Floquet Theory for Internal Gravity Waves in a Density-Stratified Fluid

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Density-Stratified Fluid Dynamics

Density-Stratified Fluids

- density of the fluid varies with altitude
  - stable stratification: heavy fluids below light fluids, internal waves
  - unstable stratification: heavy fluids above light fluids, convective dynamics

Buoyancy-Gravity Restoring Dynamics

- uniform stable stratification: $d\rho/dz < 0$ constant

- vertical displacements $\Rightarrow$ oscillatory motions
Internal Gravity Waves

evidence of internal gravity waves in the atmosphere
  - left: lenticular clouds near Mt. Ranier, Washington
  - right: uniform flow over a mountain $\Rightarrow$ oscillatory wave motions

scientific significance of studying internal gravity waves
  - internal waves are known to be unstable
  - a major suspect of clear-air-turbulence
Gravity Wave Instability: Three Approaches

Triad resonant instability (Davis & Acrivos 1967, Hasselmann 1967)

▷ primary wave + 2 infinitesimal disturbances ⇒ exponential growth
▷ perturbation analysis

Direct Numerical Simulation (Lin 2000)

▷ primary wave + weak white-noise modes
▷ stability diagram
  ▷ unstable Fourier modes

Linear Stability Analysis & Floquet-Fourier method (Mied 1976, Drazin 1977)

▷ linearized Boussinesq equations & stability via eigenvalue computation
My Thesis Goal

- Floquet-Fourier computation: over-counting of instability in wavenumber space
- Lin’s DNS: two branches of disturbance Fourier modes
- goal: to identify all physically unstable modes from Floquet-Fourier computation
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Boussinesq Equations in Vorticity-Buoyancy Form

\[
\nabla \cdot \vec{u} = 0 ; \; \frac{D\eta}{Dt} = -b_x ; \; \frac{Db}{Dt} = -N^2 w
\n
- incompressible, inviscid Boussinesq Fluid
  - Euler equations + weak density variation (the Boussinesq approximation)
  - Brunt-Vaisala frequency $N$: uniform stable stratification, $N^2 > 0$
- 2D velocity: $\vec{u}(x, z, t)$; buoyancy: $b(x, z, t)$
  - streamfunction: $\vec{u} = \begin{pmatrix} u \\ w \end{pmatrix} = -\vec{\nabla} \times \psi \hat{y} = \begin{pmatrix} -\psi_z \\ \psi_x \end{pmatrix}$
  - vorticity: $\vec{\nabla} \times \vec{u} = \eta \hat{y} = \nabla^2 \psi \hat{y}$
Exact Plane Gravity Wave Solutions

\[
\frac{D\eta}{Dt} = -b_x \\
\frac{Db}{Dt} = -\mathcal{N}^2 w
\]

- dynamics of buoyancy & vorticity \(\Rightarrow\) oscillatory wave motions
- exact plane gravity wave solutions

\[
\begin{pmatrix}
\psi \\
b \\
\eta
\end{pmatrix}
= \begin{pmatrix}
-\frac{\Omega_d}{K} \\
\mathcal{N}^2 \\
\mathcal{N}^2 K/\Omega_d
\end{pmatrix} 2A \sin(Kx + Mz - \Omega_d t)
\]

- primary wavenumbers: \((K, M)\)
- dispersion relation: \(\Omega_d^2(K, M) = \frac{\mathcal{N}^2 K^2}{K^2 + M^2}\).
Linear Stability Analysis

dimensionless exact plane wave + small disturbances

\[
\begin{pmatrix}
\psi \\
b \\
\eta
\end{pmatrix}
= 
\begin{pmatrix}
-\Omega \\
1 \\
1/\Omega
\end{pmatrix}
2\epsilon \sin(x + z - \Omega t) 
+ 
\begin{pmatrix}
\tilde{\psi} \\
\tilde{b} \\
\tilde{\eta}
\end{pmatrix}
\]

