligned}$$

and $\tilde{\Lambda}(\mu) \leq \frac{1}{2} \|\Pi h\|_{\mathcal{H}^r}^2 \leq \frac{1}{2} \int_0^1 \|\widehat{E_\mu(\omega_s)} - E_\mu(\tilde{\sigma}_0^*(\omega_s))\|_{a^{-1}}^2 ds$.

4 is derived from the contraction principle. Indeed, the map

$$\begin{aligned}L^p(\tilde{W}, E_x), P_\varepsilon &\rightarrow \mathcal{M}_1(E_x) \\ Z &\mapsto \text{the law of } Z \text{ under } \tilde{P}\end{aligned}\tag{4.21}$$

is continuous when $\mathcal{M}_1(E_x)$ is endowed with the topology of weak convergence. Moreover, it transforms P_ε into \mathcal{Q}_ε . ■

Remark. Let $q_\mu(s, \cdot)$ be the conjugated quadratic form of $E_\mu(\sigma(\omega_s)) E_\mu(\sigma(\omega_s))^*$, that is,

$$q_\mu(s, x) = \inf \{ \|w\|^2, E_\mu(\sigma(\omega_s)) w = x \}.$$

Then, Proposition 6 of [7] says that

$$\tilde{\Lambda}(\mu) = \begin{cases} \frac{1}{2} \int_0^1 q_\mu(s, \widehat{E_\mu(\omega_s)} - E_\mu(\tilde{\sigma}_0^*(\omega_s))) ds & \text{if } \mu \in \{R^h, h \in \mathcal{H}^r\} \\ +\infty & \text{otherwise.} \end{cases}$$

It seems that this assertion is false. Let us consider the case

- $n = r = l = 1$;
- $\tilde{\sigma} \equiv 1$; $\sigma(x) = x$; $\tilde{\sigma}_0 = 0$;
- $x = 0$.

Then the law μ of the Ornstein–Uhlenbeck process $dz_t = z_t dt + \delta \tilde{B}_t$, can be expressed as R^h with $h_t = t$. For all t , $E(\sigma(z_t)) = E(z_t) = 0$. Thus,

$$q_\mu(s, x) = \begin{cases} 0 & \text{if } x = 0 \\ +\infty & \text{otherwise,} \end{cases}$$

$$\int_0^1 q_\mu(s, \widehat{E_\mu(\omega_s)} - E_\mu(\tilde{\sigma}_0^*(\omega_s))) ds = 0.$$

But, if $\tilde{\Lambda}(\mu) = 0$, then $\mu = R^0$; that is, μ is the law of the Brownian motion.

4.3. Proof of Proposition 5 and Theorem 7

LEMMA 9. When \mathcal{D}_b^n is provided with the $\tilde{\mathcal{C}}^{0,3}$ topology, the map D defined by (4.16) is continuous.

Proof of Lemma 9. Let ϕ^n and ϕ be flows of diffeomorphisms in \mathcal{D}_b^n such that $\phi^n \xrightarrow{\tilde{\mathcal{C}}^{0,3}} \phi$,

$$\|D(\phi^n) - D(\phi)\|_{L^p(\tilde{W}, E_x)} = \tilde{E} \left[\sup_{t \in [0, 1]} \|\phi_t^n(\tilde{z}_t^{\phi^n}) - \phi_t(\tilde{z}_t^\phi)\|^p \right]^{1/p} \leq T_1 + T_2,$$

where

$$\begin{aligned} T_1 &= \sup_{t, y} \left\| \frac{\partial \phi_t^n}{\partial x}(y) \right\| \tilde{E} \left[\sup_t \|\tilde{z}_t^{\phi^n} - \tilde{z}_t^\phi\|^p \right]^{1/p} \\ T_2 &= \tilde{E} \left[\sup_t \|\phi_t^n(\tilde{z}_t^\phi) - \phi_t(\tilde{z}_t^\phi)\|^p \right]^{1/p}. \end{aligned}$$

Treatment of T_1 . The first derivatives of ϕ_t^n converge uniformly on compact sets of $[0, 1] \times \mathbb{R}^n$ to the first derivatives of ϕ_t (which are bounded). Hence, $\sup_{n, t, y} \|(\partial \phi_t^n / \partial x)(y)\| < +\infty$. We have now to show that $\tilde{E} \left[\sup_t \|\tilde{z}_t^{\phi^n} - \tilde{z}_t^\phi\|^p \right] \xrightarrow{n \rightarrow +\infty} 0$.

For $t \in [0, 1]$, let $f_n(t) = \tilde{E} \left[\sup_{s \in [0, t]} \|\tilde{z}_s^{\phi^n} - \tilde{z}_s^\phi\|^p \right]^{1/p}$. The triangle inequality in L^p and the convexity of $x \mapsto x^p$ for $p \geq 1$ yield

$$\begin{aligned} f_n(t) &\leq \tilde{E} \left[\int_0^t \|\tilde{s}_0^{\phi^n}(s, \tilde{z}_s^{\phi^n}) - \tilde{s}_0^{\phi^n}(s, \tilde{z}_s^\phi)\|^p ds \right]^{1/p} \\ &\quad + \tilde{E} \left[\int_0^t \|\tilde{s}_0^{\phi^n}(s, \tilde{z}_s^\phi) - \tilde{s}_0^\phi(s, \tilde{z}_s^\phi)\|^p ds \right]^{1/p} \\ &\quad + \tilde{E} \left[\sup_{s \in [0, t]} \left\| \sum_{j=1}^l \int_0^s (\tilde{s}_j^{\phi^n}(u, \tilde{z}_u^{\phi^n}) - \tilde{s}_j^\phi(u, \tilde{z}_u^\phi)) \delta \tilde{B}_u^j \right\|^p \right]^{1/p}. \end{aligned}$$

Using the Bürkholder–Davies–Gundy inequality there exists a constant C such that

$$\begin{aligned} &\tilde{E} \left[\sup_{s \in [0, t]} \left\| \sum_{j=1}^l \int_0^s (\tilde{s}_j^{\phi^n}(u, \tilde{z}_u^{\phi^n}) - \tilde{s}_j^\phi(u, \tilde{z}_u^\phi)) \delta \tilde{B}_u^j \right\|^p \right]^{1/p} \\ &\leq C \tilde{E} \left[\left(\int_0^t \sum_{j=1}^l \|\tilde{s}_j^{\phi^n}(u, \tilde{z}_u^{\phi^n}) - \tilde{s}_j^\phi(u, \tilde{z}_u^\phi)\|^2 du \right)^{p/2} \right]^{1/p} \end{aligned}$$

$$\begin{aligned}
&\leq C \sum_{j=1}^l \tilde{E} \left[\left(\int_0^t \|\tilde{s}_j^{\phi^n}(u, \tilde{z}_u^{\phi^n}) - \tilde{s}_j^{\phi}(u, \tilde{z}_u^{\phi})\|^2 du \right)^{p/2} \right]^{1/p} \\
&\quad + C \sum_{j=1}^l \tilde{E} \left[\left(\int_0^t \|\tilde{s}_j^{\phi^n}(u, \tilde{z}_u^{\phi}) - \tilde{s}_j^{\phi}(u, \tilde{z}_u^{\phi})\|^2 du \right)^{p/2} \right]^{1/p} \\
&\leq C \sum_{j=1}^l \sup_{n, t, y} \left\| \frac{\partial \tilde{s}_j^{\phi^n}(t, y)}{\partial x} \right\| \tilde{E} \left[\int_0^t \|\tilde{z}_u^{\phi^n} - \tilde{z}_u^{\phi}\|^p du \right]^{1/p} \\
&\quad + C \sum_{j=1}^l \tilde{E} \left[\int_0^t \|\tilde{s}_j^{\phi^n}(u, \tilde{z}_u^{\phi}) - \tilde{s}_j^{\phi}(u, \tilde{z}_u^{\phi})\|^p du \right]^{1/p}.
\end{aligned}$$

But, $\tilde{E}[\int_0^t \|\tilde{z}_u^{\phi^n} - \tilde{z}_u^{\phi}\|^p du]^{1/p} \leq (\int_0^t f_n(u)^p du)^{1/p}$. Furthermore, the $\tilde{\mathcal{C}}^{0,3}$ convergence of ϕ^n to ϕ implies the uniform convergence on compact sets of $[0, 1] \times \mathbb{R}^n$ of $\tilde{s}_i^{\phi^n}$ and its first derivatives to \tilde{s}_i^{ϕ} and its first derivatives, which are bounded since ϕ is an element of \mathcal{D}_b^n . Therefore, $\sup_{n, t, y, j} \|\partial \tilde{s}_j^{\phi^n}(t, y) / \partial x\| < +\infty$. Thus there exists a constant C such that

$$f_n^p(t) \leq C \left[\int_0^t f_n^p(s) ds + \sum_{j=0}^l \tilde{E} \left(\int_0^1 \|\tilde{s}_j^{\phi^n}(u, \tilde{z}_u^{\phi}) - \tilde{s}_j^{\phi}(u, \tilde{z}_u^{\phi})\|^p du \right) \right].$$

Gronwall's lemma yields then

$$\forall t \in [0, 1], \quad f_n^p(t) \leq C \sum_{j=0}^l \tilde{E} \left(\int_0^1 \|\tilde{s}_j^{\phi^n}(u, \tilde{z}_u^{\phi}) - \tilde{s}_j^{\phi}(u, \tilde{z}_u^{\phi})\|^p du \right) e^{ct}$$

and we have to show that

$$\forall j \in \{0, \dots, l\}, \quad \tilde{E} \left(\int_0^1 \|\tilde{s}_j^{\phi^n}(u, \tilde{z}_u^{\phi}) - \tilde{s}_j^{\phi}(u, \tilde{z}_u^{\phi})\|^p du \right) \xrightarrow[n \rightarrow +\infty]{} 0.$$

Let R be a positive real number and $\mathcal{B}(0, R)$ the ball of radius R in \mathbb{R}^n ,

$$\begin{aligned}
&\tilde{E} \left(\int_0^1 \|\tilde{s}_j^{\phi^n}(u, \tilde{z}_u^{\phi}) - \tilde{s}_j^{\phi}(u, \tilde{z}_u^{\phi})\|^p du \right) \\
&\leq \sup_{t \in [0, 1], y \in \mathcal{B}(0, R)} \|\tilde{s}_j^{\phi^n}(t, y) - \tilde{s}_j^{\phi}(t, y)\|^p \\
&\quad + 2^{p-1} \tilde{E} [\mathbb{1}_{\sup_{t \in [0, 1]} \|\tilde{z}_t^{\phi}\| \geq R} \sup_{t, y} (\|\tilde{s}_j^{\phi^n}(t, y)\|^p + \|\tilde{s}_j^{\phi}(t, y)\|^p)].
\end{aligned}$$

The convergence of ϕ^n to ϕ implies that $\sup_{n,t,y}(\|\tilde{s}_j^{\phi^n}(t, y)\| + \|\tilde{s}_j^\phi(t, y)\|) < \infty$; thus, there is a constant C such that

$$\begin{aligned} \tilde{E} \left(\int_0^1 \|\tilde{s}_j^{\phi^n}(u, \tilde{z}_u^\phi) - \tilde{s}_j^\phi(u, \tilde{z}_u^\phi)\|^p du \right) \\ \leq \sup_{t \in [0, 1], y \in \mathcal{B}(0, R)} \|\tilde{s}_j^{\phi^n}(t, y) - \tilde{s}_j^\phi(t, y)\|^p + C\tilde{P}(\sup_{t \in [0, 1]} \|\tilde{z}_t^\phi\| \geq R). \end{aligned}$$

Let $\eta > 0$. Since \tilde{z}^ϕ is solution to a stochastic differential equation with bounded coefficients, we can find R such that $C\tilde{P}(\sup_{t \in [0, 1]} \|\tilde{z}_t^\phi\| \geq R) < \eta/2$. Let $n_0 \in \mathbb{N}$ be such that $\forall n \geq n_0$, $\sup_{t \in [0, 1], y \in \mathcal{B}(0, R)} \|\tilde{s}_j^{\phi^n}(t, y) - \tilde{s}_j^\phi(t, y)\|^p < \eta/2$. Then for $n \geq n_0$, $\tilde{E}(\int_0^1 \|\tilde{s}_j^{\phi^n}(u, \tilde{z}_u^\phi) - \tilde{s}_j^\phi(u, \tilde{z}_u^\phi)\|^p du) < \eta$.

