The two-dimensional Coulomb plasma: quasi-free approximation and central limit theorem

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Abstract

For the two-dimensional one-component Coulomb plasma, we derive an asymptotic expansion of the free energy up to order N, the number of particles of the gas, with an effective error bound $N^{1-\kappa}$ for some constant $\kappa > 0$. This expansion is based on approximating the Coulomb gas by a quasi-free Yukawa gas. Further, we prove that the fluctuations of the linear statistics are given by a Gaussian free field at any positive temperature. Our proof of this central limit theorem uses a loop equation for the Coulomb gas, the free energy asymptotics, and rigidity bounds on the local density fluctuations of the Coulomb gas, which we obtained in a previous paper.

1 Introduction and main results

1.1. One-component plasma. The two-dimensional one-component Coulomb plasma (OCP) is a Gibbs measure on the configurations of N charges $\mathbf{z} = (z_1, \dots, z_N) \in \mathbb{C}^N$. The Hamiltonian of this measure, in external potential $V : \mathbb{C} \to \mathbb{R} \cup \{+\infty\}$, is given by

$$H^G_{N,V}(\mathbf{z}) = N \sum_j V(z_j) + \sum_{j \neq k} G(z_j, z_k), \tag{1.1}$$

where $G(z_j, z_k) = \mathcal{C}(z_j - z_k)$ is the two-dimensional Coulomb potential,

$$\mathcal{C}(z_j - z_k) = \log \frac{1}{|z_j - z_k|}, \tag{1.2}$$

characterized by $\Delta \log |\cdot| = 2\pi\delta_0$ as distributions. The Coulomb plasma is our main interest, but throughout the paper we will also consider other symmetric interactions $G(z_j, z_k)$. The Gibbs measure of this plasma (OCP) at the inverse temperature $\beta > 0$ is defined by

$$P^G_{N,V,\beta}(\mathrm{d}\mathbf{z}) = \frac{1}{Z^G_{N,V,\beta}} \mathrm{e}^{-\beta H^G_{N,V}(\mathbf{z})}\, m^{\otimes N}(\mathrm{d}\mathbf{z}), \tag{1.3}$$

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where m denotes the Lebesgue measure on $\mathbb{C}$, and $Z^G_{N,V,\beta}$ the normalization constant (assuming that V has sufficient growth at infinity, so that the latter is well-defined). We will follow the convention that when $G = \mathcal{C}$ then we will omit the superscript $\mathcal{C}$ whenever there is no confusion. Similar conventions apply to other subscripts, and we will also often omit N and β.

The two-dimensional Coulomb gas $P^{\mathcal{C}}_{N,V,\beta}$ has connections with a variety of models in mathematical physics and probability theory. For $\beta = 1$, it describes the eigenvalues density for some measures on non Hermitian random matrices [16,22]. In particular for quadratic V the complex vector $\mathbf{z}$ is distributed like the spectrum of a matrix with complex Gaussian entries. Moreover, the properties of this two-dimensional gas are known to be related to the fractional quantum Hall effect: for $\beta = 2s + 1$, with s integer, $P^{\mathcal{C}}_{N,V,\beta}$ is the density obtained from Laughlin's guess for wave functions of fractional fillings of type $(2s + 1)^{-1}$ [30]. Finally, an important problem is the crystallization of the two-dimensional Coulomb gas for small temperature [2,15].

For potentials V that are lower semicontinuous and satisfy the growth condition

$$\liminf_{|z|\to\infty} \big(V(z) - (2+\varepsilon)\log|z|\big) > -\infty \tag{1.4}$$

for some $\varepsilon > 0$, it is well known (see e.g. [40]) that there exists a compactly supported equilibrium measure μ_V that is the unique minimizer of the convex energy functional

$$\mathcal{I}_V(\mu) = \iint \log\frac{1}{|z-w|}\,\mu(\mathrm{d}z)\,\mu(\mathrm{d}w) + \int V(z)\,\mu(\mathrm{d}z) \tag{1.5}$$

over the set of probability measures on $\mathbb{C}$. The unique minimizer, denoted by μ_V, is supported on a compact set S_V and, assuming that V is smooth, it has the density

$$\rho_V = \frac{1}{4\pi}\Delta V \mathbb{1}_{S_V} \tag{1.6}$$

with respect to the Lebesgue measure m. We write $I_V = \mathcal{I}_V(\mu_V)$ for the minimum of $\mathcal{I}_V$. For $\mathbf{z} \in \mathbb{C}^N$, the empirical measure is defined by

$$\hat{\mu} = \frac{1}{N}\sum_j \delta_{z_j}.$$

For arbitrary $\beta \in (0,\infty)$, it is well-known that $\hat{\mu} \to \mu_V$ vaguely in probability as $N \to \infty$, with $\hat{\mu}$ distributed under $P_{N,V}$. In [6], we have proved two stronger estimates for the Coulomb gas. The first one is a local law that asserts that for any smooth f supported in a disk of radius $b = N^{-s}$ ($s \in [0, 1/2)$) centred at z_0 in the bulk of S_V (and the f supported in the bulk when $s = 0$), we have

$$\frac{1}{N}\sum_{j=1}^N f(z_j) - \int f(z)\,\mu_V(\mathrm{d}z) = \mathrm{O}\left(\left(1+\frac{1}{\beta}\right)\log N\right)\left(N^{-1-2s}\|\Delta f\|_\infty + N^{-\frac{1}{2}-s}\|\nabla f\|_2\right), \tag{1.7}$$

with probability at least $1 - \mathrm{e}^{-(1+\beta)N^{1-2s}}$ for sufficiently large N. A stronger estimate, which we shall call rigidity, asserting that

$$\sum_{j=1}^N f(z_j) - N\int f(z)\,\mu_V(\mathrm{d}z) = \mathrm{O}(N^\varepsilon)\left(\sum_{l=1}^4 N^{-ls}\|\nabla^l f\|_\infty\right), \tag{1.8}$$

with probability at least $1-\mathrm{e}^{-\beta N^\varepsilon}$ for sufficiently large N also holds under the same assumptions.

The main result of this paper is the identification of the random error term in the above rigidity estimate. It is given by the Gaussian free field with a nonzero mean.

1.2. Main results. Our main results are the following two theorems. The global potential V is always assumed to satisfy

$$V \in \mathscr{C}^5 \text{ on a neighborhood of } S_V = \operatorname{supp} \mu_V, \quad \alpha_0 \leqslant \Delta V(z) \leqslant \alpha_0^{-1} \text{ for all } z \in S_V \tag{1.9}$$

for some $\alpha_0 > 0$, and we also assume that the boundary of S_V is piecewise $\mathscr{C}^1$; more precisely, ∂S_V is a finite union of $\mathscr{C}^1$ curves.

Theorem 1.1. *Suppose that the external potential V satisfies the conditions* (1.4) *and* (1.9). *Then there exists $\zeta_\beta^{\mathcal{C}} \in \mathbb{R}$ independent of V such that, for any $\kappa < 1/24$,*

$$\frac{1}{\beta N} \log \int \mathrm{e}^{-\beta H_V} m^{\otimes N}(\mathrm{d}\mathbf{z}) = -N I_V + \frac{1}{2}\log N + \zeta_\beta^{\mathcal{C}} + \left(\frac{1}{2} - \frac{1}{\beta}\right) \int \rho_V \log \rho_V \, \mathrm{d}m + \mathrm{O}(N^{-\kappa}).$$

A similar result, as a limiting statement instead of a quantitative error bound, and with $\zeta_\beta^{\mathcal{C}}$ characterized via a large deviation principle, was previously proved in [33]. For our application to the proof of Theorem 1.2 below, a quantitative error bound is essential. In addition, we will provide a physical interpretation of $\zeta_\beta^{\mathcal{C}}$ as the *residual free energy* of the Coulomb (or technically a long-range Yukawa) gas on the torus; see Theorems 2.1 and 3.2.

For the statement of Theorem 1.2, we need the following additional notations. For f supported in a disk of radius $b = N^{-s}$, we introduce the norms

$$\|f\|_{k,b} = \sum_{j=0}^{k} b^j \|\nabla^j f\|_\infty. \tag{1.10}$$

When $b = 1$, we denote it by $\|f\|_k$. Moreover, for any function f with support in S_V, let

$$X_V^f = \sum_j f(z_j) - N \int f \, \mathrm{d}\mu_V, \tag{1.11}$$

$$Y_V^f = \frac{1}{4\pi} \int \Delta f \log \Delta V \, \mathrm{d}m = \frac{1}{4\pi} \int \Delta f(z) \log \rho_V(z) \, m(\mathrm{d}z). \tag{1.12}$$

In the following theorem, $f : \mathbb{C} \to \mathbb{R}$ is supported on a disk with radius $b = N^{-s}$ for a fixed scale $s \in [0, 1/2)$, and $\|f\|_{5,b} < \infty$ uniformly in N. We also assume that the support of f satisfies $\operatorname{dist}(\operatorname{supp}(f), S_V^c) > \varepsilon$ for some $\varepsilon > 0$ uniformly in N. (Indeed, the last condition can be relaxed to $\varepsilon = N^{-1/4+c}$ for arbitrarily small c, i.e., f still supported in the bulk).

Theorem 1.2. *Suppose that V satisfies the condition* (1.4) *and* (1.9), *and that f has support in a ball of radius $b = N^{-s}$ with the above conditions. Then there exists $\tau_0 = \tau_0(s) > 0$ such that for any $0 < \tau < \tau_0$ and $0 < \lambda \ll (Nb^2)^{1-2\tau}$, we have*

$$\frac{1}{\beta\lambda} \log \mathbb{E}\left(\mathrm{e}^{-\beta\lambda\left(X_V^f - \left(\frac{1}{\beta} - \frac{1}{2}\right) Y_V^f\right)}\right) = \frac{\lambda}{8\pi} \int |\nabla f(z)|^2 \, m(\mathrm{d}z) + \mathrm{O}((Nb^2)^{-\tau}).$$

Here the expectation is with respect to $P_{N,V,\beta}^{\mathcal{C}}$.

Note that λ is allowed to be very large in this theorem; this provides strong error estimates for the Gaussian convergence. This central limit theorem is noteworthy due to the absence of normalization: fluctuations of X_V^f are only of order one, due to repulsion, but still Gaussian. For the purpose of establishing the central limit theorem for X_V^f, it suffices to take λ to be of order one (independent of N).

Finally, a result similar to Theorem 1.2 was obtained simultaneously and independently in [32].

