**Linear Instability of a Wave in a Density-Stratified Fluid**

Yuanxun Bill Bao, David J. Muraki
Department of Mathematics, Simon Fraser University, Burnaby, BC, Canada

---

**Introduction**

- A fluid with depth-dependent density is said to be density-stratified (ocean & atmosphere).
- Internal gravity waves can be generated by a displacement of a fluid element at the interface of stratified fluids.
- Physical realization: a strong wind flowing over a mountain range.
- One possible configuration is a steady laminar flow.

**Equations for a Density-Stratified Fluid**

**Equations of Motion**

\[ \nabla \cdot \mathbf{u} = 0 \quad (1) \]
\[ \frac{\partial \mathbf{u}}{\partial t} = -\frac{\rho_0 N^2 \mathbf{w}}{g} \quad (2) \]
\[ \frac{\partial \mathbf{p}}{\partial t} = \frac{1}{\rho_0} \nabla \times \mathbf{u} \quad (3) \]

- Zero-divergence, (2) Conservation of mass, (3) Conservation of Momentum
- Velocity \( \mathbf{u} = (u, v, w) \), density \( \rho(F,t) \), pressure \( p(F,t) \)
- Boussinesq approximation and Brunt-Väisälä frequency \( N \)

**2D Streamfunction Formulation (dimensionless)**

\[ \psi_x + \psi_z + J_0 \psi = 0 \]
\[ \psi_y = \psi_z \]

- Streamfunction \( \psi(x, z, t) \)
- Density \( \rho \), \( \gamma \)
- Viscosity \( \mu \)
- Advection from Jacobian \( J_0 \)

**Simple Nonlinear Solutions**

\[ \psi_x = \frac{\omega}{k} \sin(x + z - \omega t) \]

- Linear dispersion relation: \( \omega(k, \gamma) = \frac{\omega_0}{k} \)
- All \((k_x, k_z)\)-pairs satisfying linear dispersion relation give exact nonlinear solutions
- A simple sinusoidal one: \( k_x = k_z = 1, \omega < 0 \)

**Linearized Equations**

\[ \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} - \frac{\partial u}{\partial t} = 0 \]
\[ \frac{\partial v}{\partial t} + \frac{\partial v}{\partial y} - \frac{\partial v}{\partial t} = 0 \]

- Goal: to characterize the linear instability of a simple sinusoidal wave
- Linearize w.r.t the nonlinear wave

**Floquet Analysis for PDEs**

- Product of exponential & co-periodic Fourier series

**Floquet Stability Spectrum**

\[ \psi = e^{i \omega t} \sum_{k,m=0}^{\infty} \mathcal{F}(k,m) \]

- Floquet solution \( \psi(t) \)
- Mathieu Stability Spectrum

**Instability via Floquet Theory**

Textbook ODE example: Mathieu Equation

\[ k \psi_x + \psi = 0 \]

**References**


**Figure 1:** Streamlines of a laminar flow (A. Nenes) and a helical cloud formed over Mt. Fuji.[2]

**Figure 2:** Mathieu stability spectrum

**Figure 3:** Maximum growth rate vs "center-of-mass" uniqueness by D. J. Muraki

**Figure 4:** Triad resonant trace and unstable spectrum

**Figure 5:** Small disturbances grow to make a more complicated flow pattern