Treatment of T_2 . Since ϕ^n and ϕ are elements of \mathcal{D}_b^n , and $\phi^n \xrightarrow{\mathcal{C}^{0,3}} \phi$, one can choose constants K_n and K such that

- $\forall t \in [0, 1], \forall y \in \mathbb{R}^n$,

$$\|\phi_t^n(y)\| \leq K_n(1 + \|y\|)$$

$$\|\phi_t(y)\| \leq K(1 + \|y\|);$$

- $\sup_n K_n < +\infty$.

Therefore,

$$\begin{aligned} T_2^p &\leq \sup_{t, y \in \mathcal{B}(0, R)} \|\phi_t^n(y) - \phi_t(y)\|^p \\ &\quad + C(\sup_n K_n + K) \tilde{E}[\sup_t (1 + \|\tilde{z}_t^\phi\|^p) \mathbb{1}_{\sup_t \|\tilde{z}_t^\phi\|^p > R}] \\ &\leq \sup_{t, y \in \mathcal{B}(0, R)} \|\phi^n(t, y) - \phi(t, y)\|^p \\ &\quad + \underbrace{C\tilde{E}(\sup_t (1 + \|\tilde{z}_t^\phi\|^{2p}))^{1/2} \tilde{P}[\sup_t \|\tilde{z}_t^\phi\| > R]^{1/2}}_{< \infty}. \end{aligned}$$

Let R be such that the second term is less than $\eta/2$. From the uniform convergence of ϕ^n to ϕ , we can then find n such the first term is less than $\eta/2$. Lemma 9 follows. ■

We are now able to prove Theorem 7 and Proposition 5. The key is to write X^e in terms of the stochastic flow defined by (3.5) using the map D and, then, to apply the contraction principle. The only technical point is that the probability for the stochastic flow to be in \mathcal{D}_b^n is strictly less than

one. So we will have to truncate the vector fields σ_i . For all $R > 0$, we approximate the vector fields σ_i by some C_b^k vector fields σ_i^R such that

$$\begin{aligned}\sigma_i^R(y) &= 0 && \text{if } \|y\| \geq 2R \\ \sigma_i^R(y) &= \sigma_i(y) && \text{if } \|y\| \leq R.\end{aligned}$$

Let $X^{\varepsilon, R}$ be solution to the Stratonovich stochastic differential equation,

$$\begin{aligned}X_t^{\varepsilon, R} &= x + \varepsilon \sum_{i=1}^r \int_0^t \sigma_i^R(X_s^{\varepsilon, R}) dB_s^i \\ &\quad + \sum_{j=1}^l \int_0^t \tilde{\sigma}_j(X_s^{\varepsilon, R}) d\tilde{B}_s^j + \int_0^t \tilde{\sigma}_0(X_s^{\varepsilon, R}) ds.\end{aligned}$$

As long as X_t^{ε} stays in $\mathcal{B}(0, R)$, $X_t^{\varepsilon, R} = X_t^{\varepsilon}$. Moreover, X^{ε} and $X^{\varepsilon, R}$ are solutions to stochastic differential equations with bounded coefficients. Thereby, one can find constants $C_0, R_0 > 0$ such that for $R \geq R_0$ and $\varepsilon \leq 1$,

$$\begin{aligned}\mathbb{P}(\sup_t \|X_t^{\varepsilon}\| \geq R) &\leq C_0 \exp(-R^2/C_0) \\ \mathbb{E}(\sup_t \|X_t^{\varepsilon, R} - X_t^{\varepsilon}\|^p) &\leq C_0 \exp(-R^2/C_0).\end{aligned}\tag{4.22}$$

The advantage in considering $X^{\varepsilon, R}$ instead of X^{ε} is that if $\Phi^{\varepsilon, R}$ is the stochastic flow associated to the stochastic differential equation $d\xi_t^{\varepsilon, R} = \sum_{i=1}^r \sigma_i^R(\xi_s^{\varepsilon, R}) dB_s^i$, $\Phi^{\varepsilon, R} \in \mathcal{D}_b^n$, since $\Phi^{\varepsilon, R}(x) = x$ whenever $x \notin \mathcal{B}(0, 2R)$.

Proof of Proposition 5. The lower bound has already been proved in [7] and in [11] in the general case. So we only give the proof of the upper bound. Let A be a closed subset of E_x . Let us fix $L > 0$, and $\varepsilon_0 = L/R_0$ (where R_0 is chosen so that (4.22) holds). Then, for $\varepsilon \leq \varepsilon_0$,

$$\begin{aligned}\mathbb{P}[X^{\varepsilon} \in A] &\leq \mathbb{P}[X^{\varepsilon, L/\varepsilon} \in A] + \mathbb{P}[\sup_t \|X_t^{\varepsilon}\| \geq L/\varepsilon] \\ &\leq \mathbb{P}[D(\Phi^{\varepsilon, L/\varepsilon})(\tilde{B}) \in A] + C_0 \exp(-L^2/C_0 \varepsilon^2) \\ &\leq E[N^{\Phi^{\varepsilon, L/\varepsilon}}(A)] + C_0 \exp(-L^2/C_0 \varepsilon^2).\end{aligned}$$

We recall that N^{Φ} is the law of the process $D(\Phi)$, so that $0 \leq N^{\Phi}(A) \leq 1$. Therefore,

$$\mathbb{P}[X^{\varepsilon} \in A] \leq P[N^{\Phi^{\varepsilon, L/\varepsilon}}(A) > 0] + C_0 \exp(-L^2/C_0 \varepsilon^2).$$

Notations are the same as in Corollary 8. The vector fields $\sigma_i^{L/\varepsilon}$ converge in \mathcal{C}_b^m uniformly on compact sets to the vector fields σ_i . From Theorem 3, it results then that

$$\begin{aligned} \limsup_{\varepsilon \rightarrow 0} \varepsilon^2 \log P[N^{\Phi_{\varepsilon, L/\varepsilon}}(A) > 0] \\ \leq -\inf\left\{\frac{1}{2} \|h\|_{\mathcal{H}^r}^2, F(h) \in \overline{\{\phi \in F(W) \cap \mathcal{D}_b^n, N^\phi(A) > 0\}}\right\}. \end{aligned}$$

But $F(W) \subset \overline{F(\mathcal{H}^r)}$ (cf. [4]), and the continuity of the map D yields

$$\begin{aligned} \{\phi \in F(W) \cap \mathcal{D}_b^n, N^\phi(A) > 0\} &\subset \overline{\{\phi \in F(\mathcal{H}^r), N^\phi(A) > 0\}} \\ &\subset \bigcap_{\delta > 0} \overline{F\{h \in \mathcal{H}^r, N^{F(h)}(A_\delta) > 0\}} \\ &\subset \bigcap_{\delta > 0} \overline{F\{h \in \mathcal{H}^r, \exists \tilde{h} \in \mathcal{H}^l, G(h, \tilde{h}) \in A_\delta\}}. \end{aligned}$$

The last inclusion is given by the support theorem for diffusion, since $N^{F(h)}$ is nothing but the law of the diffusion defined by (4.18).

Let $B_\delta \equiv \{h \in \mathcal{H}^r, \exists \tilde{h} \in \mathcal{H}^l, G(h, \tilde{h}) \in A_\delta\}$. It remains to show that

$$I_1 \equiv \inf \left\{ 1/2 \|h\|_{\mathcal{H}^r}^2, h \in \bigcap_{\delta > 0} \overline{B_\delta} \right\} \leq \inf \left\{ 1/2 \|h\|_{\mathcal{H}^r}^2, F(h) \in \bigcap_{\delta > 0} \overline{F(B_\delta)} \right\} \equiv I_2.$$

This inequality is obvious when $I_2 = +\infty$. Therefore, we assume that $I_2 < \infty$. Let $h \in \mathcal{H}^r$ such that $I_2 = 1/2 \|h\|_{\mathcal{H}^r}^2$ and such that $\phi \equiv F(h) \in \bigcap_{\delta > 0} \overline{F(B_\delta)}$. For all $\delta > 0$, let (h_n^δ) be a sequence in B_δ such that $\phi_n^\delta \equiv F(h_n^\delta) \rightarrow \phi$. Let us define the map

$$M: F(\mathcal{H}^r) \rightarrow \mathcal{H}^r$$

$$\phi \mapsto \text{the unique } h \text{ such that } \begin{cases} \phi = F(h) \\ I(\phi) = 1/2 \|h\|_{\mathcal{H}^r}^2. \end{cases}$$

It is proved in Appendix 3 that M is continuous, when \mathcal{H}^r is endowed with the uniform convergence. Therefore, $M(\phi_n^\delta) \rightarrow M(\phi) = h$. But $M(\phi_n^\delta) = \Pi h_n^\delta$ (where Π is defined by (3.9)). Moreover, it is easy to check that $\Pi h_n^\delta \in B_\delta$. Thus $h \in \bigcap_{\delta > 0} \overline{B_\delta}$ and $I_1 \leq 1/2 \|h\|_{\mathcal{H}^r}^2 = I_2$. ■

Proof of Theorem 7. First of all, it results from the continuity of D and the contraction principle that A is a “good” rate function.

Proof of the upper bound. Let A be a closed subset of $L^p(\tilde{W}, E_x)$. Let us fix $\eta > 0$, $L > 0$, $\varepsilon_0 = L/R_0$ (where R_0 is chosen so that (4.22) holds). A^n will denote the subset of $L^p(\tilde{W}, E_x)$ defined by

$$\begin{aligned}
A^\eta &= \{z \in L^p(\tilde{W}, E_x), \exists y \in A \ \lVert y - z \rVert_{L^p(\tilde{W}, E_x)} \leq \eta\} \\
P_\varepsilon(A) &= P(X_\varepsilon \in A) \\
&\leq P(X^{\varepsilon, L/\varepsilon} \in A^\eta) + P(\lVert X^{\varepsilon, L/\varepsilon} - X_\varepsilon \rVert_{L^p(\tilde{W}, E_x)} > \eta) \\
&\leq P(D(\Phi^{\varepsilon, L/\varepsilon}) \in A^\eta) + P(\tilde{E}(\sup_t \lVert X_t^{\varepsilon, L/\varepsilon} - X_t^\varepsilon \rVert^p) > \eta^p) \\
&\leq P(D(\Phi^{\varepsilon, L/\varepsilon}) \in A^\eta) + \frac{1}{\eta^p} \mathbb{E}(\sup_t \lVert X_t^{\varepsilon, L/\varepsilon} - X_t^\varepsilon \rVert^p).
\end{aligned}$$

For $\varepsilon \leq \varepsilon_0 \wedge 1, L/\varepsilon \geq R_0$. We derive from (4.22) that the second term is bounded up by $C_0 \exp(-L^2/C_0 \varepsilon^2)$. From Theorem 3, the law of $\Phi^{\varepsilon, L/\varepsilon}$ satisfies a large deviations principle with rate function I . The map D being continuous, we just have to apply the contraction principle to derive

$$\limsup_{\varepsilon \rightarrow 0} \varepsilon^2 \log P(D(\Phi^{\varepsilon, L/\varepsilon}) \in A^\eta) \leq -I(A^\eta).$$

Therefore, $\forall L > 0, \forall \eta > 0$,

$$\limsup_{\varepsilon \rightarrow 0} \varepsilon^2 \log P_\varepsilon(A) \leq -\inf(A(A^\eta), L^2/C_0).$$

Letting L go to infinity, we derive that $\forall \eta > 0, \limsup_{\varepsilon \rightarrow 0} \varepsilon^2 \log P_\varepsilon(A) \leq -I(A^\eta)$. I being a good rate function, $I(A^\eta) \xrightarrow{\eta \rightarrow 0} I(A)$.