1.3. Related results. The study of one- and two-dimensional Coulomb and log-gases has attracted considerable attention recently, see e.g. [21] for many aspects of these probability measures in connection with statistical physics. The subject of our work, abnormally small Gaussian charge fluctuations of the one-component plasma, was first predicted in the late 1970s (see [26] and the references therein).

In dimension two, in the special case $\beta = 1$, the central limit theorem was first proved for the Ginibre ensemble, i.e. for quadratic external potential V [36, 37]. These results were extended to more general V by combining tools from determinantal point processes and the loop equation approach [4, 5]. In particular, in the latter works the determinantal structure was used to prove local isotropy of the point process, an important a priori estimate necessary to the loop equation approach. For general inverse temperature β, the determinantal structure does not hold; nevertheless an expansion of the partition function and correlation functions was predicted in [45–47]. The expansion of the partition function up to order N was rigorously obtained in [33] (along with a corresponding large deviation principle for a tagged point process); see also the related earlier works [38, 41, 42]; in addition, see also [23]. Still for the two-dimensional Coulomb gas at any temperature, a local density [31, 34] was recently proved, together with abnormally small charge fluctuations in the sense of rigidity [31], see (1.8). Other recent results in this direction include [**?**, 3, 34, 35, 39].

For the log-gas on the line, much more is known. Indeed, in dimension one the Selberg integrals are often a good starting point to evaluate partition functions, and anisotropy does not cause any trouble in the analysis of loop equations. For general β and V, full expansions of the partition function and correlators were predicted in [19], proved at first orders in [43] and at all orders in [8, 9]. A natural analogue of the rigidity (1.8) is also known to hold for log-gases on the real line [10]. Still for the log-gas in dimension 1, the central limit theorem was first discovered on the circle for $\beta = 2$ in [27], and on the real line for any β in [28]. For test functions supported on a mesoscopic scale, the local central limit theorem was proved on the circle for some compact groups in [44], for general β ensembles with quadratic V in [11] and for general V in [7].

For expansions at high temperatures, and exponential decay of microscopic correlations, in closely related models of Coulomb gases, see [13, 25]. For results on crystallization in the one-dimensional one-component Coulomb plasma, see [1, 12, 29]. Further results on Coulomb systems in statistical mechanics are reviewed in [14, 21].

1.4. Strategy. In Section 2, we first prove that an extended version of Theorem 1.1 holds for Yukawa gases on a torus. The essence is to show that the constant $\zeta_\beta^{\mathcal{C}}$ can be identified independently of the range of the Yukawa interaction. This fact and interpretation is then used in Section 3 to establish an expansion of the free energy of the Coulomb gas up to order $N^{1-\kappa}$. The main idea is to approximate the Coulomb gas first by a short-range Yukawa gas, and then by a quasi-free Yukawa gas. Roughly speaking, a Yukawa gas with range $\ell \ll 1$ can be viewed, for the purpose of computing free energy, as an idea gas consisting of independent squares of size b satisfying $1 \gg b \gg \ell$ and with the gas inside each square being a Yukawa gas with range ℓ. Since this gas is an ideal gas over a distance longer than a mesoscopic scale b, we call it a quasi-free approximation. The Yukawa approximation to the Coulomb gas is a well-known tool in the study of the quantum Coulomb gas, see, e.g., [17, 18]. However, the precision needed here is far beyond the previous results. Due to the rigidity estimate established in [6], we are able to show that the Yukawa approximation yields very mild errors.

A key difficulty in establishing Theorem 1.1 is the surface energy of a Coulomb gas. The typical inter-particle distance of this gas is $N^{-1/2}$, therefore the total Coulomb energy for particles within a distance $N^{-1/2}$ to the boundary of the support of the equilibrium measure is of order N. Theorem 1.1 requires to capture these interaction energies up to order $N^{1-\kappa}$. In other words, the leading term in the "energy associated with the charges near the boundary of the support of the Coulomb gas" has to be identified. Our idea is to use an ideal gas approximation for a boundary layer and then switch to a Yukawa approximation for interior particles. We will explain this idea in Section 3.

In Section 4, we first prove that the central limit theorem holds after subtracting a random term, the *local angle term*. From this result and the asymptotic expansion of the free energy for the Coulomb gas, Theorem 1.1, we obtain that the angle term does in fact vanish in a large deviation sense. We thus prove Theorem 1.2 for a test function f with macroscopic support. For test functions with support on a mesoscopic scale b, we proceed via conditioning to a disk of radius $2b$. This conditioning procedure was used in [6]; it has the advantage of reformulating the question into a problem on the natural scale b.

Throughout the paper, we will extensively use the local density and rigidity estimates for the Yukawa gas and Coulomb gas with additional angular interaction, in a form similar to (1.7) and (1.8). In Appendices A–B, we therefore extend the estimates of [6] to the Yukawa gas and the Coulomb gas with angle term. In Appendix C, we prove an important estimate related to the energy distortion from embedding torus into the Euclidean space.

Notation. *We use the usual Landau* O*-notation. For N-dependent quantities $A, B \geqslant 0$, we write $A \ll B$ when there exists $\varepsilon > 0$ and $N_0 \geqslant 0$ such that $A \leqslant N^{-\varepsilon} B$ for $N \geqslant N_0$. For an event E, we say that E holds with high probability if there is $\delta > 0$ and $N_0 \geqslant 0$ such that $\mathbb{P}(E) \geqslant 1 - \mathrm{e}^{-N^{\delta}}$ for $N \geqslant N_0$. For random variables A and B, we write $A \prec B$ if for any $\varepsilon > 0$ the event $|A| \leqslant N^{\varepsilon}|B|$ holds with high probability. Finally, m denotes the Lebesgue measure on $\mathbb{C}$ or on the torus, and $m^{\otimes N}$ will be denoted by m when there is no ambiguity about the dimension.*

2 Free energy of the torus

We start with proving a version of Theorem 1.1 for the Yukawa gas on the torus. This outlines the strategy for the proof of Theorem 1.1 in a simplified context and also constructs the nontrivial contribution to the constant ζ in Theorem 1.1.

2.1. Two-dimensional Yukawa gas on the torus. For $z \in \mathbb{C}$, which we also identify with $\mathbb{R}^2$, the two-dimensional Yukawa potential with range ℓ is defined by the formula

$$Y^{\ell}(z) := \frac{1}{4\pi} \int_{\mathbb{R}^2} \mathrm{e}^{-\mathrm{i}p\cdot z} \int_0^{\infty} \mathrm{e}^{-t(p^2+1/\ell^2)/2}\,\mathrm{d}t\,\mathrm{d}p = \int_1^{\infty} \mathrm{e}^{-a(s+1/s)}\,\frac{\mathrm{d}s}{s} =: g(a), \quad a = \frac{|z|}{2\ell}, \tag{2.1}$$

where $p \cdot z$ denotes the Euclidean inner product on $\mathbb{R}^2$. Denoting $m = 1/\ell$, thus $(-\Delta + m^2)Y^{\ell} = 2\pi\delta_0$ as distributions, $Y^{\ell}(z)$ is pointwise positive and positive definite, and there is an absolute constant Y_0 such that

$$Y^{\ell}(z) \begin{cases} \sim -\log|z| + \log \ell + Y_0 + \mathrm{O}(|z|/\ell) & \text{if } |z|/\ell \leqslant 1 \\ \leq C_1 \mathrm{e}^{-C_2|z|/\ell} & \text{if } |z|/\ell \geq 1. \end{cases} \tag{2.2}$$

The first equation can be checked with $Y_0 = \log 2 + \gamma$ from

$$g(a) = \gamma - \log a + \mathrm{O}(a), \quad \gamma = \int_0^{\infty} \left(\mathrm{e}^{-s} - 1_{s<1}\right) \frac{\mathrm{d}s}{s}. \tag{2.3}$$

In particular, up to the constant $Y_0 + \log \ell$, the two-dimensional Coulomb potential $-\log|z|$ is the limit $\ell \to \infty$ of $Y^\ell(z)$. We denote by $\mathbb{T}$ the two-dimensional unit torus $(\mathbb{R}/\mathbb{Z})^2$. For $\ell > 0$, the Yukawa interaction of range ℓ on $\mathbb{T}$ is given by

$$U^\ell(z) = \sum_{n\in\mathbb{Z}^2} Y^\ell(z+n). \tag{2.4}$$

The Hamiltonian of the periodic Yukawa gas on $\mathbb{T}$ with N particles is defined by

$$H_N^\ell(\mathbf{z}) = \sum_{j\neq k} U^\ell(z_j - z_k), \quad (\mathbf{z} \in \mathbb{T}^N). \tag{2.5}$$

The corresponding Gibbs measure and variational functional on probability measures are defined as in Section 1. The minimum energy of the corresponding variational functional is given by

$$\inf_\mu \int U^\ell(z-w)\,\mu(\mathrm{d}z)\,\mu(\mathrm{d}w) = 2\pi\ell^2 \tag{2.6}$$

where the infimum is over the probability measures on $\mathbb{T}$. The equality follows since the unique minimizer in (2.6) is the uniform measure on the torus, which follows from translation invariance. We denote the partition function of the Yukawa gas on the unit torus with range ℓ by

$$Z_N^{(\ell)} = \int_{\mathbb{T}^N} \mathrm{e}^{-\beta H_N^\ell(\mathbf{z})}\, m(\mathrm{d}\mathbf{z}).$$

The main result of this section is the following theorem, a version of Theorem 1.1 for the Yukawa gas on the torus.

Theorem 2.1. *There is a constant $\zeta = \zeta(\beta)$, the residual free energy of the long-range torus Yukawa gas, such that for any $\sigma > 0$ there is $\kappa > 0$ such that if $N^{-1/2+\sigma} \leqslant \ell \ll 1$,*

$$\frac{1}{\beta}\log Z_N^{(\ell)} = -2\pi\ell^2N^2 + N\log\ell + \frac{1}{2}N\log N + N\zeta + \mathrm{O}(N^{1-\kappa}). \tag{2.7}$$

More precisely, $\mathrm{O}(N^{1-\kappa})$ is $N^\varepsilon\mathrm{O}(N^{7/8} + N^{1-2\sigma})$.