\(\epsilon: \) dimensionless amplitude & dimensionless frequency: \(\Omega^2 = \frac{1}{1 + \delta^2}\)

linearized Boussinesq equations

\[
\delta^2 \tilde{\psi}_{xx} + \tilde{\psi}_{zz} = \tilde{\eta}
\]
\[
\tilde{\eta}_t + \tilde{b}_x - 2\epsilon J(\Omega\tilde{\eta} + \tilde{\psi}/\Omega, \sin(x + z - \Omega t)) = 0
\]
\[
\tilde{b}_t - \tilde{\psi}_x - 2\epsilon J(\Omega\tilde{b} + \tilde{\psi}, \sin(x + z - \Omega t)) = 0
\]

\(\delta = K/M: \) related to the wave propagation angle (Lin: \(\delta = 1.7\))

Jacobian determinant

\[
J(f, g) = \begin{vmatrix}
f_x & g_x \\
f_z & g_z
\end{vmatrix} = f_x g_z - g_x f_z
\]
Linear Stability Analysis

\( \text{dimensionless exact plane wave + small disturbances} \)

\[
\begin{pmatrix}
\psi \\
b \\
\eta
\end{pmatrix} = 
\begin{pmatrix}
-\Omega \\
1 \\
1/\Omega
\end{pmatrix} \cdot 2\epsilon \sin(x + z - \Omega t) + 
\begin{pmatrix}
\tilde{\psi} \\
\tilde{b} \\
\tilde{\eta}
\end{pmatrix}
\]

- \( \epsilon \): dimensionless amplitude & dimensionless frequency: \( \Omega^2 = \frac{1}{1 + \delta^2} \)
- Linearized Boussinesq equations

\[
\begin{align*}
\delta^2 \tilde{\psi}_{xx} + \tilde{\psi}_{zz} &= \tilde{\eta} \\
\tilde{\eta}_t + \tilde{b}_x - 2\epsilon J(\Omega \tilde{\eta} + \tilde{\psi}/\Omega, \sin(x + z - \Omega t)) &= 0 \\
\tilde{b}_t - \tilde{\psi}_x - 2\epsilon J(\Omega \tilde{b} + \tilde{\psi}, \sin(x + z - \Omega t)) &= 0
\end{align*}
\]

- System of linear PDEs with non-constant, but periodic coefficients
- Analyzed by Floquet theory
- Classical textbook example: Mathieu equation (Chapter 3)
Floquet Theory: Mathieu Equation

Mathieu Equation:

\[
\frac{d^2 u}{dt^2} + \left[ k^2 - 2\epsilon \cos(t) \right] u = 0
\]

- second-order linear ODE with periodic coefficients
- Floquet theory: \( u = e^{-i\omega t} \cdot p(t) = \text{exponential part} \times \text{co-periodic part} \)
- Floquet exponent \( \omega(k; \epsilon) \): \( \text{Im} \ \omega > 0 \rightarrow \text{instability} \)

- goal: to identify all unstable solutions in \((k, \epsilon)\)-space
Floquet Theory: Mathieu Equation

Mathieu Equation:

\[ \frac{d^2 u}{dt^2} + [k^2 - 2\epsilon \cos(t)] u = 0 \]

Two perspectives:

- perturbation analysis ⇒ two branches of Floquet exponent
  - away from resonances: \( \omega(k; \epsilon) \sim \pm k \)
  - resonant instability at primary resonance \( \left( k = \frac{1}{2} \right) \): \( \omega(k; \epsilon) \sim \pm \frac{1}{2} + i \epsilon \)

- Floquet-Fourier computation of \( \omega(k; \epsilon) \)
  - a Riemann surface interpretation of \( \omega(k; \epsilon) \) with \( k \in \mathbb{C} \)
Mathieu equation in system form:
\[
\frac{d}{dt} \begin{pmatrix} u \\ v \end{pmatrix} = i \begin{bmatrix} k^2 - 2\epsilon \cos(t) & 1 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} u \\ v \end{pmatrix}
\]

Floquet-Fourier representation:
\[
\begin{pmatrix} u \\ v \end{pmatrix} = e^{-i\omega t} \cdot \sum_{m=-\infty}^{\infty} \vec{c}_m e^{-imt}
\]

\(\omega(k; \epsilon)\) as eigenvalues of Hill’s bi-infinite matrix:

\(2 \times 2\) real blocks: \(S_m\) and \(M\)

Truncated Hill’s matrix: \(-N \leq m \leq N\)

- real-coefficient characteristic polynomial
- compute \(4N + 2\) eigenvalues: \(\{\omega_n(k; \epsilon)\}\)
  - \(\epsilon = 0\), eigenvalues from \(S_n\) blocks: \(\omega_n(k; 0) = -n \pm k\) & all real-valued
  - \(\epsilon \ll 1\), complex eigenvalues may arise from \(\epsilon = 0\) double eigenvalues
For each $k$, how many Floquet exponents are associated with the unstable solutions of Mathieu equation? **two** or **∞**? Both!

- **two** is understood from perturbation analysis
- **∞** will be understood from the Riemann surface of $\omega(k; \epsilon)$ with $k \in \mathbb{C}$
- Floquet-Fourier Computation

- \( \omega_n(k; \epsilon) \): real \( \bullet \); complex \( \bullet \)

- \( \epsilon = 0.1 \)

- \( \omega_n(k; 0) = -n \pm k \)

- \( \omega_0(k; 0) = \pm k \)

- \( \omega_n(k; \epsilon) \) curves are close to \( \omega_n(k; 0) \)

- two continuous curves close to \( \pm k \)

- the rest are shifted due to

\[
\begin{pmatrix}
  u \\
  v
\end{pmatrix} = e^{-i(\omega_0+n)t} \sum_{m=-\infty}^{\infty} \vec{c}_{m+n} e^{-imt}
\]

- For each \( k \), how many Floquet exponents are associated with the unstable solutions of Mathieu equation? two or \( \infty \)? Both!

  - two is understood from perturbation analysis

  - \( \infty \) will be understood from the Riemann surface of \( \omega(k; \epsilon) \) with \( k \in \mathbb{C} \)
A Riemann Surface Interpretation of $\omega(k; \epsilon)$

- Floquet-Fourier computation with $k \in \mathbb{C}$ → the Riemann surface of $\omega(k; \epsilon)$
  - surface height: real $\omega$; surface colour: imag $\omega$
  - layers of curves for $k \in \mathbb{R}$ become layers of sheets for $k \in \mathbb{C}$
  - the two physical branches belong to two primary Riemann sheets
- How to identify the two primary Riemann sheets?
  - more understanding of how sheets are connected
A Riemann Surface Interpretation of $\omega(k; \epsilon)$

- zoomed view near $\text{Re } k = 1/2$ shows Riemann sheet connection
- branch points: end points of instability intervals
  - loop around the branch points $\Rightarrow \sqrt{\text{type}}$
- branch cuts coincide with instability intervals (McKean & Trubowitz 1975)
A Riemann Surface Interpretation of $\omega(k; \epsilon)$

- zoomed view near $\text{Re} \, k = 0$ shows Riemann sheet connection
- branch points: two on imaginary axis
  - loop around the branch points $\Rightarrow \sqrt{\cdot}$ type
- branch cuts to $\pm i\infty$ give V-shaped sheets
A Riemann Surface Interpretation of $\omega(k; \epsilon)$

- branch cuts: instability intervals & two cuts to $\pm i\infty$
- two primary sheets: upward & downward V-shaped sheets
  - associated with the two physically-relevant Floquet exponents
- the other sheets are integer-shifts of primary sheets
A Riemann Surface Interpretation of $\omega(k; \epsilon)$

- branch cuts: instability intervals & two cuts to $\pm i\infty$
- two primary sheets: upward & downward V-shaped sheets
  - associated with the two physically-relevant Floquet exponents
- the other sheets are integer-shifts of primary sheets
Recap of Mathieu Equation

- Floquet-Fourier: \[
\begin{pmatrix} u \\ v \end{pmatrix} = e^{-i\omega t} \cdot \sum_{m=-N}^{N} \bar{c}_n e^{-imt}
\]

- 4\(N\) + 2 computed Floquet exponents \(\omega_n(k; \epsilon)\)

- Perturbation analysis: \(\omega(k; \epsilon) \sim \pm k\)

- Riemann surface has two primary Riemann sheets (physically-relevant)
Chapter 4, 5, 6 of My Thesis

- Floquet-Fourier: \( \left( \tilde{\psi} \right)_{\tilde{b}} = e^{i(kx + mz - \omega t)} \cdot \left\{ \sum_{n=-N}^{N} \left( \tilde{\psi}_n \tilde{b}_n \right) e^{in(x + z - \Omega t)} \right\} \).