Proof of the lower bound. Let A be an open subset of $L^p(\tilde{W}, E_x)$. When $I(A) = +\infty$, the lower bound is trivial. So we assume that $I(A) < +\infty$. Let $g \in A$ be such that $I(g) < +\infty$. Let L_1 be such that $I(g) < L_1$. A being open, we can choose $\eta > 0$ such that $\mathcal{B}(g, \eta) \subset A$. Let us fix $L > \sqrt{C_0 L_1}$ and $\varepsilon_0 = L/R_0$ (where R_0, C_0 are constants such that (4.22) holds).

$$\begin{aligned}
P_\varepsilon(A) &\geq P_\varepsilon(\mathcal{B}(g, \eta)) \\
&\geq P(X^{\varepsilon, L/\varepsilon} \in \mathcal{B}(g, \eta/2); \lVert X^{\varepsilon, L/\varepsilon} - X_\varepsilon \rVert_{L^p(\tilde{W}, E_x)} < \eta/2) \\
&\geq P(X^{\varepsilon, L/\varepsilon} \in \mathcal{B}(g, \eta/2)) - P(\lVert X^{\varepsilon, L/\varepsilon} - X_\varepsilon \rVert_{L^p(\tilde{W}, E_x)} \geq \eta/2) \\
&\geq P(D(\Phi^{\varepsilon, L/\varepsilon}) \in \mathcal{B}(g, \eta/2)) - \frac{2^p}{\eta^p} \mathbb{E}(\sup_t \lVert X_t^{\varepsilon, L/\varepsilon} - X_t^\varepsilon \rVert^p).
\end{aligned}$$

By the contraction principle,

$$\liminf_{\varepsilon \rightarrow 0} \varepsilon^2 \log P(D(\Phi^{\varepsilon, L/\varepsilon}) \in \mathcal{B}(g, \eta/2)) \geq -I(\mathcal{B}(g, \eta/2)) \geq -I(g)$$

For $\varepsilon \leq \varepsilon_0$,

$$\begin{aligned} \mathbb{E}(\sup_t \|X_t^{\varepsilon, L/\varepsilon} - X_t^\varepsilon\|^p) &\leq C_0 e^{-L^2/C_0\varepsilon^2} \\ &\leq \frac{\eta^p}{2^p} e^{-L_1/\varepsilon^2} \quad \text{for sufficiently small } \varepsilon. \end{aligned}$$

From $A(g) < L_1$, it follows that

$$\liminf_{\varepsilon \rightarrow 0} \varepsilon^2 \log P_\varepsilon(A) \geq -A(g)$$

Taking the supremum over A , we obtain the result. \blacksquare

5. LARGE DEVIATIONS IN A NONLINEAR FILTERING PROBLEM

This section deals with another application of flow decomposition. It concerns a nonlinear filtering problem that has been first studied by Doss [6] and then by Rabeherimana [11]. The problem can be stated as follows. Let us consider the couple signal-observation $(\mathcal{X}^\varepsilon, \mathcal{Y}^\varepsilon)$ solution to the system of stochastic differential equations,

$$\begin{aligned} d\mathcal{X}_t^\varepsilon &= \varepsilon \sum_{i=1}^r \sigma_i(\mathcal{X}_t^\varepsilon) dB_t^i + \varepsilon^2 \tilde{\sigma}_0(\mathcal{X}_t^\varepsilon) dt + \sum_{j=1}^l \tilde{\sigma}_j(\mathcal{X}_t^\varepsilon) d\mathcal{Y}_t^{\varepsilon, j} \\ d\mathcal{Y}_t^\varepsilon &= \Gamma(\mathcal{X}_t^\varepsilon) dt + d\tilde{B}_t \\ \mathcal{X}_0^\varepsilon &= x; \quad \mathcal{Y}_0^\varepsilon = 0, \end{aligned} \tag{5.23}$$

where

- $B, \tilde{B}, \sigma_i, \tilde{\sigma}_j$ satisfy the same assumptions as in Section 4.
- Γ is a sufficiently smooth function from \mathbb{R}^n to \mathbb{R}^l .

We want to obtain a large deviations principle for the conditional law of the signal \mathcal{X}^ε relative to the observation \mathcal{Y}^ε . Such a principle has been obtained in [6, 11] under some nilpotence assumptions for the vector fields. As in Section 4, we would like to free ourselves of these assumptions by using flow decomposition, and the large deviations for stochastic flows.

As done in [6, 11], the first step to obtain such a principle is to make a change of probability, in such a way that the new law of \mathcal{Y}^ε is the law of a Brownian motion independent of B . So we are led to obtain a large deviations principle for the conditional law of the process X^ε defined by (1.1) relative to the Brownian motion \tilde{B} .

5.1. Large Deviations for the Conditional Law of X^e Relative to B

We begin by decomposing the stochastic differential equation (4.10) (or (1.1)). Let ϕ be in \mathcal{D}^n . We associate to ϕ the vector fields

$$\begin{aligned} s_i^\phi(t, y) &= \phi_t^{-1} * \sigma_i(y) \quad \forall i \in \{1, \dots, r\} \\ s_0^\phi(t, y) &= \phi_t^{-1} * \sigma_0^*(y) - \frac{1}{2} \sum_{i=1}^r \left(\frac{\partial \phi_t}{\partial x}(y) \right)^{-1} \\ &\quad \times \left[\frac{\partial^2 \phi_t}{\partial x^2}(y) (s_i^\phi(t, y), s_i^\phi(t, y)) \right]. \end{aligned} \quad (5.24)$$

We consider then the Itô stochastic differential equation,

$$\begin{aligned} dz_t^{\phi, \varepsilon} &= \varepsilon^2 s_0^\phi(t, z_t^{\phi, \varepsilon}) dt + \varepsilon \sum_{i=1}^r s_i^\phi(t, z_t^{\phi, \varepsilon}) \delta B_t^i \\ z_0^{\phi, \varepsilon} &= x. \end{aligned} \quad (5.25)$$

s_i^ϕ are continuous and locally Lipschitz. Thus, the trajectories of $z^{\phi, \varepsilon}$ may explode. Nevertheless, (5.25) defines a map from W to the space of explosive trajectories $\mathcal{E}_x(\mathbb{R}^n)$ (see Azencott [1]),

$$\begin{aligned} \mathcal{E}_x(\mathbb{R}^n) &= \{f: [0, 1] \rightarrow \mathbb{R}^n \cup \infty, f(0) = x, f \text{ continuous:} \\ &\quad f(t_0) = \infty \Rightarrow \forall t \in [t_0, 1], f(t) = \infty\}. \end{aligned}$$

When $f \in \mathcal{E}_x(\mathbb{R}^n)$, we define the explosion time of f as

$$\tau(f) \equiv \inf\{s, f(s) = \infty\}.$$

We will say that a sequence $(f_n)_n$ in $\mathcal{E}_x(\mathbb{R}^n)$ converges to $f \in \mathcal{E}_x(\mathbb{R}^n)$ if and only if $(f_n)_n$ converges to f uniformly on compact subsets of $[0, \tau(f)]$. In particular, this means that $\tau(f) \leq \liminf \tau(f_n)$. Similarly as in Section 4, we define then the map

$$\begin{aligned} \tilde{D}: \mathcal{D}^n &\rightarrow L^0(W, \mathcal{E}_x(\mathbb{R}^n)) \\ \phi &\mapsto (\omega \mapsto \phi_\cdot(z_\cdot^{\phi, \varepsilon})) \end{aligned}$$

(with the convention $\phi_\cdot(\infty) = \infty$).

Let $T^e(\phi, d\omega)$ denote the law of the process $\tilde{D}(\phi)(T^e(\phi, d\omega))$ is a probability measure on $\mathcal{E}_x(\mathbb{R}^n)$. We will show a large deviations principle for the family $(T^e(\phi, d\omega))_e$. As in Section 3, we begin with the “quasicontinuity” of the map $e\omega \mapsto \tilde{D}(\phi)(e\omega)$.

PROPOSITION 10. *Given $h \in \mathcal{H}^r$ and $\phi \in \mathcal{D}^n$, we define the process $x^\phi(h)$ as the solution to the ordinary differential equation*

$$x_t^\phi(h) = x + \int_0^t \sum_{i=1}^r s_i^\phi(s, x_s^\phi(h)) \dot{h}_s^i ds. \quad (5.26)$$

Then, $\forall \phi \in \mathcal{D}^n$, $\forall K$ compact sets of \mathbb{R}^n , $\forall a > 0$, $\forall L > 0$, $\forall R > 0$, $\forall T \in]0, 1]$, $\exists b, \varepsilon_0$ such that $\forall \varepsilon \leq \varepsilon_0$, $\forall h \in \mathcal{H}^r$ such that $\|h\|_{\mathcal{H}^r} \leq a$, $x^\phi(h)([0, T]) \subset K$,

$$P\left[\sup_{t \in [0, T]} \|\phi_t(z_t^{\phi, \varepsilon}) - \phi_t(x_t^\phi(h))\| \geq R; \|\varepsilon B - h\| \leq b\right] \leq e^{-L/\varepsilon^2}.$$

Proof of Proposition 10. $\forall \eta > 0$, we will denote by K^η the set $K^\eta = \{y \in \mathbb{R}^n, \exists z \in K \ |y - z| \leq \eta\}$. Let $\eta \in]0, 1[$ be such that

$$\sup_{t \in [0, 1], y, z \in K^1, \|y - z\| \leq \eta} \|\phi_t(y) - \phi_t(z)\| < R.$$

Then,

$$\begin{aligned} & P\left[\sup_{t \in [0, T]} \|\phi_t(z_t^{\phi, \varepsilon}) - \phi_t(x_t^\phi(h))\| \geq R; \|\varepsilon B - h\| \leq b\right] \\ & \leq P\left[\sup_{t \in [0, T]} \|\phi_t(z_t^{\phi, \varepsilon}) - \phi_t(x_t^\phi(h))\| \geq R; \sup_{t \in [0, T]} \|z_t^{\phi, \varepsilon} - x_t^\phi(h)\| \leq \eta\right] \\ & \quad + P\left[\sup_{t \in [0, T]} \|z_t^{\phi, \varepsilon} - x_t^\phi(h)\| \geq \eta; \|\varepsilon B - h\| \leq b\right] \\ & \leq P\left[\sup_{\substack{t \in [0, 1] \\ y, z \in K^1 \\ \|y - z\| \leq \eta}} \|\phi_t(y) - \phi_t(z)\| \geq R\right] \\ & \quad + P\left[\sup_{t \in [0, T]} \|z_t^{\phi, \varepsilon} - x_t^\phi(h)\| \geq \eta; \|\varepsilon B - h\| \leq b\right] \\ & = P\left[\sup_{t \in [0, T]} \|z_t^{\phi, \varepsilon} - x_t^\phi(h)\| \geq \eta; \|\varepsilon B - h\| \leq b\right]. \end{aligned}$$

So we are led to show that $\forall K$ compact subsets of \mathbb{R}^n , $\forall T \in]0, 1]$, $\forall a, L > 0$, $\forall R > 0$, there exists $\varepsilon_0, b > 0$ such that $\forall \varepsilon \leq \varepsilon_0$ and $\forall h \in \mathcal{H}^r$, $\|h\|_{\mathcal{H}^r} \leq a$, $x^\phi(h)([0, T]) \subset K$,

$$P\left[\sup_{t \in [0, T]} \|z_t^{\phi, \varepsilon} - x_t^\phi(h)\| \geq R; \|\varepsilon B - h\| \leq b\right] \leq e^{-L/\varepsilon^2}.$$

Case $h = 0$. We will need the following lemma, which states the quasicontinuity in the case $h = 0$.