Remark 2.2. *The above statement holds without the assumption $\ell \ll 1$. However, for our application, this generalization is not needed, and we thus restrict to this slightly simplified case.*

To prove Theorem 2.1, we define

$$\zeta^{(\ell)}(N) = \frac{1}{N}\xi^{(\ell)}(N) - \frac{1}{2}\log N, \quad \xi^{(\ell)}(N) = \frac{1}{\beta}\log Z_N^{(\ell)} + 2\pi\ell^2N^2 - N\log\ell. \tag{2.8}$$

In this notation, Theorem 2.1 asserts that $\zeta^\ell(N) = \zeta + \mathrm{O}(N^{1-\kappa})$ whenever $\ell \geqslant N^{-1/2+\sigma}$.

Along this section and in Section 3, we will repeatedly use the Jensen inequality in the form

$$\log\int \mathrm{e}^{-B} + \mathbb{E}^B(B-A) \leq \log\int \mathrm{e}^{-A} \leq \log\int \mathrm{e}^{-B} + \mathbb{E}^A(B-A), \tag{2.9}$$

where $\mathbb{E}^A X = \frac{\int \mathrm{e}^{-A}X}{\int \mathrm{e}^{-A}}$ and integration is with respect to a fixed measure.

2.2. Continuity and scaling relation of the residual free energy on a torus. In Lemmas 2.3–2.4 below, it is proved that $\zeta^{(\ell)}(N)$ is almost independent of the range ℓ, provided that $\ell \gg N^{-1/2}$, and that $\zeta^{(\ell)}(N)$ depends only weakly on the number of particles N. In Lemma 2.5, we further state a scaling relation for the Yukawa gas on a torus of side length b.

Lemma 2.3. *For any $\sigma > 0$, if $N^{-1/2+\sigma} \leqslant \nu \leqslant \omega \ll 1$ then*

$$\mathrm{O}(N^{-2\sigma+\varepsilon}) \leqslant \zeta^{(\omega)}(N) - \zeta^{(\nu)}(N) \leqslant 0. \tag{2.10}$$

Proof. We start with the upper bound on $\zeta^{(\omega)} - \zeta^{(\nu)}$. By Jensen's inequality,

$$\frac{1}{\beta}\log\int \mathrm{e}^{-\beta H_N^{\omega}(\mathbf{z})}\, m(\mathrm{d}\mathbf{z}) \leq \frac{1}{\beta}\log\int \mathrm{e}^{-\beta H_N^{\nu}(\mathbf{z})}\, m(\mathrm{d}\mathbf{z}) - \mathbb{E}^{H_N^{\omega}}[H_N^{\omega} - H_N^{\nu}]. \tag{2.11}$$

Let $L_\omega^\nu(z) = U^\omega(z) - U^\nu(z)$. Then, since $\int U^\ell(z)\, m(\mathrm{d}z) = 2\pi\ell^2$ and $L_\omega^\nu(0) = \log(\omega/\nu)$,

$$\sum_{j\neq k} L_\omega^\nu(z_j - z_k) = 2\pi(\omega^2 - \nu^2)N^2 - N\log(\omega/\nu) + N^2\mathbf{L}_\omega^\nu, \tag{2.12}$$

where

$$\mathbf{L}_\omega^\nu = \int L_\omega^\nu(z-w)\, \tilde{\mu}(\mathrm{d}w)\, \tilde{\mu}(\mathrm{d}z) \geqslant 0, \tag{2.13}$$

and the last inequality follows since L is positive definite, as can be verified by representating it in Fourier space. This completes the proof that $\zeta^{(\omega)}(N) - \zeta^{(\nu)}(N) \leqslant 0$.

For the lower bound, we use the Jensen inequality in the reverse direction, by which we have

$$\frac{1}{\beta}\log\int \mathrm{e}^{-\beta H_N^{\omega}(\mathbf{z})}\, m(\mathrm{d}\mathbf{z}) \geq \frac{1}{\beta}\log\int \mathrm{e}^{-\beta H_N^{\nu}(\mathbf{z})}\, m(\mathrm{d}\mathbf{z}) - 2\pi(\omega^2-\nu^2)N^2 + N\log(\omega/\nu) + N^2\mathbb{E}^{H_N^{\nu}}\mathbf{L}_\nu^\omega. \tag{2.14}$$

In the case that $\omega = \nu N^{c\varepsilon}$, the rigidity estimate for the Yukawa gas on the torus, Proposition B.3 in the form (B.5) with $G(z,w) = L_\omega^\nu(z,w)$, implies that

$$N^2\mathbb{E}^{H_N^{\nu}}\mathbf{L}_\omega^\nu = N^\varepsilon\mathrm{O}(\nu^{-2}). \tag{2.15}$$

Therefore with $\nu_1 = \omega$, $\nu_k = \nu$ and $N^{-c\varepsilon} \geqslant \nu_{j+1}/\nu_j \geqslant 1$, iterating the previous estimate and using that $\sum_{j=1}^{k-1}\nu_j^2\nu_{j+1}^{-4} = N^\varepsilon\mathrm{O}(\nu^{-2})$,

$$\frac{1}{\beta}\log\int \mathrm{e}^{-\beta H_N^{\omega}(\mathbf{z})}\, m(\mathrm{d}\mathbf{z}) \geq \frac{1}{\beta}\log\int \mathrm{e}^{-\beta H_N^{\nu}(\mathbf{z})}\, m(\mathrm{d}\mathbf{z}) - 2\pi(\omega^2-\nu^2)N^2 + N\log(\omega/\nu) + N^\varepsilon\mathrm{O}(\nu^{-2}).$$

Since $\nu^{-2} \leqslant N^{1-2\sigma}$ by assumption, this completes the proof. □

Lemma 2.4. *The torus residual free energy satisfies*

$$\zeta^{(\gamma)}(n) - \zeta^{(\gamma)}(m) = \mathrm{O}\left(|m-n|\frac{\log(n+m)}{n+m}\right). \tag{2.16}$$

Proof. We first note the bound $\xi_1^{(\gamma)}(n) = \mathrm{O}(n\log n)$. This bound follows exactly as in [6, Proposition 4.1] or (A.57), by smearing out the point charges into densities and positive definiteness (for the upper bound) and by Jensen's inequality (for the lower bound). Using this bound, we will now prove the following more precise version:

$$\xi_1^{(\gamma)}(n) + 2\pi\gamma^2 - \log\gamma \le \xi_1^{(\gamma)}(n+1) \le \xi_1^{(\gamma)}(n) + \mathrm{O}(\log n).$$

On the unit torus, by Jensen's inequality,

$$\log \frac{\int e^{-\beta \sum_{i\neq j; i,j=1}^{n+1} U^\gamma(z_i - z_j)} m(\mathrm{d}\mathbf{z})}{\int \mathrm{e}^{-\beta \sum_{i\neq j; i,j=1}^{n} U^\gamma(z_i - z_j)} m(\mathrm{d}\mathbf{z})} \ge -2\beta \mathbb{E}_n^\gamma \sum_{j=1}^n U^\gamma(z_{n+1} - z_j).$$

Integrating both sides over z_{n+1}, and again using Jensen's inequality, we get

$$\log Z_{n+1} \ge \int m(\mathrm{d}z_{n+1}) \log \int \mathrm{e}^{-\beta \sum_{i\neq j; i,j=1}^{n+1} U^\gamma(z_i - z_j)} m(\mathrm{d}\mathbf{z}) \ge \log Z_n - (2n\beta)(2\pi\gamma^2).$$

By the definition of $\xi_1^{(\gamma)}(n)$, it follows that

$$\begin{aligned} \xi_1^{(\gamma)}(n+1) &= 2\pi\gamma^2(n+1)^2 - (n+1)\log\gamma + \frac{1}{\beta}\log Z_{n+1}(\beta) \\ &\ge 2\pi\gamma^2(n+1)^2 - 2n(2\pi\gamma^2) + \frac{1}{\beta}\log Z_n(\beta) - (n+1)\log\gamma = \xi_1^{(\gamma)}(n) + 2\pi\gamma^2 - \log\gamma. \end{aligned}$$

For the other direction, set $\hat{H}_k = \sum_{i\neq j, i,j\neq k}^{n+1} U^\gamma(z_i - z_j)$. Then, by Hölder's inequality,

$$Z_{n+1}(\beta) = \int \exp\Big[-\frac{\beta}{n-1}\sum_{k=1}^{n+1}\hat{H}_k\Big] m(\mathrm{d}\mathbf{z}) \le \int \mathrm{e}^{-\beta\frac{n+1}{n-1}\hat{H}_k} m(\mathrm{d}\mathbf{z}) = Z_n(\beta\frac{n+1}{n-1}).$$

Since $\xi_1^{(\gamma)}(n) = \mathrm{O}(n\log n)$, we have

$$\frac{1}{\beta}\log\int \mathrm{e}^{-\beta H_n} m(\mathrm{d}\mathbf{z}) = -2\pi\gamma^2 n^2 + \mathrm{O}(n\log n), \quad H_n = \sum_{i\neq j}^n U^\gamma(z_i - z_j).$$

By convexity of the function $t \to \log\int \mathrm{e}^{-tH_n} m(\mathrm{d}\mathbf{z})$, we have

$$-\mathbb{E}_n^{\gamma,\beta} H_n \le \log\int \mathrm{e}^{-(\beta+1)H_n} m(\mathrm{d}\mathbf{z}) - \log\int \mathrm{e}^{-\beta H_n} m(\mathrm{d}\mathbf{z}) \le -2\pi\gamma^2 n^2 + \mathrm{O}(n\log n).$$

Integrating the relation $\partial_\beta \log Z_n(\beta) = -\mathbb{E}_n^{\gamma,\beta} H_n$, we therefore get

$$\begin{aligned} \log Z_{n+1}(\beta) \le \log Z_n\left(\beta\frac{n+1}{n-1}\right) &= \log Z_n(\beta) - \int_\beta^{\beta\frac{n+1}{n-1}} \mathbb{E}_n^{\gamma,s} H_n \,\mathrm{d}s \\ &\le \log Z_n(\beta) - 2\pi\gamma^2\frac{2n^2\beta}{n-1} + \mathrm{O}(n\log n). \end{aligned}$$

In summary, we have proved that

$$\begin{aligned} \xi_1^{(\gamma)}(n+1) &= 2\pi\gamma^2(n+1)^2 - (n+1)\log\gamma + \frac{1}{\beta}\log Z_{n+1}(\beta) \\ &\le 2\pi\gamma^2(n+1)^2 + \frac{1}{\beta}\log Z_n(\beta) - 2\pi\gamma^2\frac{2n^2}{n-1} - n\log\gamma - \log\gamma + \mathrm{O}(\log n) \\ &= \xi_1^{(\gamma)}(n) + \mathrm{O}(\log n). \end{aligned}$$