- \( 4N + 2 \) computed Floquet exponents \( \omega_n(k, m; \epsilon, \delta) \)

- Perturbation analysis: \( \omega(k, m; \epsilon, \delta) \sim \pm \frac{|k|}{\sqrt{\delta^2 k^2 + m^2}} \)

- Riemann surface analysis \( \Rightarrow \) physically-relevant Floquet exponents

\[\begin{array}{cc}
\epsilon = 0.1, \delta = 1.7 \\
\hline
-2 & -1 \\
-0.5 & 0 \\
0 & 1 \\
1 & 2 \\
-2 & -1 \\
\end{array}\]

\[\begin{array}{cc}
\delta = 1.7, \epsilon = 0.1 \\
\hline
-2 & -1 \\
0 & 1 \\
1 & 2 \\
-2 & -1 \\
\end{array}\]
Gravity Wave Stability Problem

- four parameters of $\omega(k, m; \epsilon, \delta)$
  - $\epsilon, \delta = 1.7$ (Lin)
  - wavevector, $(k, m)$; $k \in \mathbb{C}$ with $k - m = 2.5$
- over-counting of Floquet-Fourier computation
  - vertical & horizontal shifts $\rightarrow$ instability bands
Gravity Wave Stability Problem

- four parameters of $\omega(k, m; \epsilon, \delta)$
  - $\epsilon, \delta = 1.7$ (Lin)
  - wavevector, $(k, m)$ ; $k \in \mathbb{C}$ with $k - m = 2.5$

- over-counting of Floquet-Fourier computation
  - vertical & horizontal shifts $\rightarrow$ instability bands

- physically-relevant Floquet exponents solves over-counting problem
Fixing the Gap along $k - m = 1$

- new feature: four-sheet collision (only two for Mathieu!)
- physically corresponds to near-resonance of four fourier modes (section 5.3)
Fixing the Gap along $k - m = 1$

▷ zoomed view near $\text{Re } k = 0$ with Riemann surface
Fixing the Gap along $k - m = 1$

continuation algorithm for $\omega(k, m; \epsilon = 0.1)$ starts from $\epsilon = 0$ values
Fixing the Gap along $k - m = 1$

- continuation algorithm for $\omega(k, m; \epsilon = 0.1)$ starts from $\epsilon = 0$ values
- $\epsilon = 0.02$: shows $\epsilon = 0$ limit incorrect
Fixing the Gap along $k - m = 1$

- continuation algorithm for $\omega(k, m; \epsilon = 0.1)$ starts from $\epsilon = 0$ values
- $\epsilon = 0.02$: suggests redefining $\epsilon = 0$ branch values (continuous)
Fixing the Gap along $k - m = 1$

\[ \varepsilon = 0.06, \delta = 1.7 \]

\[ \varepsilon = 0, \delta = 1.7 \]

- Continuation algorithm for $\omega(k, m; \varepsilon = 0.1)$ starts from $\varepsilon = 0$ values
- $\varepsilon = 0.06$: instability bands are about to merge
Fixing the Gap along $k - m = 1$

Continuation algorithm for $\omega(k, m; \epsilon = 0.1)$ starts from $\epsilon = 0$ values

$\epsilon = 0.1$: the gap is fixed
Instabilities from Two Primary Sheets

- stability diagram is a superposition of instabilities from the two primary sheets
- both primary sheets are continuous in $\text{Re}\, \omega$ & $\text{Im}\, \omega$
- over-counting problem is solved by complex analysis!
In Closing: What I Have Learned

- density-stratified fluid dynamics & internal gravity waves
- linear stability analysis
- the Mathieu equation, Floquet theory & Floquet-Fourier computation
- perturbation analysis (near & away from resonance)
- understanding the Riemann surface structure & computation
Four Sheets: $\epsilon = 0.1$