LEMMA 11. *Given $c: [0, 1] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ in $\mathcal{C}^{0, k}$, we define the processes $\bar{z}_t^{\phi, \varepsilon}$ and \bar{x}_t^ϕ as the solutions (in $\mathcal{C}_x(\mathbb{R}^n)$) to the equations*

$$\begin{aligned}\bar{z}_t^{\phi, \varepsilon} &= x + \int_0^t c(s, \bar{z}_s^{\phi, \varepsilon}) ds + \varepsilon^2 \int_0^t s_0^\phi(s, \bar{z}_s^{\phi, \varepsilon}) ds + \varepsilon \sum_{i=1}^r \int_0^t s_i^\phi(s, \bar{z}_s^{\phi, \varepsilon}) \delta B_s^i \\ \bar{x}_t^\phi &= x + \int_0^t c(s, \bar{x}_s^\phi) ds.\end{aligned}$$

Then, $\forall \phi \in \mathcal{D}^n$, $\forall L, R > 0$, $\forall T \in]0, 1]$, $\forall K$ compact subset of \mathbb{R}^n such that $\bar{x}^\phi([0, T]) \subset K$, $\exists b, \varepsilon_0$ such that $\forall \varepsilon \leq \varepsilon_0$,

$$P\left[\sup_{t \in [0, T]} \|\bar{z}_t^{\phi, \varepsilon} - \bar{x}_t^\phi\| \geq R; \|\varepsilon B\| \leq b\right] \leq e^{-L/\varepsilon^2}.$$

Proof of Lemma 11. Let θ_R^ε be the stopping time,

$$\theta_R^\varepsilon = \inf\{s \text{ such that } \|\bar{z}_s^{\phi, \varepsilon} - \bar{x}_s^\phi\| \geq R\}.$$

When $t \leq \theta_R^\varepsilon \wedge T$, $\bar{z}_t^{\phi, \varepsilon} \in K^R$. Therefore $\tau(\bar{z}^{\phi, \varepsilon}) > \theta_R^\varepsilon \wedge T$ P -a.e. Furthermore,

$$\begin{aligned}P\left[\sup_{t \in [0, T]} \|\bar{z}_t^{\phi, \varepsilon} - \bar{x}_t^\phi\| \geq R; \|\varepsilon B\| \leq b\right] \\ = P\left[\sup_{t \leq \theta_R^\varepsilon \wedge T} \|\bar{z}_t^{\phi, \varepsilon} - \bar{x}_t^\phi\| \geq R; \|\varepsilon B\| \leq b\right].\end{aligned}$$

For all $t \leq \theta_R^\varepsilon \wedge T$,

$$\begin{aligned}\|\bar{z}_t^{\phi, \varepsilon} - \bar{x}_t^\phi\| &\leq \int_0^t \|c(s, \bar{z}_s^{\phi, \varepsilon}) - c(s, \bar{x}_s^\phi)\| ds + \varepsilon^2 \int_0^t \|s_0^\phi(s, \bar{z}_s^{\phi, \varepsilon})\| ds \\ &\quad + \varepsilon \left\| \sum_{i=1}^r \int_0^t s_i^\phi(s, \bar{z}_s^{\phi, \varepsilon}) \delta B_s^i \right\| \\ &\leq \sup_{t, y \in K^R} \left\| \frac{\partial c}{\partial x}(t, y) \right\| \int_0^t \|\bar{z}_s^{\phi, \varepsilon} - \bar{x}_s^\phi\| ds + \varepsilon^2 \sup_{t, y \in K^R} \|s_0^\phi(t, y)\| \\ &\quad + \left\| \varepsilon \sum_{i=1}^r \int_0^t s_i^\phi(s, \bar{z}_s^{\phi, \varepsilon}) \delta B_s^i \right\|.\end{aligned}$$

By Gronwall's lemma, we obtain that for some constant C (depending on K and R),

$$\sup_{t \leq \theta_R^\varepsilon \wedge T} \|\bar{z}_t^{\phi, \varepsilon} - \bar{x}_t^\phi\| \leq C \left(\varepsilon^2 + \sup_{t \leq \theta_R^\varepsilon \wedge T} \left\| \varepsilon \sum_{i=1}^r \int_0^t s_i^\phi(s, \bar{z}_s^{\phi, \varepsilon}) \delta B_s^i \right\| \right).$$

Therefore,

$$\begin{aligned} P\left[\sup_{t \leq \theta_R^\varepsilon \wedge T} \|\bar{z}_t^{\phi, \varepsilon} - \bar{x}_t^\phi\| \geq R; \|\varepsilon B\| \leq b\right] \\ \leq P\left[\varepsilon^2 C \geq \frac{R}{2}\right] + P\left[\sup_{t \leq \theta_R^\varepsilon \wedge T} \left\| \varepsilon \sum_{i=1}^r \int_0^t s_i^\phi(s, \bar{z}_s^{\phi, \varepsilon}) \delta B_s^i \right\| \geq \frac{R}{2C}; \|\varepsilon B\| \leq b\right]. \end{aligned}$$

The first term vanishes when $\varepsilon < (R/2C)^{1/2}$. Thus, we are led to show that $\forall L, R, R' > 0, \exists b, \varepsilon_0$ such that

$$\varepsilon \leq \varepsilon_0 \Rightarrow P\left[\sup_{t \leq \theta_R^\varepsilon \wedge T} \left\| \varepsilon \sum_{i=1}^r \int_0^t s_i^\phi(s, \bar{z}_s^{\phi, \varepsilon}) \delta B_s^i \right\| \geq R'; \|\varepsilon B\| \leq b\right] \leq e^{-L/\varepsilon^2}.$$

For all integer n , we define

- $t_k = k/n$ ($k = 0, \dots, n$).
- $\phi^n(t, y) = \phi(t_k, y) \quad \forall y \in \mathbb{R}^n, \forall t \in [t_k; t_{k+1}[.$
- $\bar{z}_t^{\phi, n, \varepsilon} = \bar{z}_{t_k}^{\phi, \varepsilon} \quad \forall t \in [t_k; t_{k+1}[.$

Let \mathcal{L} be a compact set in \mathbb{R}^n such that $\{\phi(t, y); t \in [0, 1], y \in K^R\} \subset \mathcal{L}$:

$$P\left[\sup_{t \leq \theta_R^\varepsilon \wedge T} \left\| \varepsilon \sum_{i=1}^r \int_0^t s_i^\phi(s, \bar{z}_s^{\phi, \varepsilon}) \delta B_s^i \right\| \geq R'; \|\varepsilon B\| \leq b\right] \leq P_1 + P_2 + P_3$$

with

- $P_1 = P\left[\sup_{t \leq \theta_R^\varepsilon \wedge T} \|\bar{z}_t^{\phi, n, \varepsilon} - \bar{z}_t^{\phi, \varepsilon}\| + \|\phi^n - \phi\|_{\tilde{\mathcal{C}}^{0,1}(\mathcal{L})} > \gamma\right]$
- $P_2 = P\left[\sup_{t \leq \theta_R^\varepsilon \wedge T} \|\bar{z}_t^{\phi, n, \varepsilon} - \bar{z}_t^{\phi, \varepsilon}\| + \|\phi^n - \phi\|_{\tilde{\mathcal{C}}^{0,1}(\mathcal{L})} \leq \gamma; \sup_{t \leq \theta_R^\varepsilon \wedge T} \left\| \varepsilon \sum_{i=1}^r \int_0^t (s_i^\phi(s, \bar{z}_s^{\phi, \varepsilon}) - s_i^{\phi^n}(s, \bar{z}_s^{\phi, n, \varepsilon})) \delta B_s^i \right\| \geq R'/2\right]$
- $P_3 = P\left[\sup_{t \leq \theta_R^\varepsilon \wedge T} \left\| \varepsilon \sum_{i=1}^r \int_0^t s_i^{\phi^n}(s, \bar{z}_s^{\phi, n, \varepsilon}) \delta B_s^i \right\| \geq R'/2; \|\varepsilon B\| \leq b\right].$

Treatment of P_2 . $\varepsilon \sum_{i=1}^r \int_0^{t \wedge \theta_R^\varepsilon \wedge T} (s_i^\phi(s, \bar{z}_s^{\phi, \varepsilon}) - s_i^{\phi^n}(s, \bar{z}_s^{\phi, n, \varepsilon})) \delta B_s^i$ is a martingale with quadratic variation

$$\begin{aligned} \varepsilon^2 \sum_{i=1}^r \int_0^{t \wedge \theta_R^\varepsilon \wedge T} \|s_i^\phi(s, \bar{z}_s^{\phi, \varepsilon}) - s_i^{\phi^n}(s, \bar{z}_s^{\phi, n, \varepsilon})\|^2 ds \\ \leq C\varepsilon^2 \left(\sup_{s \leq \theta_R^\varepsilon \wedge T} \|\bar{z}_s^{\phi, \varepsilon} - \bar{z}_s^{\phi, n, \varepsilon}\| + \|\phi^n - \phi\|_{\tilde{\mathcal{C}}^{0,1}(\mathcal{L})} \right)^2 \\ \leq C\varepsilon^2 \gamma^2. \end{aligned}$$

Therefore, $P_2 \leq C_1 e^{-C_2 R'^2/\varepsilon^2 \gamma^2} \leq \frac{1}{2} e^{-L/\varepsilon^2}$ for γ and ε sufficiently small.

Treatment of P_1 .

$$\begin{aligned} \|\phi^n - \phi\|_{\tilde{\mathcal{C}}^{0,1}(\mathcal{L})} &\leq \sup_{\substack{t, t' \in [0, 1] \\ |t - t'| < 1/n \\ y \in \mathcal{L}}} \sum_{\substack{\alpha \in \mathbb{N}^n \\ |\alpha| \leq 1}} \left(\left\| \frac{\partial^{|\alpha|} \phi_t}{\partial x^\alpha}(y) - \frac{\partial^{|\alpha|} \phi_{t'}}{\partial x^\alpha}(y) \right\| \right. \\ &\quad \left. + \left\| \frac{\partial^{|\alpha|} \phi_t^{-1}}{\partial x^\alpha}(y) - \frac{\partial^{|\alpha|} \phi_{t'}^{-1}}{\partial x^\alpha}(y) \right\| \right). \end{aligned}$$

Thus, the continuity in (t, y) of the functions $\phi_t(y)$, $\phi_t^{-1}(y)$, $(\partial \phi_t / \partial x)(y)$, $((\partial \phi_t / \partial x)(y))^{-1}$, shows that $\|\phi^n - \phi\|_{\tilde{\mathcal{C}}^{0,1}(\mathcal{L})} \rightarrow_{n \rightarrow \infty} 0$. Once γ is fixed, it is possible to choose n_1 such that for $n \geq n_1$, $\|\phi^n - \phi\|_{\tilde{\mathcal{C}}^{0,1}(\mathcal{L})} < \gamma/2$. Thus, for $n \geq n_1$,

$$P_1 \leq P \left[\sup_{t \leq \theta_R^\varepsilon \wedge T} \|\bar{z}_t^{\phi, n, \varepsilon} - \bar{z}_t^{\phi, \varepsilon}\| > \frac{\gamma}{2} \right].$$

Now, $\forall \varepsilon \leq 1$,

$$\begin{aligned} \sup_{t \leq \theta_R^\varepsilon \wedge T} \|\bar{z}_t^{\phi, n, \varepsilon} - \bar{z}_t^{\phi, \varepsilon}\| &= \sup_{\substack{k; t \in [\tau_k, \tau_{k+1}[\\ t \leq \theta_R^\varepsilon \wedge T}} \left\| \int_{\tau_k}^t c(s, \bar{z}_s^{\phi, \varepsilon}) ds + \int_{\tau_k}^t \varepsilon^2 s_0^\phi(s, \bar{z}_s^{\phi, \varepsilon}) ds \right. \\ &\quad \left. + \sum_{i=1}^r \int_{\tau_k}^t s_i^\phi(s, \bar{z}_s^{\phi, \varepsilon}) \varepsilon \delta B_s^i \right\| \\ &\leq \sup_{t, y \in K^R} (\|c(t, y)\| + \|s_0^\phi(t, y)\|) \frac{1}{n} \\ &\quad + \sup_{\substack{k; t \in [\tau_k, \tau_{k+1}[\\ t \leq \theta_R^\varepsilon \wedge T}} \left\| \sum_{i=1}^r \int_{\tau_k}^t s_i^\phi(s, \bar{z}_s^{\phi, \varepsilon}) \varepsilon \delta B_s^i \right\|. \end{aligned}$$

So,

$$\begin{aligned} P \left[\sup_{t \leq \theta_R^\varepsilon \wedge T} \|\bar{z}_t^{\phi, n, \varepsilon} - \bar{z}_t^{\phi, \varepsilon}\| > \frac{\gamma}{2} \right] &\leq P \left[\sup_{t, y \in K^R} (\|c(t, y)\| + \|s_0^\phi(t, y)\|) \frac{1}{n} > \gamma/4 \right] \\ &\quad + \sum_{k=0}^{n-1} P \left[\sup_{\substack{t \in [\tau_k, \tau_{k+1}[\\ t \leq \theta_R^\varepsilon \wedge T}} \left\| \int_{\tau_k}^t \sum_{i=1}^r s_i^\phi(s, \bar{z}_s^{\phi, \varepsilon}) \varepsilon \delta B_s^i \right\| > \frac{\gamma}{4} \right]. \end{aligned}$$

The first term of the summation vanishes for sufficiently large n ($n \geq n_2$). Furthermore, the quadratic variation of $\int_{t_k}^t \varepsilon \sum_{i=1}^r s_i^\phi(s, \bar{z}_s^{\phi, \varepsilon}) \delta B_s^i$ is bounded up by $C\varepsilon^2/n$. Therefore for $n \geq \sup(n_1, n_2)$

$$P_1 \leq C_1 e^{-C_2 \gamma^2 n/\varepsilon^2} \leq \frac{1}{2} e^{-L/\varepsilon^2}$$

for large n and small ε .