The claim now follows from the definition of $\zeta^{(\gamma)}$ in (2.8). □

We also record the following scaling relation for the Yukawa gas. On the torus of side length b, the Yukawa interaction is given by

$$U_b^\ell(z) = U^{\ell/b}(z/b). \tag{2.17}$$

Here and below we denote the relative interaction range by $\gamma = \ell/b$ and write

$$\xi_b^{(\gamma)}(n) = \frac{1}{\beta}\log Z_{b,n}^{(\gamma)} + 2\pi\gamma^2 n^2 - n\log\ell, \quad Z_{b,n}^{(\gamma)} = \int_{\mathbb{T}_b^n} \mathrm{e}^{-\beta\sum_{i\neq j} U_b^\ell(w_i-w_j)}\, m(\mathrm{d}w). \tag{2.18}$$

Lemma 2.5. *For any $K>0$,*

$$\xi_{Kb}^{(\gamma)}(n) = \Big(\frac{1}{\beta}-\frac{1}{2}\Big) n\log K^2 + \xi_b^{(\gamma)}(n). \tag{2.19}$$

In particular, by choosing $K = b^{-1}$, with the definition of ζ from (2.8)*,*

$$\xi_b^{(\gamma)}(n) = n\zeta^{(\gamma)}(n) + \frac{n}{2}\log n + n\Big(\frac{1}{2}-\frac{1}{\beta}\Big)\log b^{-2}. \tag{2.20}$$

Proof. By definition of the Yukawa potential (2.17), $U_{Kb}^{K\ell}(Kr) = U_b^\ell(r)$. Therefore, by changing variables to $z = wK$,

$$\begin{aligned}\frac{1}{\beta}Z_{n,Kb}^{(\gamma)} &= \frac{1}{\beta}\log\int_{|z_i|\leq Kb/2} \mathrm{e}^{-\beta\sum_{i\neq j}^n U_{Kb}^{K\ell}(z_i-z_j)}\, m(\mathrm{d}\mathbf{z})\\ &= \frac{1}{\beta}\log\int_{|w_i|\leq b/2} \mathrm{e}^{-\beta\sum_{i\neq j} U_b^\ell(w_i-w_j)}\, m(\mathrm{d}\mathbf{w}) + \frac{1}{\beta}n\log K^2 = \frac{1}{\beta}\log Z_{n,b}^{(\gamma)} + \frac{1}{\beta}n\log K^2,\end{aligned}$$

where the term with $\log K^2$ comes from the scaling factor in the Jacobian. With $\gamma = \ell/b$ and using the definition (2.18) of ξ, we have the rescaling identity

$$\xi_{Kb}^{(\gamma)}(n) = 2\pi\gamma^2 n^2 - n\log K\ell + \frac{1}{\beta}n\log K^2 + \frac{1}{\beta}\log Z_{n,b}^{(\gamma)} = \Big(\frac{1}{\beta}-\frac{1}{2}\Big)n\log K^2 + \xi_b^{(\gamma)}(n)$$

as claimed. □

2.3. Quasi-free approximation. To prove Theorem 2.1, by Lemma 2.3, we may assume that the interaction range ℓ is replaced by $\ell \geqslant N^{-1/2+\sigma}$ for an arbitrary fixed $\sigma > 0$.

In the following, we identify the unit torus with the square $[-1/2, 1/2)^2$. For b with $\ell \ll b \ll 1$ and such that $1/b$ and Nb^2 are both integers, we then divide the unit torus into squares α of side length b. We consider the quasi-free Yukawa interaction defined by removing the interaction between particles in a square with particles outside that square and replacing the interaction between particles in the same square by a periodic one. To state the resulting estimate, denote by $\mathbf{n} = (n_\alpha)$ a particle profile, i.e., an assignment of the number of particles in each square, satisfying $\sum_\alpha n_\alpha = N$. We define the quasi-free free energy for particle profile $\mathbf{n}$ by

$$F(\mathbf{n}) = \frac{1}{\beta}\log\binom{N}{\mathbf{n}} + \frac{1}{\beta}\sum_\alpha \log\int_{\mathbb{T}_\alpha^{n_\alpha}} \mathrm{e}^{-\beta\hat{H}_\alpha(\mathbf{u})}\, m(\mathrm{d}\mathbf{u}), \tag{2.21}$$

where $\mathbb{T}_\alpha$ is a torus of side length b associated to the square α. The name quasi-free stems from the fact that particles in different squares do not interact. Here $\hat{H}_\alpha$ is a Hamiltonian on the torus $\mathbb{T}_\alpha$ defined by

$$\hat{H}_\alpha(\mathbf{u}) = \sum_{i \neq j} U_\alpha^\ell(u_i - u_j),$$

where U_α^ℓ is the periodic Yukawa interaction on $\mathbb{T}_\alpha$. The term

$$\binom{N}{\mathbf{n}} = \frac{N!}{\prod_\alpha n_\alpha!}$$

arises as the number of ways to distribute N particles into groups of sizes (n_α) with $\sum_\alpha n_\alpha = N$. Moreover, we denote by $\bar{\mathbf{n}} = (\bar{n}_\alpha)$ with $\bar{n}_\alpha = \bar{n} = Nb^2$ the mean number of particles in α. For $z \in \mathbb{C}^N$, we define $\mathbf{n}(\mathbf{z}) = (n_\alpha(\mathbf{z}))$ where $n_\alpha(\mathbf{z})$ is the number of particles $z_j \in \alpha$.

Proposition 2.6 (Upper Bound)**.**

$$\frac{1}{\beta} \log \int \mathrm{e}^{-\beta H^\ell(\mathbf{z})}\, m(\mathrm{d}\mathbf{z}) \leqslant \frac{1}{\beta} \log \sum_{\mathbf{n}} \mathrm{e}^{\beta F(\mathbf{n})} + N^\varepsilon \mathrm{O}(N^2 \ell^3 b^{-1}). \tag{2.22}$$

Proposition 2.7 (Lower Bound)**.**

$$\frac{1}{\beta} \log \int \mathrm{e}^{-\beta H^\ell(\mathbf{z})}\, m(\mathrm{d}\mathbf{z}) \geq F(\bar{\mathbf{n}}) + N^\varepsilon \mathrm{O}(N^2 \ell^3 b^{-1}). \tag{2.23}$$

These two propositions and their proofs are simplified versions of Propositions 3.5, 3.6 and their proofs. To avoid duplication, we only sketch their proofs and the modifications here.

Proof of Proposition 2.6. Denote by

$$\Phi_\alpha : \alpha \to \mathbb{T}_\alpha, \quad \text{the natural embedding from the square } \alpha \text{ into the torus } \mathbb{T}_\alpha \tag{2.24}$$

(mapping the boundary of α to a vertical and a horizontal line in $\mathbb{T}_\alpha$). More precisely, the division of the (original unit) torus $\mathbb{T}$ into squares requires a choice of origin, which we parametrize by $u \in [-b/2, b/2)^2$, thus defining a map $\Phi = \Phi^u$. By translation-invariance in $\mathbb{T}$, we may in the end average over the choice of u. For $z, w \in \mathbb{T}$, we define the Hamiltonian for configurations on $\mathbb{C}$ through the embedding Φ by

$$\tilde{Y}_u^\ell(z, w) = \sum_\alpha U_\alpha^\ell(\Phi_\alpha^u(z), \Phi_\alpha^u(w)) \mathbf{1}_{z \in \alpha} \mathbf{1}_{w \in \alpha}, \tag{2.25}$$

and consider the Hamiltonian $\tilde{H}_u^\ell$ with pair interaction $\tilde{Y}_u^\ell$,

$$\tilde{H}_u^\ell(\mathbf{z}) = \sum_{i \neq j} \tilde{Y}_u^\ell(z_i, z_j). \tag{2.26}$$

Thus the interaction between particles is periodic in each square α and vanishes if the two particles are in different squares. Finally, we define the function $\bar{Y}$ as the average of $\tilde{Y}$ over the choice of origin u of the division into squares: for $z, w \in \mathbb{T}$,

$$\bar{Y}(z, w) = \frac{1}{b^2} \int_{[-b/2, b/2]^2} \mathrm{d}u\, \tilde{Y}_u^\ell(z, w). \tag{2.27}$$

In fact, $\bar{Y}(z,w)$ is a function of $z-w$, which is given by $g(z-w)+\mathrm{O}(\mathrm{e}^{-cb/\ell})$ with g as in (3.20); see the proof of Lemma 3.9 (i). By Jensen's inequality and then averaging over u,

$$\frac{1}{\beta}\log\int \mathrm{e}^{-\beta H^\ell(\mathbf{z})}\,m(\mathrm{d}\mathbf{z}) \leq \frac{1}{\beta}\mathbb{E}^u \log\int \mathrm{e}^{-\beta \tilde{H}^\ell_u(\mathbf{z})}\,m(\mathrm{d}\mathbf{z}) + \mathbb{E}^u\mathbb{E}^{H^\ell}(\tilde{H}^\ell_u - H^\ell). \tag{2.28}$$

The second term on the right-hand side is $\mathbb{E}^{H^\ell}\sum_{i\neq j}[\bar{Y}(z_i,z_j)-Y^\ell(z_i-z_j)]$. Exactly as in the first bound in Lemma 3.10, this term is bounded by $N^\varepsilon \mathrm{O}(N^2\ell^3 b^{-1})$. Since

$$\int \mathrm{e}^{-\beta\tilde{H}^\ell_u(\mathbf{z})}\,m(\mathrm{d}\mathbf{z}) = \sum_{\mathbf{n}}\binom{N}{\mathbf{n}}\prod_\alpha \int_{\mathbb{T}_\alpha^{n_\alpha}} \mathrm{e}^{-\beta\hat{H}_\alpha(\mathbf{u})}\,m(\mathrm{d}\mathbf{u}),$$

this completes the sketch of the proof. □

Proof of Proposition 2.7. To obtain a lower bound on the partition function, we can restrict the particle numbers in all squares α to their mean $\bar{n}_\alpha = Nb^2$. Thus define the indicator function

$$\hat{\chi}(\mathbf{z}) = \prod_\alpha \mathbf{1}\Big(n_\alpha(\mathbf{z}) = \bar{n}_\alpha\Big). \tag{2.29}$$

