Probability, homework 4 due October 4.

Exercise 1. Let n and m be random numbers chosen independently and uniformly on $[\![1, N]\!]$. What are $\Omega, \mathcal{A}$ and $\mathbb{P}$ (which all implicitly depend on N) ? Prove that $\mathbb{P}(n \wedge m = 1) \underset{N\to\infty}{\longrightarrow} \zeta(2)^{-1}$ where $\zeta(2) = \prod_{p\in\mathcal{P}}(1-p^{-2})^{-1} = \sum_{n\geq 1} n^{-2} = \frac{\pi^2}{6}$ (you don't have to prove these equalities). Here $\mathcal{P}$ is the set of prime numbers and $n \wedge m = 1$ means that their greatest common divisor is 1.

Exercise 2. Let X be a random variable with density $f_X(x) = (1-|x|)\mathbb{1}_{(-1,1)}(x)$. Show that its characteristic function is

$$\phi_X(u) = \frac{2(1-\cos u)}{u^2}.$$

Exercise 3.

(1) Prove that $\hat{\mu}$ is real-valued if and only if μ is symmetric, i.e. $\mu(A) = \mu(-A)$ for any Borel set A

(2) If X and Y are i.i.d., prove that $X - Y$ has a symmetric distribution.

Exercise 4. Let X_λ be a real random variable, with Poisson distribution with parameter λ. Calculate the characteristic function of X_λ. Conclude that $(X_\lambda - \lambda)/\sqrt{\lambda}$ converges in distribution to a standard Gaussian, as $\lambda \to \infty$.

Exercise 5. Assume that the sequence of random variables $(X_n)_{n\geq1}$ satisfies $\mathbb{E}\, X_n \to 1$ and $\mathbb{E}\, X_n^2 \to 1$. Prove that $(X_n)_{n\geq1}$ converges in distribution. What is the limit?

Exercise 6. Let $(X_n)_{n\geq1}$, $(Y_n)_{n\geq1}$ be real random variables, with X_n and Y_n independent for any $n \geq 1$, and assume that X_n converges in distribution to X and Y_n to Y, with X and Y independent defined on the same probability space. Prove that $X_n + Y_n$ converges in distribution to $X + Y$.

Exercise 7. Let X, Y be independent and assume that for some constant α we have $\mathbb{P}(X + Y = \alpha) = 1$. Prove that X and Y are both constant random variables.

Exercise 8. Let $f, g : \mathbb{R} \to \mathbb{R}$ be nondecreasing measurable functions. Let μ be a probability measure on $\mathbb{R}$ and assume $f, g, fg \in \mathrm{L}^1(\mu)$. prove that

$$\int fg\mathrm{d}\mu \geq \int f\mathrm{d}\mu \cdot \int g\mathrm{d}\mu.$$

Exercise 9. Let $(X_n)_{n\geq1}$ be a sequence of i.i.d. random variables with standard Cauchy distribution and let $M_n = \max(X_1, \dots, X_n)$. Prove that $(nM_n^{-1})_{n\geq1}$ converges in distribution and identify the limit.

Exercise 10. Let $(X_i)_{i\geq1}$ be a sequence of independent random variables, with X_i uniform on $[-i, i]$. Let $S_n = X_1 + \cdots + X_n$. Prove that $S_n/n^{3/2}$ converges in distribution and describe the limit.

Exercise 11. Find a probability distribution μ of a $\mathbb{Z}$-valued random variable which is symmetric ($\mu(\{i\}) = \mu(\{-i\})$ for any $i \in \mathbb{Z}$), not integrable, but such that its characteristic function is differentiable at 0.

Exercise 12. Let X, Y be i.i.d., with characteristic functions denoted φ_X, φ_Y, and suppose $\mathbb{E}(X) = 0$, $\mathbb{E}(X^2) = 1$. Assume also that $X + Y$ and $X - Y$ are independent.

(1) Prove that
$$\varphi_X(2u) = (\varphi_X(u))^3 \varphi_X(-u)$$
(2) Prove that X is a standard Gaussian random variable.