Treatment of P_3 . For $s \in [t_k, t_{k+1}]$, $s_i^{\phi^n}(s, \bar{z}_s^{\phi, n, \varepsilon}) = s_i^\phi(t_k, \bar{z}_{t_k}^{\phi, \varepsilon})$. This yields that $\forall t \leq \theta_R^\varepsilon \wedge T$,

$$\begin{aligned} \left\| \int_0^t \varepsilon \sum_{i=1}^r s_i^{\phi^n}(s, \bar{z}_s^{\phi, n, \varepsilon}) \delta B_s^i \right\| &= \left\| \varepsilon \sum_{k=0}^{n-1} \sum_{i=1}^r s_i^\phi(t_k, \bar{z}_{t_k}^{\phi, \varepsilon}) (B_{t \wedge t_{k+1}}^i - B_{t \wedge t_k}^i) \right\| \\ &\leq \sup_{i, t, y \in K^R} \|s_i^\phi(t, y)\| \sum_{i, k} \|\varepsilon B_{t_{k+1}}^i - \varepsilon B_{t_k}^i\| \\ &\leq Cbn. \end{aligned}$$

Therefore, $P_3 \leq P(Cbn \geq R'/2) = 0$ for b sufficiently small. And the proof of Lemma 11 is complete. ■

We return now to the proof of Proposition 10, that is, to the case $h \neq 0$.

Case $h \neq 0$. Given $h \in \mathcal{H}^r$, $\|h\|_{\mathcal{H}^r} \leq a$, $x^\phi(h)([0, T]) \subset K$, we define

- the process $W_t^\varepsilon = B_t - h_t/\varepsilon$
- the probability P^ε on W by

$$\frac{dP^\varepsilon}{dP} \Big|_{\sigma(B_s, s \leq t)} = \exp \left[\frac{1}{\varepsilon} \int_0^t (\dot{h}_s, \delta B_s) - \frac{1}{2\varepsilon^2} \int_0^t \|\dot{h}_s\|^2 ds \right]$$

- $A \equiv \{\sup_{t \leq T} \|z_t^{\phi, \varepsilon}(\varepsilon B) - x_t^\phi(h)\| \geq R\} \cap \{\|\varepsilon B - h\| \leq b\}$.
- $B \equiv \{\int_0^1 (\dot{h}_s, \delta B_s) < -\lambda/\varepsilon\}$.

Then, $P(A) \leq P(B) + E^\varepsilon[(dP/dP^\varepsilon) \mathbb{1}_A \mathbb{1}_B]$. Since $\int_0^1 (\dot{h}_s, \delta B_s)$ is gaussian with mean 0 and variance $\|h\|_{\mathcal{H}^r}^2$,

$$P(B) \leq \frac{\varepsilon \|h\|_{\mathcal{H}^r}}{\lambda \sqrt{2\pi}} \exp \left(-\frac{\lambda^2}{2\varepsilon^2 \|h\|_{\mathcal{H}^r}^2} \right) \leq \frac{1}{2} \exp \left(-\frac{L}{\varepsilon^2} \right)$$

for large λ and small ε :

$$\begin{aligned} E^\varepsilon \left[\frac{dP}{dP^\varepsilon} \mathbb{1}_A \mathbb{1}_B \right] &\leq \exp \left(\frac{\lambda}{\varepsilon^2} \right) \exp \left(\frac{a}{2\varepsilon^2} \right) \\ &\times P^\varepsilon \left[\sup_{t \leq T} \|z_t^{\phi, \varepsilon} - x_t^\phi(h)\| \geq R; \|\varepsilon W^\varepsilon\| \leq b \right] \end{aligned}$$

with

$$dz_t^{\phi, \varepsilon} = \varepsilon^2 s_0^\phi(t, z_t^{\phi, \varepsilon}) dt + \sum_{i=1}^r s_i^\phi(t, z_t^{\phi, \varepsilon}) \dot{h}_t^i dt + \varepsilon \sum_{i=1}^r s_i^\phi(t, z_t^{\phi, \varepsilon}) \delta W_t^{e, i}$$

$$dx_t^\phi(h) = \sum_{i=1}^r s_i^\phi(t, x_t^\phi(h)) \dot{h}_t^i dt.$$

Under P^ε , W^ε is a standard Brownian motion. Applying Lemma 11 with $c(t, y) = \sum_{i=1}^r s_i^\phi(t, y) \dot{h}_t^i$, we obtain an exponential bound for $E^\varepsilon[(dP/dP^\varepsilon) \mathbb{1}_A \mathbb{1}_{B^\varepsilon}]$, and the proof of Proposition 10 follows. ■

From Proposition 10, we derive as usually a large deviations principle for the family $(T^\varepsilon(\phi, d\omega))_\varepsilon$.

PROPOSITION 12. *For ϕ fixed in \mathcal{D}^n , we define the rate function L_ϕ on $\mathcal{E}_x(\mathbb{R}^n)$ by*

$$\forall z \in \mathcal{E}_x(\mathbb{R}^n), \quad L_\phi(z) = \inf \left\{ \frac{1}{2} \|h\|_{\mathcal{H}^r}^2, h \in \mathcal{H}^r \text{ such that } z = \phi(x^\phi(h)) \right\},$$

where $x^\phi(h)$ is solution to (5.26). Then we have

- L_ϕ is a “good” rate function.
- $\forall A \subset \mathcal{E}_x(\mathbb{R}^n)$,

$$-L_\phi(\bar{A}) \leq \liminf_{\varepsilon \rightarrow 0} \varepsilon^2 \log T^\varepsilon(\phi, A) \leq \varepsilon^2 \limsup_{\varepsilon \rightarrow 0} \log T^\varepsilon(\phi, A) \leq -L_\phi(\bar{A}).$$

Moreover, if $\tau(z^{\phi, \varepsilon}) > 1$, P -a.e., and if $\forall h \in \mathcal{H}^r$, $\tau(x^\phi(h)) > 1$, the result remains true when the topological space $\mathcal{E}_x(\mathbb{R}^n)$ is replaced by the topological space E_x .

Proof of Proposition 12. It results as usual from the quasicontinuity and the continuity of the map $h \in \mathcal{H}_a^r \mapsto \phi(x^\phi(h)) \in \mathcal{E}_x(\mathbb{R}^n)$. ■

We derive from Proposition 12 a large deviations principle for a particular version of the conditional law of X^ε relative to \tilde{B} . As in Section 4, we define:

$$\begin{aligned} \tilde{F}: \mathcal{H}^l &\rightarrow \mathcal{D}^n \\ \tilde{h} &\mapsto \text{flow of diffeomorphisms associated to the} \\ &\quad \text{ordinary differential equation} \end{aligned} \tag{5.27}$$

$$dx_t = \tilde{\sigma}_0(x_t) dt + \sum_{j=1}^l \tilde{\sigma}_j(x_t) \dot{\tilde{h}}_t^j.$$

According to the results of Bismut [4], \tilde{F} can be extended in a measurable way to \tilde{W} . This extension will still be denoted by \tilde{F} . We define then for \tilde{P} -almost all $\tilde{\omega}$:

- the probability $N^e(\tilde{\omega}, d\omega) \equiv T^e(\tilde{F}(\tilde{\omega}), d\omega)$;
- $\forall h \in \mathcal{H}^l$, $\tilde{\xi}(h)(\tilde{\omega}) \equiv \tilde{F}(\tilde{\omega})(x^{\tilde{F}(\tilde{\omega})}(h))$;
- the rate function $l_{\tilde{\omega}} = L_{\tilde{F}(\tilde{\omega})}$.

We have then the following result.

PROPOSITION 13. 1. $\forall h \in \mathcal{H}^r$, $\tilde{\xi}_t(h)$ is a solution to

$$\begin{aligned} d\tilde{\xi}_t(h) = x + \sum_{i=1}^r \int_0^t \sigma_i(\tilde{\xi}_s(h)) \dot{h}_s^i ds + \int_0^t \tilde{\sigma}_0(\tilde{\xi}_s(h)) ds \\ + \sum_{j=1}^l \tilde{\sigma}_j(\tilde{\xi}_s(h)) d\tilde{B}_s^j \end{aligned}$$

and $l_{\tilde{\omega}}(z) = \inf\{1/2 \|h\|_{\mathcal{H}^r}^2, h \in \mathcal{H}^l \text{ such that, } z = \tilde{\xi}(h)\}$. \tilde{P} -a.e., $l_{\tilde{\omega}}$ is a good rate function.

2. $N^e(\tilde{\omega}, d\omega)$ is a probability measure on E_x , which is a version of the conditional law of X^e relative to \tilde{B} . \tilde{P} -a.e. $\forall A \subset E_x$:

$$-l_{\tilde{\omega}}(\mathring{A}) \leq \liminf_{\varepsilon \rightarrow 0} \varepsilon^2 \log N^e(\tilde{\omega}, A) \leq \limsup_{\varepsilon \rightarrow 0} \varepsilon^2 \log N^e(\tilde{\omega}, A) \leq -l_{\tilde{\omega}}(\bar{A}).$$

Proof of Proposition 13. Point 1 is a consequence of Theorem 2 and of the definition of $l_{\tilde{\omega}}$. We derive also from Theorem 2 that $\mathbb{P}(\tau(z^{\tilde{F}(\tilde{\omega}), e}(\omega)) > 1) = 1$ and \mathbb{P} -a.e. $X^e(\omega, \tilde{\omega}) = \tilde{D}(\tilde{F}(\tilde{\omega}))(\omega)$. Therefore, $N^e(\tilde{\omega}, d\omega)$ is version of the law of X^e relative to \tilde{B} . Large deviation inequalities are the same as in Proposition 12. ■

5.2. Application to Nonlinear Filtering

We consider now the original problem, that is, large deviations for the conditional law of \mathcal{X}^e relative to \mathcal{Y}^e , where $(\mathcal{X}^e, \mathcal{Y}^e)$ is a solution to (5.23). To begin with, we introduce some notations:

- An element y of \mathbb{R}^{n+1} will be decomposed into (y_1, y_2) , where $y_1 \in \mathbb{R}^n$ and $y_2 \in \mathbb{R}$.
- We define the following vector fields in \mathbb{R}^{n+1} : $\forall y \in \mathbb{R}^{n+1}$,

$$\begin{aligned} & \text{--- } \forall i \in \{1, \dots, r\}, \lambda_i(y) \equiv \begin{pmatrix} \sigma_i(y_1) \\ 0 \end{pmatrix}; \\ & \text{--- } \forall j \in \{1, \dots, l\}, \tilde{\lambda}_j(y) \equiv \begin{pmatrix} \tilde{\sigma}_j(y_1) \\ \Gamma_j(y_1) \end{pmatrix}; \end{aligned}$$

$$\begin{aligned}
- \tilde{\lambda}_0(y) &\equiv \left(\tilde{\sigma}_0(y_1) \right. \\
&\quad \left. - \frac{1}{2} \sum_{j=1}^l \left(\frac{\partial \Gamma_j}{\partial x}(y_1), \tilde{\sigma}_j(y_1) \right) \right) \\
- \lambda_0^*(y) &\equiv \frac{1}{2} \sum_{i=1}^r \left(\frac{\partial \lambda_i}{\partial x}(y), \lambda_i(y) \right).
\end{aligned}$$