Trivially,

$$\frac{1}{\beta}\log\int \mathrm{e}^{-\beta H^\ell(\mathbf{z})}\,m(\mathrm{d}\mathbf{z}) \geq \frac{1}{\beta}\log\int \mathrm{e}^{-\beta H^\ell(\mathbf{z})}\,\hat{\chi}(\mathbf{z})\,m(\mathrm{d}\mathbf{z}). \tag{2.30}$$

Next we break the permutation symmetry. Ordering the squares α arbitrarily as $\alpha_1,\alpha_2,\dots$, we write $\tilde{\chi}(\mathbf{z})$ for $\hat{\chi}(\mathbf{z})$ multiplied by the indicator function of the event in which the particles $z_1,\dots,z_{\bar{n}}$ are in α_1, the particles $z_{\bar{n}+1},\dots,z_{2\bar{n}}$ are in α_2, and so forth. Then

$$\frac{1}{\beta}\log\int \mathrm{e}^{-\beta H^\ell(\mathbf{z})}\,\hat{\chi}(\mathbf{n}(\mathbf{z}))\,m(\mathrm{d}\mathbf{z}) = \frac{1}{\beta}\log\binom{N}{\mathbf{n}} + \frac{1}{\beta}\log\int \mathrm{e}^{-\beta H^\ell(\mathbf{z})}\,\tilde{\chi}(\mathbf{z})\,m(\mathrm{d}\mathbf{z}).$$

Let $\Psi_\alpha : \mathbb{T}_\alpha \to \alpha$ be the flat embedding from the torus into the square α that is smooth except along a horizontal and a vertical line; similarly as in the upper bound, the definition of this map requires a choice of origin (of the two lines in $\mathbb{T}_\alpha$ along which the embedding is discontinuous), over which we will average in the end. Defining $\hat{H}(\mathbf{u}) = \sum_\alpha \hat{H}_\alpha(\mathbf{u}^\alpha)$, by Jensen's inequality and averaging over the choice of origin in $\mathbb{T}_\alpha$, we have

$$\frac{1}{\beta}\log\int \mathrm{e}^{-\beta H^\ell(\mathbf{z})}\tilde{\chi}(\mathbf{z})\,m(\mathrm{d}\mathbf{z}) \geq \frac{1}{\beta}\log\int \mathrm{e}^{-\beta \hat{H}^\ell(\mathbf{z})}\tilde{\chi}(\mathbf{z})\,m(\mathrm{d}\mathbf{z}) + \hat{\mathbb{E}}(\hat{H}^\ell(\mathbf{u}) - H^\ell(\Psi\mathbf{u})), \tag{2.31}$$

where $\hat{\mathbb{E}}$ denotes the expectation of independent Coulomb gases on the tori $\mathbb{T}_\alpha$ and the independent choices of origin. Exactly as in (3.41), omitting the errors from the nonconstant density and the boundary, and using the alternative bound (3.57) for E, the last term on the right-hand side is bounded by

$$\hat{\mathbb{E}}(\hat{H}(\mathbf{u}) - H^\ell(\Psi(\mathbf{u}))) = \mathrm{O}(N^2\ell^3 b^{-1}). \tag{2.32}$$

This concludes the sketch of the proof. □

2.4. Quasi-free approximation concluded. The main consequence of the quasi-free approximation for the torus is Proposition 2.10 below. In preparation, we need two elementary lemmas.

Lemma 2.8. *Let*

$$h_\alpha(\mathbf{n}) = 2\pi\gamma^2(n_\alpha - \bar{n})^2 - n_\alpha \zeta^{(\gamma)}(n_\alpha) - \frac{1}{2} n_\alpha \log n_\alpha - \Big(\frac{1}{2} - \frac{1}{\beta}\Big) n_\alpha \log b^{-2}. \tag{2.33}$$

Then

$$F(\mathbf{n}) = \frac{1}{\beta} \log \binom{N}{\mathbf{n}} + 2\pi\ell^2 N^2 + \sum_\alpha h_\alpha(\mathbf{n}) - N \log \ell. \tag{2.34}$$

Proof. From (2.21) and (2.18), recall that $F(\mathbf{n}) = \frac{1}{\beta}\log\binom{N}{\mathbf{n}} - \sum_\alpha T_\alpha(n_\alpha)$, where

$$T_\alpha(n_\alpha) := -\frac{1}{\beta} \log \int_{\mathbb{T}_\alpha^{n_\alpha}} \mathrm{e}^{-\beta \sum_{j \neq k} U_\alpha^\ell (z_j - z_k)}\, m(\mathrm{d}\mathbf{z}) = 2\pi\gamma^2 n_\alpha^2 - n_\alpha \log \ell - \xi_b^{(\gamma)}(n_\alpha).$$

By the scaling relation (2.20), we also have $h_\alpha(\mathbf{n}) = 2\pi\gamma^2(n_\alpha - \bar{n})^2 - \xi_b^{(\gamma)}(n_\alpha)$. The equality

$$\sum_\alpha 2\pi\gamma^2 n_\alpha^2 = 2\pi\gamma^2 \sum_\alpha (n_\alpha - \bar{n})^2 + 2\pi\ell^2 N^2$$

therefore implies

$$\sum_\alpha T_\alpha(n_\alpha) = 2\pi\ell^2 N^2 + \sum_\alpha h_\alpha(\mathbf{n}) - N \log \ell.$$

This completes the proof. □

Lemma 2.9. *Assume that $|\mathcal{E}_\alpha(n) - \mathcal{E}_\alpha(m)| \leqslant |n-m|(n+m)^\varepsilon$. Then (recall that $\gamma = \ell/b$)*

$$\frac{1}{\beta} \log \sum_{\mathbf{n}} \mathrm{e}^{\beta \mathcal{E}(\mathbf{n})} \leq \mathcal{E}(\bar{\mathbf{n}}) + N^\varepsilon \mathrm{O}(\ell^{-2}), \qquad \mathcal{E}(\mathbf{n}) := \sum_\alpha \Big[-2\pi\gamma^2(n_\alpha - \bar{n})^2 + \mathcal{E}_\alpha(n_\alpha) \Big]. \tag{2.35}$$

Proof. The lemma is a special case of Lemma 3.21 in Section 3, so we omit the proof here. □

Proposition 2.10. *For any $\sigma > 0$, there is $\tau > 0$ such that if $\ell \geqslant N^{-1/2+\sigma}$ and $1 \geqslant b \geqslant N^{1+\sigma}\ell^3$,*

$$\zeta^{(\ell)}(N) = \zeta^{(\ell/b)}(Nb^2) + \mathrm{O}(N^{-\tau}). \tag{2.36}$$

More precisely, $\mathrm{O}(N^{-\tau})$ *is* $N^\varepsilon \mathrm{O}(N\ell^3/b + 1/(N\ell^2))$.

Proof. The assumptions on ℓ and b imply that the error terms in (2.22), (2.23) are $\mathrm{O}(N^{1-\tau})$. By Propositions 2.6, 2.7, together with Lemma 2.8, therefore

$$\frac{1}{\beta} \log \int \mathrm{e}^{-\beta H^\ell(\mathbf{z})}\, m(\mathrm{d}\mathbf{z}) \geqslant -2\pi\ell^2 N^2 + N \log \ell + \frac{1}{\beta} \log \binom{N}{\bar{\mathbf{n}}} \mathrm{e}^{-\beta \sum_\alpha h_\alpha(\bar{\mathbf{n}})} - \mathrm{O}(N^{1-\tau}), \tag{2.37}$$

$$\frac{1}{\beta} \log \int \mathrm{e}^{-\beta H^\ell(\mathbf{z})}\, m(\mathrm{d}\mathbf{z}) \leqslant -2\pi\ell^2 N^2 + N \log \ell + \frac{1}{\beta} \log \sum_{\mathbf{n}} \binom{N}{\mathbf{n}} \mathrm{e}^{-\beta \sum_\alpha h_\alpha(\mathbf{n})} + \mathrm{O}(N^{1-\tau}). \tag{2.38}$$

We compute the sums on the right-hand sides of (2.37), (2.38). By Stirling's formula,

$$\log \binom{N}{\mathbf{n}} = N \log N - \sum_\alpha n_\alpha \log n_\alpha + \mathrm{O}(\log N). \tag{2.39}$$

With $\mathcal{E}_\alpha(n_\alpha)=(\frac{1}{2}-\frac{1}{\beta})n_\alpha\log(n_\alpha b^{-2})+n_\alpha\zeta^{(\gamma)}(n_\alpha)$ and $\mathcal{E}$ of (2.35), we rewrite (2.37), (2.38) as

$$\frac{1}{\beta}\log\int \mathrm{e}^{-\beta H^\ell(\mathbf{z})}\,m(\mathrm{d}\mathbf{z})+2\pi\ell^2N^2-N\log\ell\geqslant \mathcal{E}(\bar{\mathbf{n}})+\frac{1}{\beta}N\log N+\mathrm{O}(N^{1-\tau}),$$
$$\frac{1}{\beta}\log\int \mathrm{e}^{-\beta H^\ell(\mathbf{z})}\,m(\mathrm{d}\mathbf{z})+2\pi\ell^2N^2-N\log\ell\leqslant \frac{1}{\beta}\log\sum_{\mathbf{n}}\mathrm{e}^{\beta\mathcal{E}(\bar{\mathbf{n}})}+\frac{1}{\beta}N\log N+\mathrm{O}(N^{1-\tau}),$$

By Lemma 2.4, $\mathcal{E}_\alpha$ satisfies the assumption of Lemma 2.9. Lemma 2.9 then shows that the sum over $\mathbf{n}$ can be estimated by its dominant term $\bar{\mathbf{n}}$ with error $N^\varepsilon\mathrm{O}(\ell^{-2})=\mathrm{O}(N^{1-\tau})$. Since

$$\mathcal{E}(\bar{\mathbf{n}})=\left(\frac{1}{2}-\frac{1}{\beta}\right)\sum_\alpha \bar{n}\log(\bar{n}b^{-2})+\sum_\alpha\bar{n}\zeta^{(\gamma)}(\bar{n})=\left(\frac{1}{2}-\frac{1}{\beta}\right)N\log N+N\zeta^{(\gamma)}(Nb^2),$$

this replacement yields

$$\frac{1}{\beta}\log\int \mathrm{e}^{-\beta H_V^\ell(\mathbf{z})}\,m(\mathrm{d}\mathbf{z})+2\pi\ell^2N^2-N\log\ell=\frac{1}{2}N\log N+N\zeta^{(\gamma)}(Nb^2)+\mathrm{O}(N^{1-\tau}),$$

which completes the proof of (2.36). □

2.5. Existence of torus residual free energy: proof of Theorem 2.1. We now prove Theorem 2.1. For this, we need the next lemma which uses (2.36) to improve the estimate (2.16).