- A flow ϕ of diffeomorphisms in \mathbb{R}^{n+1} transports these vector fields into

$$\begin{aligned}
- \hat{s}_i^\phi(t, y) &= \phi_t^{-1} * \lambda_i(y), \quad i = 1, \dots, r, \\
- \hat{s}_0^\phi(t, y) &= \phi_t^{-1} * \lambda_0^*(y) - \frac{1}{2} \sum_{i=1}^r \left(\frac{\partial \phi_t}{\partial x}(y) \right)^{-1} \\
&\quad \times \left[\frac{\partial^2 \phi_t}{\partial x^2}(y) (\hat{s}_i^\phi(t, y), \hat{s}_i^\phi(t, y)) \right].
\end{aligned}$$

- For all $\phi \in \mathcal{D}^{n+1}$, we will denote by $\hat{z}^{\phi, \varepsilon}$ the process in $\mathcal{E}_x(\mathbb{R}^{n+1})$ solution to

$$\begin{aligned}
d\hat{z}_t^{\phi, \varepsilon} &= \varepsilon^2 \hat{s}_0^\phi(t, \hat{z}_t^{\phi, \varepsilon}) dt + \varepsilon \sum_{i=1}^r \hat{s}_i^\phi(t, \hat{z}_t^{\phi, \varepsilon}) \delta B_t^i \\
\hat{z}_0^{\phi, \varepsilon} &= (x, 0).
\end{aligned}$$

- We map then \mathcal{D}^{n+1} to $L^0(W, \mathcal{E}_x(\mathbb{R}^{n+1}))$ by

$$\begin{aligned}
\hat{D}: \mathcal{D}^{n+1} &\rightarrow L^0(W, \mathcal{E}_x(\mathbb{R}^{n+1})) \\
\phi &\mapsto (\omega \mapsto \phi(\hat{z}^{\phi, \varepsilon}(\omega))).
\end{aligned}$$

- The processes $\mathcal{C}_{(1)}^{\varepsilon, \phi} \in \mathcal{E}_x(\mathbb{R}^n)$ and $\mathcal{C}_{(2)}^{\varepsilon, \phi} \in \mathcal{E}_x(\mathbb{R}^1)$ are defined by

$$\hat{D}(\phi) \equiv (\mathcal{C}_{(1)}^{\varepsilon, \phi}, \mathcal{C}_{(2)}^{\varepsilon, \phi}).$$

- Finally, we will denote by F the “flow” map

$$\hat{F}: \tilde{W} \rightarrow \mathcal{D}^{n+1}$$

$\tilde{\omega} \mapsto$ stochastic flow associated to the stochastic differential

$$\text{equation } d\hat{\xi}_t = \tilde{\lambda}_0(\hat{\xi}_t) dt + \sum_{j=1}^l \tilde{\lambda}_j(\hat{\xi}_t) d\tilde{B}_t^j.$$

We then have then the following result.

THEOREM 14. 1. $\forall A \in \mathcal{B}(\mathcal{E}_x(\mathbb{R}^n))$, we define

$$M^e(\tilde{\omega}, A) \equiv \frac{E[\mathbb{1}_A(\mathcal{C}_{(1)}^{e, \tilde{F}(\tilde{\omega})}) \exp(\mathcal{C}_{(2), 1}^{e, \tilde{F}(\tilde{\omega})} - \frac{1}{2} \int_0^1 \|\Gamma(\mathcal{C}_{(1), s}^{e, \tilde{F}(\tilde{\omega})})\|^2 ds)]}{E[\exp(\mathcal{C}_{(2), 1}^{e, \tilde{F}(\tilde{\omega})} - \frac{1}{2} \int_0^1 \|\Gamma(\mathcal{C}_{(1), s}^{e, \tilde{F}(\tilde{\omega})})\|^2 ds)]}$$

$M^e(\tilde{\omega}, \bullet)$ is a probability measure on E_x , which is a version of the conditional law of \mathcal{X}^e relative to \mathcal{Y}^e .

2. \tilde{P} -a.e., $\forall A \subset E_x$,

$$-l_{\tilde{\omega}}(\mathring{A}) \leq \liminf_{\varepsilon \rightarrow 0} \varepsilon^2 \log M^e(\tilde{\omega}, A) \leq \limsup_{\varepsilon \rightarrow 0} \varepsilon^2 \log M^e(\tilde{\omega}, A) \leq -l_{\tilde{\omega}}(\bar{A})$$

with $l_{\tilde{\omega}}$ as in Proposition 13.

Proof of Proposition 14. $\forall \varepsilon > 0$, we define a new probability \hat{P}^e on $W \otimes \tilde{W}$ by

$$\begin{aligned} \frac{d\hat{P}^e}{d\mathbb{P}} \Big|_{\sigma(B_s, \tilde{B}_s, s \leq t)} &= \exp \left[- \int_0^t \Gamma(\mathcal{X}_s^e) \delta \tilde{B}_s - \frac{1}{2} \int_0^t \|\Gamma(\mathcal{X}_s^e)\|^2 ds \right] \\ &= \exp \left[- \int_0^t \Gamma(\mathcal{X}_s^e) \delta \mathcal{Y}_s^e + \frac{1}{2} \int_0^t \|\Gamma(\mathcal{X}_s^e)\|^2 ds \right]. \end{aligned}$$

Under \hat{P}^e , \mathcal{Y}^e has the law of a Brownian motion independent of B . Therefore, $(\mathcal{X}^e, \mathcal{Y}^e)$ has the same law under \hat{P}^e as (X^e, \tilde{B}) under \mathbb{P} . Let G and H be two measurable functions respectively defined on E_x and E_0 :

$$\begin{aligned} \mathbb{E}(G(\mathcal{X}^e) H(\mathcal{Y}^e)) &= \hat{E}^e \left(G(\mathcal{X}^e) H(\mathcal{Y}^e) \frac{d\mathbb{P}}{d\hat{P}^e} \right) \\ &= \mathbb{E} \left[G(X^e) H(\tilde{B}) \exp \left(\int_0^1 \Gamma(X_s^e) \delta \tilde{B}_s - \frac{1}{2} \int_0^1 \|\Gamma(X_s^e)\|^2 ds \right) \right]. \end{aligned}$$

We deduce then that \tilde{P} -a.e.

$$\mathbb{E}(G(\mathcal{X}^e)|_{\mathcal{Y}^e = \tilde{\omega}}) = \frac{\mathbb{E}[G(X^e) \exp(\int_0^1 \Gamma(X_s^e) \delta \tilde{B}_s - \frac{1}{2} \int_0^1 \|\Gamma(X_s^e)\|^2 ds)|_{\tilde{B} = \tilde{\omega}}]}{\mathbb{E}[\exp(\int_0^1 \Gamma(X_s^e) \delta \tilde{B}_s - \frac{1}{2} \int_0^1 \|\Gamma(X_s^e)\|^2 ds)|_{\tilde{B} = \tilde{\omega}}]}.$$

The process $\mathcal{S}_t^e = (X_t^e, \int_0^t \Gamma(X_s^e) \delta \tilde{B}_s)$ satisfies the stochastic differential equation

$$d\mathcal{S}_t^e = \varepsilon \sum_{i=1}^r \lambda_i(\mathcal{S}_t^e) dB_t^i + \sum_{j=1}^l \tilde{\lambda}_j(\mathcal{S}_t^e) d\tilde{B}_t^j + \tilde{\lambda}_0(\mathcal{S}_t^e) dt$$

$$\mathcal{S}_0^e = (x, 0).$$

Thus, the decomposition of stochastic differential equations yields

- \mathbb{P} -a.e., $\tau(\hat{z}^{\hat{F}(\tilde{\omega})}(\omega)) > 1$.
- \mathbb{P} -a.e., $\mathcal{S}^e = \hat{D}(\hat{F}(\tilde{\omega}))(\omega)$.

By independence of B and \tilde{B} , it results that \tilde{P} -a.e., $\forall A \in \mathcal{B}(E_x)$,

$$\begin{aligned} \mathbb{E}(\mathbb{1}_A(\mathcal{X}^e)|_{\mathcal{Y}^e = \tilde{\omega}}) &= \frac{E[\mathbb{1}_A(\mathcal{C}_{(1)}^{e, \hat{F}(\tilde{\omega})}) \exp(\mathcal{C}_{(2), 1}^{e, \hat{F}(\tilde{\omega})} - \frac{1}{2} \int_0^1 \|\Gamma(\mathcal{C}_{(1), s}^{e, \hat{F}(\tilde{\omega})})\|^2 ds)]}{E[\exp(\mathcal{C}_{(2), 1}^{e, \hat{F}(\tilde{\omega})} - \frac{1}{2} \int_0^1 \|\Gamma(\mathcal{C}_{(1), s}^{e, \hat{F}(\tilde{\omega})})\|^2 ds)]} \\ &= M^e(\tilde{\omega}, A) \end{aligned}$$

This proves 1.

Proof of the lower bound. Let A be an open set of E_x . When $l_{\tilde{\omega}}(A) = +\infty$, the lower bound is trivial, let us suppose that $l_{\tilde{\omega}}(A) < +\infty$. Let $h \in \mathcal{H}^r$ be such that

$$z \equiv \tilde{F}(\tilde{\omega})(x^{\tilde{F}(\tilde{\omega})}(h)) \in A \quad (\text{where } x^\phi(h) \text{ is defined by (5.26)})$$

$$\frac{1}{2} \|h\|_{\mathcal{H}^r}^2 = l_{\tilde{\omega}}(z) = l_{\tilde{\omega}}(A).$$

let us introduce

- the random variable,

$$U^e \equiv \exp \left(\mathcal{C}_{(2), 1}^{e, \hat{F}(\tilde{\omega})} - \frac{1}{2} \int_0^1 \|\Gamma(\mathcal{C}_{(1), s}^{e, \hat{F}(\tilde{\omega})})\|^2 ds \right) \quad (5.28)$$

- the process $W_t^e = B_t - h_t/\varepsilon$
- the probability measure P^e on W ,

$$\frac{dP^e}{dP} \Big|_{\sigma(B_s, s \leq t)} = \exp \left[\frac{1}{\varepsilon} \int_0^t (\dot{h}_s, \delta W_s^e) + \frac{1}{2\varepsilon^2} \int_0^t \|\dot{h}_s\|^2 ds \right].$$

Then, $M^e(\tilde{\omega}, A) = E(\mathbb{1}_A(\mathcal{C}_{(1)}^{e, \hat{F}(\tilde{\omega})}) U^e) / E(U^e)$. By Girsanov's transformation,

$$\begin{aligned} E(\mathbb{1}_A(\mathcal{C}_{(1)}^{e, \hat{F}(\tilde{\omega})}) U^e) &= E^e \left(\mathbb{1}_A(\mathcal{C}_{(1)}^{e, \hat{F}(\tilde{\omega})}) U^e \frac{dP}{dP^e} \right) \\ &= E \left(\mathbb{1}_A(\tilde{\mathcal{C}}_{(1)}^{e, \hat{F}(\tilde{\omega})}) \tilde{U}^e \exp \left(-\frac{1}{\varepsilon} \int_0^t (\dot{h}_s, \delta B_s) - \frac{1}{2\varepsilon^2} \int_0^t \|\dot{h}_s\|^2 ds \right) \right), \end{aligned}$$

where

$$-\tilde{\mathcal{C}}^{\varepsilon, \phi}(\omega) = \hat{D}(\phi) \left(\omega + \frac{h}{\varepsilon} \right)$$

$$-\tilde{U}^\varepsilon = \exp \left(\tilde{\mathcal{C}}^{\varepsilon, \hat{F}(\tilde{\omega})}_{(2), 1} - \frac{1}{2} \int_0^1 \left\| \Gamma(\tilde{\mathcal{C}}^{\varepsilon, \hat{F}(\tilde{\omega})}_{(1), s}) \right\|^2 ds \right).$$

Therefore,

$$E(\mathbb{1}_A(\mathcal{C}^{\varepsilon, \hat{F}(\tilde{\omega})}_{(1)}) U^\varepsilon) \geq E(\mathbb{1}_A(\tilde{\mathcal{C}}^{\varepsilon, \hat{F}(\tilde{\omega})}_{(1)}) \tilde{U}^\varepsilon \mathbb{1}_{\int_0^1 \dot{h}_s \delta B_s \leq K})$$

$$\times \exp \left(-\frac{I_{\tilde{\omega}}(A)}{\varepsilon^2} \right) \exp \left(-\frac{K}{\varepsilon} \right).$$

Now, using well-known results about the asymptotic behavior of perturbed dynamic systems, P -a.e., $\tilde{\mathcal{C}}^{\varepsilon, \phi} \xrightarrow[\varepsilon \rightarrow 0]{} \phi(z^\phi)$ in $\mathcal{E}_x(\mathbb{R}^n)$, where z^ϕ is a solution to the ordinary differential equation,

$$z_t^\phi = \begin{pmatrix} x \\ 0 \end{pmatrix} + \sum_{i=1}^r \int_0^t \hat{s}_i^\phi(s, z_s^\phi) \dot{h}_s^i ds.$$

Thus, P -a.e., $\tilde{U}^\varepsilon \xrightarrow[\varepsilon \rightarrow 0]{} \tilde{U}^0$, with $\tilde{U}^0 > 0$ and independent of B .