Lemma 2.11. *For any $\sigma>0$ there exists $\tau>0$ such that for ν with $n^{-1/2+\sigma}\leqslant\nu\leqslant n^{-1/3-\sigma}$,*

$$\max_{\tilde{n}\in[n,2n]}|\zeta^{(\nu)}(n)-\zeta^{(\nu)}(\tilde{n})|=\mathrm{O}(n^{-\tau}). \tag{2.40}$$

More precisely, $\mathrm{O}(n^{-\tau})$ *is* $n^\varepsilon\mathrm{O}(n\nu^3+1/(\nu\sqrt{n}))$.

Proof. For $u>0$, we set $b=n^{-u}$ and assume that $B=1/b$ is an integer; we further set $N=B^2n$ and $\ell=b\nu$. With $u>0$ sufficiently small (depending on σ), the assumptions of Proposition 2.10 are satisfied (with parameters ℓ,b and N). Similarly, we define $\tilde{b},\tilde{B},\tilde{N},\tilde{\ell}$ with n replaced by $\tilde{n}$ and u replaced by $\tilde{u}$. As a consequence, with $N^\varepsilon\mathrm{O}(N\ell^3/b+1/(N\ell^2))=\mathrm{O}(n^{-\tau})$,

$$\zeta^{(\ell)}(B^2n)=\zeta^{(\nu)}(n)+\mathrm{O}(n^{-\tau}), \tag{2.41}$$
$$\zeta^{(\tilde{\ell})}(\tilde{B}^2\tilde{n})=\zeta^{(\nu)}(\tilde{n})+\mathrm{O}(n^{-\tau}), \tag{2.42}$$

Further, by (2.10), with $n^\varepsilon\mathrm{O}(1/(N\ell^2))=n^\varepsilon\mathrm{O}(1/(n\nu^2))=n^\varepsilon\mathrm{O}(1/(\sqrt{n}\nu))=\mathrm{O}(n^{-\tau})$,

$$|\zeta^{(\tilde{\ell})}(\tilde{B}^2\tilde{n})-\zeta^{(\ell)}(\tilde{B}^2\tilde{n})|=O(n^{-\tau}). \tag{2.43}$$

Writing $\tilde{n}=Mn$ and choosing B and $\tilde{B}$ such that $|\tilde{B}\sqrt{M}-B|\leq 1$, we have $|\tilde{B}^2M-B^2|\leq\mathrm{O}(B)$ and (2.16) implies

$$|\zeta^{(\ell)}(B^2n)-\zeta^{(\ell)}(\tilde{B}^2\tilde{n})|\leq\frac{|\tilde{B}^2\tilde{n}-B^2n|}{|\tilde{B}^2\tilde{n}+B^2n|^{1-\varepsilon}}=\frac{|\tilde{B}^2M-B^2|n^\varepsilon}{|\tilde{B}^2M+B^2|^{1-\varepsilon}}\leq\frac{\mathrm{O}(n^\varepsilon)}{B^{1-2\varepsilon}}=n^{2\varepsilon}\mathrm{O}(b)=\mathrm{O}(n^{-\tau}), \tag{2.44}$$

where the last inequality follows from $n^\varepsilon\mathrm{O}(b)=n^\varepsilon\mathrm{O}(1/(\sqrt{n}\nu))=\mathrm{O}(n^{-\tau})$.

Finally, by combining the estimates (2.41), (2.42), (2.43), (2.44), we obtain

$$|\zeta^{(\nu)}(n)-\zeta^{(\nu)}(\tilde{n})|=|\zeta^{(\ell)}(B^2n)-\zeta^{(\tilde{\ell})}(\tilde{B}^2\tilde{n})|+\mathrm{O}(n^{-\tau})=\mathrm{O}(n^{-\tau}),$$

proving (2.40). □

Proof of Theorem 2.1. For $j \in \mathbb{N}$, define the sequences $n_j = 2^j$, $\nu_j = 2^{-cj}$ and $\zeta_j = \zeta^{(\nu_j)}(n_j)$. For any fixed $c \in (1/3, 1/2)$, the assumptions of Lemma 2.11 are satisfied with $n = n_j$ and $\nu = \nu_j$. Therefore, together with (2.10),

$$\left|\zeta_i - \zeta_k\right| \leqslant \sum_{j=i}^{k-1} |\zeta^{(\nu_j)}(n_j) - \zeta^{(\nu_j)}(n_{j+1})| + \sum_{j=i}^{k-1} |\zeta^{(\nu_j)}(n_j) - \zeta^{(\nu_{j+1})}(n_{j+1})| \leqslant \sum_{j=i}^{k-1} \mathrm{O}(n_j^{-\tau}) = \mathrm{O}(2^{-i\tau})$$

for all $k \geqslant i \geqslant i_0$ and $\tau = 1/8 + \varepsilon$, since $\sum_{j=i}^{k-1} \mathrm{O}(n_j \nu_j^3 + 1/(\nu_j \sqrt{n_j})) = \mathrm{O}(2^{-i/8})$ for the optimal $c = 3/8$. This implies the existence of the limit $\lim_{j\to\infty} \zeta_j = \zeta$ and $\zeta_j = \zeta + \mathrm{O}(2^{-j\tau})$. Finally, by (2.10) in the first equality and (2.40) in the second,

$$\zeta^{(\ell)}(N) = \zeta^{(\nu_1)}(N) + \mathrm{O}(N^{-2\sigma+\varepsilon}) = \zeta_{j_N} + \mathrm{O}(N^{-\kappa}) = \zeta + \mathrm{O}(N^{-\kappa}),$$

where j_N is the smallest integer j such that $2^j \geqslant N$, and that $N^\varepsilon \mathrm{O}(N^{-1/8} + N^{-2\sigma}) = \mathrm{O}(N^{-\kappa})$. This completes the proof. □

3 Proof of Theorem 1.1: quasi-free approximation

In this section, we prove Theorem 1.1. We follow the same strategy as in Section 2. Differences are that now we take into account that the equilibrium measure can be non-constant and effects from the boundary as well. As in the torus case, the upper bound for the partition function can be established by using the Jensen inequality and the positive definiteness of the Coulomb potential. The lower bound involving estimating the Coulomb energy near the surface, as explained in the introduction, is the main difficulty in this section. We will explain the ideas to resolve this and related issues along the proof in this section.

Remark 3.1. *The argument given in this section can be applied to a more general setting than that stated in Theorem 1.1. Assume that the equilibrium measure can be decomposed into $\mu_V(\mathrm{d}z) = \rho_V(z)\, m(\mathrm{d}z) + v(z)\,\mathrm{d}s$, where $\mathrm{d}s$ is the length measure on ∂S_V, and that there is a domain $\Omega \subset S_V$ (permitted to depend on N) such that the following conditions hold. For some constants $a > 0, A \geqslant 0, K > 0$,*

$$\sum_{k=0}^{3} \|(\nabla^k \rho_V) 1_\Omega\|_\infty \leqslant K, \quad \int_\Omega \rho_V(z)\, m(\mathrm{d}z) \geqslant 1 - N^{-a}, \quad \|v\|_\infty \leqslant N^A. \tag{3.1}$$

Further, the domain Ω is regular on any scale $1 \geqslant b \geqslant N^{-1/2}$ in the sense that for any partition of $\mathbb{C}$ into squares of side length b, Ω intersects $\mathrm{O}(b^{-2})$ of the squares and the boundary $\partial\Omega$ intersects $\mathrm{O}(b^{-1})$ squares. Under this assumption, one can follow the proof in this section to check that the error term $\mathrm{O}(N^{-\kappa})$ in Theorem 1.1 can be explicitly chosen as (with any $a' < a$)

$$C(\Omega, A)(1 + K^2) N^{-(\kappa \wedge a')}. \tag{3.2}$$

Under this more general formulation, the proof applies to the conditional Coulomb gas on mesoscopic scales; see Section 4.6. There the sets S_V and Ω will be disks of radii 1 and $1 - N^{-\tau}$ after rescaling and the regularity condition is trivial. We remind the reader that N is the number of particles in S_V which in the mesoscopic case is of order Nb^2 if we maintain the convention that N is the number of particles in the full plane.

Throughout this section, we make the standing assumption that V satisfies the asymptotic condition (1.4) and (1.9), or more generally that the conditions of the remark above hold.

3.1. Two-dimensional Yukawa gas. The Yukawa gas with range R is defined as in Section 1.1, with the two-body potential $G(z,w) = Y^R(z-w)$, the Yukawa interaction (2.1). In particular, for points $\mathbf{z} = (z_1, \dots, z_N) \in \mathbb{C}^N$, the Yukawa energy with range R in external potential V is

$$H_V^R(\mathbf{z}) = N \sum_j V(z_j) + \sum_{j \neq k} Y^R(z_j - z_k),$$

and the corresponding Gibbs measure is defined as in (1.3). Moreover, for probability measures μ on $\mathbb{C}$, the associated energy functional on probability measures of the Yukawa gas is given by

$$\mathcal{I}_V^R(\mu) := \int V(z)\,\mu(\mathrm{d}z) + \iint Y^R(z-w)\,\mu(\mathrm{d}z)\,\mu(\mathrm{d}w). \tag{3.3}$$

We denote by μ_V^R its unique minimizer (the equilibrium measure) and by $I_V^R = \inf_{\mu:\int \mathrm{d}\mu=1} \mathcal{I}_V^R(\mu)$ the corresponding minimizing energy. The existence of the minimizer and general properties are summarized in Appendix A. There we also extend the estimates on the local density established in [6] for the Coulomb gas to the Yukawa gas; these estimates will be used below.

Theorem 3.2. *For any $\sigma > 0$, there exists a constant $\kappa > 0$ such that, for all $R \geqslant N^{-1/2+\sigma}$,*

$$\frac{1}{\beta N} \log \int_{\mathbb{C}^N} \mathrm{e}^{-\beta H_V^R(\mathbf{z})}\, m(\mathrm{d}\mathbf{z}) = -N I_V^R + \log R + \frac{1}{2}\log N + \zeta + \Big(\frac{1}{2} - \frac{1}{\beta}\Big) \int_{\mathbb{C}} \rho_V^R \log \rho_V^R \,\mathrm{d}m + \mathrm{O}(N^{-\kappa}),$$

where ζ is the residual torus free energy of Theorem 2.1. For $R \geqslant 1$, any $\kappa < 1/24$ is admissible.

The remainder of Section 3 is devoted to the proof of Theorem 3.2, which is concluded in Section 3.8. Theorem 1.1 for the Coulomb gas is then a direct consequence, by taking $R \to \infty$, which we do in Section 3.9.

3.2. Short-range Yukawa approximation. For given $\ell < R$, we decompose the Yukawa potential as $Y^R = Y^\ell(z) + L_R^\ell(z)$. The formula (2.1) shows that the Fourier transform of L_R^ℓ is nonnegative so that L_R^ℓ is a positive definite function. We denote the empirical measure by $\hat{\mu} = N^{-1} \sum_j \delta_{z_j}$ and write $\tilde{\mu}_V^R = \hat{\mu} - \mu_V^R$, where μ_V^R is the equilibrium measure for Yukawa gas with range R.

Lemma 3.3. *For any $0 < \ell < R$, we have the identity*

$$\sum_{j \neq k} L_R^\ell(z_j - z_k) + N \sum_j V(z_j) = N \sum_j Q(z_j) + N^2 \mathbf{L}_R^\ell - N \log(R/\ell) - N^2 K_R^\ell, \tag{3.4}$$

where

$$Q(z) = V(z) + 2 \int L_R^\ell(z-w)\,\mu_V^R(\mathrm{d}w), \tag{3.5}$$

$$\mathbf{L}_R^\ell = \int L_R^\ell(z-w)\,\tilde{\mu}_V^R(\mathrm{d}w)\,\tilde{\mu}_V^R(\mathrm{d}z), \tag{3.6}$$

$$K_R^\ell = \int L_R^\ell(z-w)\,\mu_V^R(\mathrm{d}w)\,\mu_V^R(\mathrm{d}z). \tag{3.7}$$

In particular,

$$H_Q^\ell(\mathbf{z}) = H_V^R(\mathbf{z}) - N^2 \mathbf{L}_R^\ell + N \log(R/\ell) + N^2 K_R^\ell. \tag{3.8}$$

Moreover, the minimizers of the variational functionals $\mathcal{I}_Q^\ell$ and $\mathcal{I}_V^R$ coincide, i.e., $\mu_Q^\ell = \mu_V^R$, and their energies satisfy $I_Q^\ell = I_V^R + K_R^\ell$. The Euler–Lagrange equation for the measure μ_Q^ℓ is

$$\int Y^\ell(z-w)\,\mu_Q^\ell(\mathrm{d}w) + \tfrac{1}{2}Q(z) = c_V \quad \textit{q.e. in } S_V^R \quad \textit{and} \qquad (3.9)$$
$$\int Y^\ell(z-w)\,\mu_Q^\ell(\mathrm{d}w) + \tfrac{1}{2}Q(z) \geqslant c_V \quad \textit{q.e. in } \mathbb{C},$$

with the same constant c_V as in the Euler–Lagrange equation for μ_V^R.

Proof. The proof of (3.4) is a direct calculation, using that $L_R^\ell(0) = \log(R/\ell)$ which can be seen from (2.2), and thus

$$\int_{z\neq w} L_R^\ell(z-w)\,\tilde{\mu}_V^R(\mathrm{d}w)\,\tilde{\mu}_V^R(\mathrm{d}z) = \int L_R^\ell(z-w)\,\tilde{\mu}_V^R(\mathrm{d}w)\,\tilde{\mu}_V^R(\mathrm{d}z) - \frac{1}{N}\log(R/\ell).$$

The equilibrium measures (minimizers) of $\mathcal{I}_V^R$ and $\mathcal{I}_Q^\ell$ are characterized by the Euler–Lagrange equations (A.10), which state that in the supports of the measures, the equalities

$$\frac{1}{2}V + Y^R * \mu_V^R = c_V^R, \quad \frac{1}{2}Q + Y^\ell * \mu_Q^\ell = c_Q^\ell$$

hold, and that equality is replaced by inequality outside the supports of the equilibrium measures. By definition of Q and the Euler–Lagrange equation for μ_V^R, the solution μ_Q^ℓ satisfies (3.9). By the uniqueness of the minimizers, we thus conclude that $\mu_Q^\ell = \mu_V^R$ and $S_V^R = S_Q^\ell$, i.e., the two minimizers coincide. Moreover, a simple computation yields that $I_V^R = I_Q^\ell + K_R^\ell$. □

In view of the above lemma, from now on, we write μ_V instead of $\mu_V^R = \mu_Q^\ell$ and ρ_V for the density of the absolutely continuous part of μ_V. The next lemma gives an elementary estimate on Q that will be useful later.

Lemma 3.4. *For $z \in S_V$ with distance $\gg \ell$ to the boundary of the support,*

$$Q(z) = 2c_V - 4\pi\ell^2\rho_V(z) + N^\varepsilon \mathrm{O}(\ell^4)\|\nabla^2\rho_V\|_\infty + \mathrm{O}(\mathrm{e}^{-N^\varepsilon}). \qquad (3.10)$$

Proof. By Lemma 3.3, for $z \in S_V$, we have

$$\begin{aligned} Q(z) &= 2c_V - 2\int Y^\ell(z-w)\,\mu_V(\mathrm{d}w) \\ &= 2c_V - 2\rho_V(z)\int Y^\ell(z-w)\,m(\mathrm{d}w) + N^\varepsilon \mathrm{O}(\ell^4)\|\nabla^2\rho_V\|_\infty + \mathrm{O}(\mathrm{e}^{-N^\varepsilon}). \end{aligned}$$

Here we used that, by the exponential decay of Y^ℓ, we may restrict the integral over w a disk of radius $\mathrm{O}(\ell N^\varepsilon)$ around z, up to an error $\mathrm{O}(\mathrm{e}^{-N^{c\varepsilon}})$. Moreover, since z is in the support of μ_V with distance $\gg \ell$ to the boundary of the support, we may Taylor expand the equilibrium density to second order and use that the first-order term vanishes after integration. The definition of the Yukawa potential (2.1) implies $\int Y^\ell(z-w)\,m(\mathrm{d}w) = 2\pi\ell^2$. This implies (3.10). □

3.3. Quasi-free approximation. In this and the next subsections, we approximate the partition function of the (long-range) Yukawa gas in terms of the *quasi-free Yukawa approximation*, which we now define. The idea is the same as in Section 2.3, with the additional element that now the boundary requires a special treatment.

Given parameters $0 \ll \ell \ll b \leqslant b' \ll 1$ (chosen later), we divide $\mathbb{C}$ into a grid of squares of side length b with centers $\alpha \in (b\mathbb{Z})^2 \subset \mathbb{C}$; often we will also write α to mean the square with center α. Furthermore, it is useful to also consider the shifted grid, in which all squares are translated by $u \in [-b/2, b/2]^2$ so that their centers are $u + \alpha$. We say that the square α is in the bulk, and write $\alpha \subset D$, if it and its translates by $u \in [-b/2, b/2]^2$ have distance at least b' to the complement of S_V (respectively Ω in the situation of Remark 3.1); we write $D = D_u$ for the union of the bulk squares. Throughout Section 3, we always assume that $b' \gg N^{-1/4}$ and (to reduce the number of parameters), in the context of Remark 3.1, that $b' \geqslant N^{-a}$. Furthermore, we always assume the previously stated bounds $0 \ll \ell \ll b \ll 1$.

Given parameters as above, we consider the quasi-free Yukawa gas obtained by removing the interaction between particles in a bulk square with particles outside that square, and replacing the interactions between particles in the same square by a periodic one inside each bulk square.

Let $\mathbf{n} = (n_\alpha)$ be a particle profile with $\sum_\alpha n_\alpha = N$ where now α can take as value also the boundary B. Similarly as in (2.21), we define the quasi-free free energy for particle profile $\mathbf{n}$ by

$$F(\mathbf{n}) = \frac{1}{\beta} \log \binom{N}{\mathbf{n}} + \frac{1}{\beta} \sum_{\alpha \subset D} \log \int_{\mathbb{T}_\alpha^{n_\alpha}} \mathrm{e}^{-\beta \hat{H}_\alpha(\mathbf{u})} \, m(\mathrm{d}\mathbf{u}) - \hat{H}_B, \tag{3.11}$$

where

$$\hat{H}_\alpha(\mathbf{u}) = \sum_{i \neq j} U_\alpha^\ell(u_i - u_j) + N n_\alpha Q(\alpha), \quad \hat{H}_B = N^2 I_{Q,B} + 2c_V N(n_B - N\mu_V(B)),$$

where $I_{Q,B}$ is a constant defined in (3.25) below. We denote by $\bar{\mathbf{n}} = (\bar{n}_\alpha)$ the approximate mean number of particles in α, where α is either a square or the boundary region. More precisely, $\bar{n}_\alpha$ is an integer at distance at most 1 to $N\mu_V(\alpha)$ (here viewing α as a set); we assume that this rounded choice is such that $\sum_\alpha \bar{n}_\alpha = N$. Sums over $\mathbf{n}$ will always be over all particle profiles with $\sum_\alpha n_\alpha = N$. For $z \in \mathbb{C}^N$, we define $\mathbf{n}(\mathbf{z})$ to be the particle profile of the configuration $\mathbf{z} \in \mathbb{C}^N$, i.e., $\mathbf{n}(\mathbf{z}) = (n_\alpha(\mathbf{z}))$ where $n_\alpha(\mathbf{z})$ is the number of particles $z_j \in \alpha$ (again with α either a bulk square or the boundary region B).

Proposition 3.5 (Upper Bound)**.** *There exists u such that*

$$\frac{1}{\beta} \log \int \mathrm{e}^{-\beta H_V^R(\mathbf{z})} \, m(\mathrm{d}\mathbf{z}) - N \log(R/\ell) - N^2 K_R^\ell \leqslant \frac{1}{\beta} \log \sum_{\mathbf{n}} \mathrm{e}^{\beta F(\mathbf{n})}$$
$$+ N^\varepsilon \mathrm{O}(N^2 \ell^3 b^{-1} + N^2 \ell^2 b) \|\rho_V\|_2 + \mathrm{O}(n_B \log N). \tag{3.12}$$

The error terms can be understood as follows. The error $N^2 \ell^3 b^{-1} = (N\ell^2)(N^2 \ell b^{-1})$ is the number of pair interactions via a Yukawa gas of range ℓ for particles in neighboring squares; the error $N^2 \ell^2 b$ is the variation of the effective potential Q over a square of size b. The error terms in the following lower bound cannot be obtained by a simple counting; it relies on higher order cancellation which we will explain later on.