Moreover, it is easily seen from definition of \hat{F} that

$$\forall t \in [0, 1], \quad \forall (x_1, x_2) \in \mathbb{R}^{n+1}, \quad \hat{F}_t(\tilde{\omega})(x_1, x_2) = \begin{pmatrix} \tilde{F}_t(\tilde{\omega})(x_1) \\ x_2 + G_t(x_1, \tilde{\omega}) \end{pmatrix}$$

for some function G and \tilde{F} defined by (5.27). This yields that

- $\forall i \in \{1, \dots, r\}$,

$$\hat{s}_i^{\hat{F}(\tilde{\omega})}(t, x_1, x_2) = \begin{pmatrix} s_i^{\tilde{F}(\tilde{\omega})}(t, x_1) \\ -\frac{\partial G_t}{\partial x_1} s_i^{\tilde{F}(\tilde{\omega})}(t, x_1) \end{pmatrix}.$$

- $\forall t \in [0, 1], z_{1, t}^{\hat{F}(\tilde{\omega})} = x_t^{\tilde{F}(\tilde{\omega})}(h)$.

Therefore, $\tilde{\mathcal{C}}^{\varepsilon, \hat{F}(\tilde{\omega})}_{(1)} \rightarrow \tilde{F}(\tilde{\omega})(x_t^{\tilde{F}(\tilde{\omega})}(h)) = z$. By Fatou's lemma, we deduce then that

$$\liminf_{\varepsilon \rightarrow 0} E(\mathbb{1}_A(\tilde{\mathcal{C}}^{\varepsilon, \hat{F}(\tilde{\omega})}_{(1)}) \tilde{U}^\varepsilon \mathbb{1}_{\int_0^1 \dot{h}_s \delta B_s \leq K}) \geq E(\mathbb{1}_A(z) \tilde{U}^0 \mathbb{1}_{\int_0^1 \dot{h}_s \delta B_s \leq K})$$

$$= \tilde{U}^0 P \left(\int_0^1 \dot{h}_s \delta B_s \leq K \right)$$

$$\geq \tilde{U}^0 / 2$$

for sufficiently large K . It results then that

$$\liminf_{\varepsilon \rightarrow 0} \varepsilon^2 \log E(\mathbb{1}_A(\mathcal{C}_{(1)}^{\varepsilon, \hat{F}(\tilde{\omega})}) U^\varepsilon) \geq -l_{\tilde{\omega}}(A).$$

Now, $\varepsilon^2 \log M^\varepsilon(\tilde{\omega}, A) = \varepsilon^2 \log E(\mathbb{1}_A(\mathcal{C}_{(1)}^{\varepsilon, \hat{F}(\tilde{\omega})}) U^\varepsilon) - \varepsilon^2 \log E(U^\varepsilon)$. The lower bound follows then from the fact that \tilde{P} -a.e. $\forall p \geq 1$, $\sup_{\varepsilon \leq 1} E((U^\varepsilon)^p) < \infty$, which is proved in Appendix 4.

Proof of the upper bound. Let A be a closed subset of E_x . Using Hölder's inequality, $\forall p > 1$,

$$\begin{aligned} \varepsilon^2 \log M^\varepsilon(\tilde{\omega}, A) &\leq \frac{1}{p} \varepsilon^2 \log E(\mathbb{1}_A(\mathcal{C}_{(1)}^{\varepsilon, \hat{F}(\tilde{\omega})})) + \frac{1}{q} \varepsilon^2 \log E((U^\varepsilon)^q) \\ &\quad - \varepsilon^2 \log E(U^\varepsilon) \end{aligned}$$

(with $1/p + 1/q = 1$).

The particular form of $\hat{F}(\tilde{\omega})$ implies that $\mathcal{C}_{(1)}^{\varepsilon, \hat{F}(\tilde{\omega})} = \tilde{F}(\tilde{\omega})(z^{\tilde{F}(\tilde{\omega}), \varepsilon})$, where $z^{\phi, \varepsilon}$ is defined by (5.25). Since \tilde{P} -a.e. $\forall p \geq 1$, $\sup_{\varepsilon \leq 1} E((U^\varepsilon)^p) < \infty$, we have

$$\begin{aligned} \limsup_{\varepsilon \rightarrow 0} \varepsilon^2 \log M^\varepsilon(\tilde{\omega}, A) &\leq \frac{1}{p} \limsup_{\varepsilon \rightarrow 0} \varepsilon^2 \log P(\tilde{F}(\tilde{\omega})(z^{\tilde{F}(\tilde{\omega}), \varepsilon}) \in A) \\ &\leq -\frac{1}{p} l_{\tilde{\omega}}(A) \quad \text{by Proposition 13.} \end{aligned}$$

Letting p decrease to 1, we obtain the result. ■

APPENDIX 1

Let F and G be two functions in $\mathcal{C}_b^m(\mathbb{R}^n, \mathbb{R})$. Let Ω be a bounded domain in \mathbb{R}^n . Let $\phi, \psi: \Omega \rightarrow \mathbb{R}^n$ be in $W^{m, 2}(\Omega)$. We assume that there is a constant $R > 0$ such that

$$\|\phi\|_{W^{m, 2}(\Omega)} \leq R$$

$$\|\psi\|_{W^{m, 2}(\Omega)} \leq R.$$

Then, if $m \geq n + 1$,

1. $F \circ \phi \in W^{m, 2}(\Omega)$, and there is a constant $C > 0$ such that

$$\sup_{\phi, \|\phi\|_{W^{m, 2}(\Omega)} \leq R} \|F \circ \phi\|_{W^{m, 2}(\Omega)} \leq C.$$

2. When $F \in \mathcal{C}_b^{m+1}$, $\|F \circ \phi - F \circ \psi\|_{W^{m, 2}(\Omega)} \leq C \|\phi - \psi\|_{W^{m, 2}(\Omega)}$.
3. $\|F \circ \phi - G \circ \phi\|_{W^{m, 2}(\Omega)} \leq C \|F - G\|_{\mathcal{C}_b^m}$.

Proof. $\forall \alpha \in \mathbb{N}^n$, $|\alpha| \leq m$,

$$D^\alpha(F \circ \phi)(x) = \sum_{\substack{\beta \in \mathbb{N}^n \\ |\beta| \leq |\alpha|}} D^\beta F(\phi(x)) P_\beta(x),$$

where P_β is a polynomial in the derivatives of ϕ , whose each monomials $\prod_{i=1}^l D^{\gamma_i} \phi(x)$ satisfies $\sum_{i=1}^l \gamma_i = \alpha$. Therefore,

$$\int_{\Omega} \|D^\alpha(F \circ \phi)(x)\|^2 dx \leq C \sum_{\substack{\beta \in \mathbb{N}^n \\ |\beta| \leq |\alpha|}} \sup_x \|D^\beta F(x)\|^2 \int_{\Omega} \|P_\beta(x)\|^2 dx.$$

Thus, we have to control terms such as $\int_{\Omega} \prod_{i=1}^l \|D^{\gamma_i} \phi(x)\|^2 dx$ with $\sum_i \gamma_i = \alpha$. When $|\gamma_i| \leq m - [n/2] - 1$, the Sobolev embedding theorem ensures that

$$\sup_x \|D^{\gamma_i} \phi(x)\| \leq C \|\phi\|_{W^{m,2}(\Omega)} \leq CR.$$

But there is at most one term with $|\gamma_i| > m - [n/2] - 1$. Indeed, if $|\gamma_{i_0}| \geq m - [n/2]$,

$$\sum_{i \neq i_0} |\gamma_i| = |\alpha| - |\gamma_{i_0}| \leq \left[\frac{n}{2} \right] \leq m - \left[\frac{n}{2} \right] - 1,$$

since $m \geq n + 1$. Therefore,

$$\begin{aligned} \int_{\Omega} \prod_{i=1}^l \|D^{\gamma_i} \phi(x)\|^2 dx &\leq \prod_{i \neq i_0} \sup_x \|D^{\gamma_i} \phi(x)\|^2 \int_{\Omega} \|D^{\gamma_{i_0}} \phi(x)\|^2 dx \\ &\leq CR^{2l-2} \|\phi\|_{W^{m,2}(\Omega)}^2 \leq CR^{2l}. \end{aligned}$$

This proves 1, 2 and 3 are obtained in a similar way:

$$\begin{aligned} &\int_{\Omega} \|D^\alpha(F \circ \phi)(x) - D^\alpha(F \circ \psi)(x)\|^2 dx \\ &\leq C \sum_{\substack{\beta \in \mathbb{N}^n \\ |\beta| \leq |\alpha|}} \sup_x \|D^\beta F(x)\|^2 \int_{\Omega} \|P_\beta(x) - Q_\beta(x)\|^2 dx \\ &+ C \sum_{\substack{\beta \in \mathbb{N}^n \\ |\beta| \leq |\alpha|}} \sup_x \|D^\beta F(\phi(x)) - D^\beta F(\psi(x))\|^2 \int_{\Omega} \|Q_\beta(x)\|^2 dx. \end{aligned}$$

Q_β is the same polynomial as P_β , for the derivatives of ψ . Since F is in \mathcal{C}_b^{m+1} ,

$$\begin{aligned} \sup_x \|D^\beta F(\phi(x)) - D^\beta F(\psi(x))\|^2 &\leq C \sup_x \|\phi(x) - \psi(x)\|^2 \\ &\leq C \|\phi - \psi\|_{W^{m,2}(\Omega)}^2. \end{aligned}$$

We obtain an upper bound (depending on R) to $\int_\Omega \|Q_\beta(x)\|^2 dx$ as previously. To control $\int_\Omega \|P_\beta(x) - Q_\beta(x)\|^2 dx$, we have to bound up terms such as

$$\begin{aligned} \int_\Omega \left\| \prod_{i=1}^l D^{\gamma_i} \phi(x) - \prod_{i=1}^l D^{\gamma_i} \psi(x) \right\|^2 dx \\ \leq C \sum_{i=1}^l \int_\Omega \left\| \prod_{j < i} D^{\gamma_j} \phi(x) (D^{\gamma_i} \phi(x) - D^{\gamma_i} \psi(x)) \prod_{j > i} D^{\gamma_j} \psi(x) \right\|^2 dx. \end{aligned}$$

The same argument as for 1 implies 2. Finally,

$$\begin{aligned} \int_\Omega \|D^\alpha (F \circ \phi)(x) - D^\alpha (G \circ \phi)(x)\|^2 dx \\ \leq C \sum_{\substack{\beta \in \mathbb{N}^n \\ |\beta| \leq |\alpha|}} \int_\Omega \|(D^\beta F(\phi(x)) - D^\beta G(\phi(x))) P_\beta(x)\|^2 dx \\ \leq C \|F - G\|_{\mathcal{C}_b^m}^2 \sum_{\substack{\beta \in \mathbb{N}^n \\ |\beta| \leq |\alpha|}} \int_\Omega \|P_\beta(x)\|^2 dx. \quad \blacksquare \end{aligned}$$

APPENDIX 2

Let H be a Hilbert space, and let $(e_n)_n$ be an orthonormal basis of H . Let $(\Omega, \mathcal{A}, (\mathcal{F}_t)_{t \in [0,1]}, P)$ be a probability space. Let M_t be a \mathcal{F}_t -adapted process with value in H .

We assume that for all n , $M_t^n \equiv (M_t, e_n)$ is a real \mathcal{F}_t -martingale and that there exists constants $K, L > 0$ such that $\forall N \in \mathbb{N}^*, \forall t \in [0, 1]$,

$$\begin{aligned} \sum_{k=1}^N \langle M^k \rangle_t &\leq K \quad P\text{-a.e.} \\ \sum_{k,l=1}^N \int_0^t M_s^k M_s^l d\langle M^k, M^l \rangle_s &\leq L \quad P\text{-a.e.} \end{aligned}$$

Then, $\forall R$ such that $R^2 \geq K$,

$$P\left[\sup_{t \in [0, 1]} \|M_t\|_H \geq R\right] \leq \exp\left[-\frac{(R^2 - K)^2}{8L}\right].$$

Proof. Let $V_N \equiv \text{Span}(e_1, \dots, e_N)$, and let P_N denote the orthogonal projection on V_N :

$$\begin{aligned} P\left[\sup_{t \in [0, 1]} \|P_N M_t\|_H \geq R\right] &= P\left[\sup_{t \in [0, 1]} \|P_N M_t\|_H^2 \geq R^2\right] \\ &\leq P\left[\sup_{t \in [0, 1]} \left(\sum_{k=1}^N (M_t^k)^2 - \langle M^k \rangle_t\right) \geq R^2 - K\right]; \end{aligned}$$

$\sum_{k=1}^N (M_t^k)^2 - \langle M^k \rangle_t$ is a real martingale, whose quadratique variation is

$$4 \sum_{k, l=1}^N \int_0^t M_s^k M_s^l d\langle M^k, M^l \rangle_s \leq 4L.$$

As a result:

$$P\left[\sup_{t \in [0, 1]} \|P_N(M_t)\|_H \geq R\right] \leq \exp\left[-\frac{(R^2 - K)^2}{8L}\right].$$

Letting N go to infinity, Beppo Levi's lemma yields the result. \blacksquare

APPENDIX 3

The map

$$M: F(\mathcal{H}^r) \rightarrow \mathcal{H}^r$$

$$\phi \mapsto \text{the unique } h \text{ such that } \begin{cases} \phi = F(h) \\ I(\phi) = 1/2 \|h\|_{\mathcal{H}^r}^2 \end{cases}$$

is continuous, when \mathcal{H}^r is endowed with the uniform convergence.

Proof. Let (ϕ^n) and ϕ be such that $\phi^n \rightarrow \tilde{\phi}^{0, k} \phi$. Let $h^n = M(\phi^n)$ and $h = M(\phi)$. By Theorem 3, dt -a.e. \dot{h}_t^n and $\dot{h}_t \in V^\perp$. Moreover,

$$\begin{aligned} \phi_t^n(x) - \phi_t(x) &= \int_0^t (\sigma(\phi_s^n(x)) - \sigma(\phi_s(x))) \dot{h}_s ds \\ &\quad + \int_0^t \sigma(\phi_s^n(x))(\dot{h}_s^n - \dot{h}_s) ds. \end{aligned}$$

Using the properties of σ , we derive that for all K compact subset of \mathbb{R}^n ,

$$\sup_{t, x \in K} \left\| \int_0^t \sigma(\phi_s^n(x))(\dot{h}_s^n - \dot{h}_s) ds \right\| \xrightarrow{n \rightarrow \infty} 0. \quad (5.29)$$

Now let us note that $V = \bigcap_{x \in \mathbb{R}^n} \text{Ker } \sigma(x) = \bigcap_{x \in I} \text{Ker } \sigma(x)$, where I is a finite set. Let K_0 be a compact set of \mathbb{R}^n such that $I \subset K_0$. The convergence of ϕ_n to ϕ yields

$$\sup_{s, x \in K_0} \|(\phi_s^n)^{-1}(x)\| \rightarrow \sup_{s, x \in K_0} \|(\phi_s)^{-1}(x)\|.$$

Thus, there is a compact set K of \mathbb{R}^n such that

$$\{(\phi_s^n)^{-1}(x), x \in K_0, s \in [0, 1], n \in \mathbb{N}\} \subset K.$$

Now, $\sup_{t, x \in I} \|\int_0^t \sigma(x)(\dot{h}_s^n - \dot{h}_s) ds\| \leq \sup_{t, x \in K} \|\int_0^t \sigma(\phi_s^n(x))(\dot{h}_s^n - \dot{h}_s) ds\|$. Using (5.29), we obtain that

$$\sup_{t, x \in I} \|\sigma(x)(h_t^n - h_t)\| \rightarrow 0.$$

Let $I = \{x_1, \dots, x_n\}$, $\sigma_i \equiv \sigma(x_i)$, $V_i \equiv \text{Ker } \sigma_i$, and P_i be the orthogonal projection on V_i^\perp :

$$\begin{aligned} \sup_t \|\sigma_i(h_t^n - h_t)\| &= \sup_t \|\sigma_i(P_i(h_t^n - h_t))\| \\ &\geq \lambda_{\min}((\sigma_i^{V_i^\perp})^* \sigma_i^{V_i^\perp})^{1/2} \sup_t \|P_i(h_t^n - h_t)\|, \end{aligned}$$

where

- for all matrix S symmetric and positive, $\lambda_{\min}(S)$ is the lowest eigenvalue of S .

$$\begin{aligned} \bullet \quad \sigma_i^{V_i^\perp} &: V_i^\perp \rightarrow \text{Im}(\sigma_i) \\ &y \mapsto \sigma_i(y). \end{aligned}$$

By definition, $\sigma_i^{V_i^\perp}$ is invertible; thus $\lambda_{\min}((\sigma_i^{V_i^\perp})^* \sigma_i^{V_i^\perp}) > 0$. Therefore, $\forall i \in \{1, \dots, n\}$, $\sup_t \|P_i(h_t^n - h_t)\| \rightarrow 0$. But, $h_t^n - h_t \in V^\perp = (\bigcap V_i)^\perp = \sum V_i^\perp$ and we obtain that $\sup_t \|h_t^n - h_t\| \rightarrow 0$. ■

APPENDIX 4

Let U^e be defined in Section 5 by (5.28). Then $\forall p, \tilde{P}$ -a.e. $\sup_{e \leq 1} E[(U^e)^p] < \infty$.

Proof. We will in fact show that $\tilde{E}[\sup_{e \leq 1} E[(U^e)^p]] < \infty$:

$$\begin{aligned}
\tilde{E}[\sup_{\varepsilon \leq 1} E[(U^\varepsilon)^p]] &\leq \mathbb{E}[\sup_{\varepsilon \leq 1} (U^\varepsilon)^p] \\
&\leq \mathbb{E} \left[\sup_{\varepsilon \leq 1} \exp \left(p \int_0^1 \Gamma(X_s^\varepsilon) \delta \tilde{B}_s - \frac{p}{2} \int_0^1 \|\Gamma(X_s^\varepsilon)\|^2 ds \right) \right] \\
&\leq \mathbb{E} \left[\sup_{\varepsilon \leq 1} \exp \left(p \int_0^1 \Gamma(X_s^\varepsilon) \delta \tilde{B}_s \right) \right].
\end{aligned}$$

Let

$$M_t^\varepsilon \equiv p \int_0^t \Gamma(X_s^\varepsilon) \delta \tilde{B}_s; \quad V_t^\varepsilon \equiv \begin{pmatrix} X_t^\varepsilon \\ M_t^\varepsilon \\ \varepsilon \end{pmatrix}$$

is a solution to

$$\begin{aligned}
dV_t^\varepsilon &= \lambda_i(V_t^\varepsilon) dB_t^i + \tilde{\lambda}_j(V_t^\varepsilon) d\tilde{B}_t^j + \lambda_0(V_t^\varepsilon) dt \\
V_0^\varepsilon &= \begin{pmatrix} x \\ 0 \\ \varepsilon \end{pmatrix}
\end{aligned}$$

where

$$\begin{aligned}
\lambda_i \begin{pmatrix} x \\ m \\ \varepsilon \end{pmatrix} &= \begin{pmatrix} \varepsilon \sigma_i(x) \\ 0 \\ 0 \end{pmatrix}, \quad \tilde{\lambda}_j \begin{pmatrix} x \\ m \\ \varepsilon \end{pmatrix} = \begin{pmatrix} \tilde{\sigma}_j(x) \\ p \Gamma_j(x) \\ 0 \end{pmatrix}, \\
\lambda_0 \begin{pmatrix} x \\ m \\ \varepsilon \end{pmatrix} &= \begin{pmatrix} \tilde{\sigma}_0(x) \\ 1/2 \sum_j \Gamma_j'(x) \Gamma_j(x) \\ 0 \end{pmatrix}.
\end{aligned}$$

Thus all these vector fields are \mathcal{C}_b^m on $\mathbb{R}^n \times \mathbb{R}^+ \times [0, 1]$. Therefore, there is a version of V^ε which is continuously differentiable in (x, ε) . Moreover, $\partial V^\varepsilon / \partial \varepsilon$ is a solution to the variational equation

$$\begin{aligned}
d \left(\frac{\partial V_t^\varepsilon}{\partial \varepsilon} \right) &= \sum_i \lambda'_i(V_t^\varepsilon) \frac{\partial V_t^\varepsilon}{\partial \varepsilon} dB_t^i + \sum_j \tilde{\lambda}'_j(V_t^\varepsilon) \frac{\partial V_t^\varepsilon}{\partial \varepsilon} d\tilde{B}_t^j \\
&\quad + \lambda'_0(V_t^\varepsilon) \frac{\partial V_t^\varepsilon}{\partial \varepsilon} dt \\
\frac{\partial V_0^\varepsilon}{\partial \varepsilon} &= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.
\end{aligned}$$

Therefore,

$$\begin{aligned}\mathbb{E}[\sup_{\varepsilon \leq 1} \exp(M_t^\varepsilon)] &\leq \mathbb{E}\left[\int_0^1 \exp(M_t^\varepsilon) \left|\frac{\partial M_t^\varepsilon}{\partial \varepsilon}\right| d\varepsilon\right] + \mathbb{E}(\exp(M_t^0)) \\ &\leq \int_0^1 \mathbb{E}[\exp(2M_t^\varepsilon)]^{1/2} \mathbb{E}\left[\left(\frac{\partial M_t^\varepsilon}{\partial \varepsilon}\right)^2\right]^{1/2} d\varepsilon + \mathbb{E}(\exp(M_t^0)).\end{aligned}$$

- Since $\langle M^\varepsilon \rangle_t = p^2 \int_0^t \|\Gamma(X_s^\varepsilon)\|^2 ds$ is bounded by some constant K , for all $\varepsilon \leq 1$, $\mathbb{E}[\exp(2M_t^\varepsilon)] \leq \exp(2K)$.

$$\begin{aligned}\mathbb{E}\left[\left(\frac{\partial M_t^\varepsilon}{\partial \varepsilon}\right)^2\right] &= \mathbb{E}\left[\left\langle \frac{\partial M^\varepsilon}{\partial \varepsilon} \right\rangle_t\right] \\ &= p^2 \mathbb{E}\left[\sum_j \int_0^t \left(\Gamma'_j(X_s^\varepsilon), \frac{\partial X_s^\varepsilon}{\partial \varepsilon}\right)^2 ds\right] \\ &\leq C \int_0^t \mathbb{E}\left[\left\|\frac{\partial X_s^\varepsilon}{\partial \varepsilon}\right\|^2\right] ds.\end{aligned}$$

Applying Itô formula, it is easy to check that there is a constant C such that $\forall \varepsilon \leq 1$.

$$\begin{aligned}\mathbb{E}\left[\left\|\frac{\partial X_t^\varepsilon}{\partial \varepsilon}\right\|^2\right] &\leq C + C\mathbb{E}\left[\sum_i \int_0^t \left\|\sigma_i(X_s^\varepsilon) + \varepsilon\sigma'_i(X_s^\varepsilon) \frac{\partial X_s^\varepsilon}{\partial \varepsilon}\right\|^2 ds\right] \\ &\quad + C\mathbb{E}\left[\sum_j \int_0^t \left\|\tilde{\sigma}'_j(\tilde{X}_s^\varepsilon) \frac{\partial X_s^\varepsilon}{\partial \varepsilon}\right\|^2 ds\right] \\ &\leq C\left(1 + \int_0^t \mathbb{E}\left[\left\|\frac{\partial X_s^\varepsilon}{\partial \varepsilon}\right\|^2\right] ds\right).\end{aligned}$$

Another application of the Gronwall lemma completes the proof. ■

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