Proposition 3.6 (Lower Bound)**.** *Assume $\ell/b \gg (Nb^2)^{-1/4}$. Then there is $\tau > 0$ such that for all u,*

$$\frac{1}{\beta}\log\int e^{-\beta H_V^R(\mathbf{z})} m(d\mathbf{z}) - N\log(R/\ell) - N^2 K_R^\ell \geq F(\bar{\mathbf{n}}) + N^\varepsilon O(N^{1-\tau} + b^2\ell^{-4}) \\ + O(N^2(b^4+\ell^2 b))(\|\rho_V\|_3 + \|\rho_V\|_3^2) + O(b^{-2}\log N + \bar{n}_B \log N). \tag{3.13}$$

More precisely, $O(N^{1-\tau})$ *is* $N^\varepsilon O(N^{4/5}/\ell^{2/5} + Nb)$.

Propositions 3.5, 3.6 assert that the free energy of a (long-range) Yukawa gas with Hamiltonian H_V^R can be approximated by that of the (short-range) quasi-free Yukawa gases, for appropriate choices of the parameters b, b' and ℓ. These propositions are analogous to Propositions 2.6, 2.7, with the additional treatment of the boundary and taking into account that the density of the equilibrium measure is not constant.

In the following, we usually omit the subscript u from D_u and write $B = S_V \setminus D$ to denote the boundary region. Moreover, we write $\bigcap D = \bigcap_u D_u$ and $\bigcup D = \bigcup_u D_u$.

Lemma 3.7. *The number of bulk squares is $O(b^{-2})$, the number of bulk squares touching the boundary region is $O(b^{-1})$, and the equilibrium mass covered by the squares is $\mu_V(\bigcap D) \geqslant 1 - O(b')$. In addition, for any $\alpha \subset D$,*

$$\bar{n}_\alpha = O(Nb^2)\|\rho_V\|_\infty, \quad \bar{n}_\alpha = Nb^2\rho(\alpha) + O(Nb^3)\|\nabla\rho_V\|_\infty, \quad \bar{n}_B = O(Nb'). \tag{3.14}$$

Proof. The claim about the number of boundary squares follows immediately from the fact that the support of S_V has diameter of order 1. The statements about the number of squares touching the boundary region and the mass not covered by the squares follow from the assumption that the the boundary of S_V is piecewise C^1. In the more general situation of Remark 3.1, the estimates hold by the assumption stated in the remark. Finally, (3.14) follows immediately from the fact that, by construction, ρ_V is C^1 on the squares α. ☐

3.4. Upper bound: proof of Proposition 3.5. As in the proof of Proposition 2.6, to each square $u+\alpha$ we associate a torus $\mathbb{T}_\alpha$ of the same side length, and denote their identification by $\Phi_\alpha^u : u+\alpha \to \mathbb{T}_\alpha$; see (2.24). Then, analogously to (2.25), we define

$$\tilde{Y}_u^\ell(z,w) = \sum_{\alpha\subset D} U^\ell(\Phi_\alpha^u(z), \Phi_\alpha^u(w)) \mathbb{1}_{z\in u+\alpha, w\in u+\alpha} + Y^\ell(z-w)\mathbb{1}_{z\notin D_u, w\notin D_u}, \tag{3.15}$$

and $\tilde{Q}_u$ by replacing Q in the bulk squares $\alpha \subset D$ by its value at the centers of the squares, and outside D by adding the equilibrium contribution from the pair interaction with the bulk, i.e.,

$$\tilde{Q}_u(z) = \sum_{\alpha\subset D} Q(\alpha)\mathbb{1}_{z\in u+\alpha} + \left(Q(z) + 2N\int_D Y^\ell(z-w)\,\mu_V(dw)\right) 1_{z\notin D_u}. \tag{3.16}$$

Denote by $\tilde{H}_u^\ell$ the corresponding Hamiltonian

$$\tilde{H}_u^\ell(\mathbf{z}) = N\sum_j \tilde{Q}_u(z_j) + \sum_{i\neq j}\tilde{Y}_u^\ell(z_i,z_j). \tag{3.17}$$

The main work in the proof of Proposition 3.5 is contained in the proof of Proposition 3.8 below.

Proposition 3.8. *There exists u such that*

$$\frac{1}{\beta}\log\int e^{-\beta H_V^R(\mathbf{z})}\,m(\mathrm{d}\mathbf{z}) \leqslant \frac{1}{\beta}\log\int e^{-\beta \tilde{H}_u^\ell(\mathbf{z})}\,m(\mathrm{d}\mathbf{z}) + N\log(R/\ell) + N^2 K_R^\ell \\ + N^\varepsilon \mathrm{O}(N^2\ell^3 b^{-1} + N^2\ell^2 b)(\|\rho_V\|_\infty + \|\nabla\rho_V\|_\infty + \|\nabla^2\rho_V\|_\infty). \quad (3.18)$$

In preparation of the proof, for $z,w\in\mathbb{C}$, we define the averages over u of $\tilde{Y}$ and $\tilde{Q}$ by

$$\bar{Y}(z,w) = \frac{1}{b^2}\int_{[-b/2,b/2]^2} \tilde{Y}_u^\ell(z,w)\,\mathrm{d}u, \qquad \bar{Q}(z) = \frac{1}{b^2}\int_{[-b/2,b/2]^2}\tilde{Q}_u(z)\,\mathrm{d}u. \quad (3.19)$$

The following lemma provides estimates on $\bar{Y}$, which are stated in terms of the function

$$g(x,y) = \frac{(b-|x|)_+(b-|y|)_+}{b^2} Y^\ell\left(\sqrt{|x|_b^2+|y|_b^2}\right), \quad |x|_b = |x|\wedge(b-|x|). \quad (3.20)$$

Here we write $Y(r)$ for $Y(z)$ with $|z|=r$.

Lemma 3.9. *Assume that $\ell \ll b$ and write $(x,y) = z-w$. Then*

(i) Inside the bulk, i.e., for $z,w\in\bigcap D$, we have $\bar{Y}(z,w) = g(z-w) + \mathrm{O}(e^{-cb/\ell})$.

(ii) Away from the bulk, i.e., for $z,w\notin\bigcup D$, by definition we have $\bar{Y}(z,w) = Y^\ell(z-w)$.

(iii) In general, and in particular near the boundary, we have the inequalities

$$g(z-w) + \mathrm{O}(e^{-cb/\ell}) \leqslant \bar{Y}(z,w) \leqslant Y^\ell(z-w) + \mathrm{O}(e^{-cb/\ell}) \quad \text{if } |z-w|_\infty \leqslant b/2, \\ \bar{Y}(z,w) \leqslant Y^\ell(z-w) + \mathrm{O}(e^{-cb/\ell}).$$

Proof. (i) The probability that a fixed x in $\mathbb{R}$ is contained in a uniformly random interval of length b containing 0 is $(b-|x|)_+/b$. In the case $|x|\leqslant b/2$, $|y|\leqslant b/2$, if both of z,w are contained in a square, their periodic distance is also given by $\sqrt{|x|^2+|y|^2}$. Thus

$$g(z-w) = \frac{1}{b^2}\int_0^b \mathrm{d}p\int_0^b \mathrm{d}q\,\tilde{Y}_{(p,q)}^\ell(z,w) = \frac{1}{b^2}\int_0^b\mathrm{d}p\int_0^b\mathrm{d}q\,Y^\ell(z-w)1_{\{p\geqslant|x|,q\geqslant|y|\}} + \mathrm{O}(e^{-c/\gamma}) \\ = \frac{(b-|x|)(b-|y|)}{b^2}Y^\ell\big(\sqrt{x^2+y^2}\big) + \mathrm{O}(e^{-c/\gamma}).$$

The other cases are entirely analogous.

(ii) In this case, especially $z,w\notin D_u$ which by the definition (3.15) implies $\bar{Y}(z,w) = Y^\ell(z,w)$.

(iii) By the exponential decay of Y^ℓ, using that U^ℓ is the periodization of Y^ℓ, we can have $\tilde{Y}_u^\ell(z,w) > Y^\ell(z-w) + \mathrm{O}(e^{-cb/\ell})$ for some u only if $|z-w|_\infty > b/2$. This implies the bound $\bar{Y}(z,w)\leqslant Y^\ell(z-w)+\mathrm{O}(e^{-cb/\ell})$ for $|z-w|_\infty\leqslant b/2$. For the lower bound on $\bar{Y}$ for $|z-w|_\infty\leqslant b/2$, we notice that $\tilde{Y}_u^\ell(z,w) = Y^\ell(z-w)+\mathrm{O}(e^{-cb/\ell})$ if and only if either z and w belong to the same square $\alpha\subset D_u$ or $z,w\notin D_u$, and in other cases $\tilde{Y}_u^\ell(z,w) = 0$. The probability of first event, in the u-average, may be bounded from below by the event that z and w are both in the same square, irregardless of whether the square is in D_u or not. This probability is $(b-x)(b-y)/b^2$, and therefore

$$\tilde{Y}(z,w) \geqslant \frac{(b-x)(b-y)}{b^2}Y^\ell(z-w) + \mathrm{O}(e^{-cb/\ell}) = g(z-w) + \mathrm{O}(e^{-cb/\ell}).$$

For the remaining cases, consider for example the case $b/2 < x \leqslant b, 0 \leqslant y \leqslant b/2$. Then, similarly,

$$\bar{Y}(z,w) - Y^\ell(z-w) = \mathbb{P}_u(z,w \in \alpha \subset D_u) Y^\ell(\sqrt{x^2+(b-y)^2}) \\ + (\mathbb{P}_u(z,w \notin D_u) - 1) Y^\ell(z-w) + \mathrm{O}(\mathrm{e}^{-cb/\ell}) \leqslant \mathrm{O}(\mathrm{e}^{-cb/\ell}).$$

The cases $0 \leqslant x \leqslant b/2, b/2 < y \leqslant b$ and $b/2 < x \leqslant b, b/2 \leqslant y \leqslant b$ are entirely analogous. This completes the proof. □

Proof of Proposition 3.8. By Jensen's inequality,

$$\frac{1}{\beta} \log \int \mathrm{e}^{-\beta H_V^R(\mathbf{z})} \, m(\mathrm{d}\mathbf{z}) \leq \frac{1}{\beta} \log \int \mathrm{e}^{-\beta \tilde{H}_u^\ell(\mathbf{z})} \, m(\mathrm{d}\mathbf{z}) + \mathbb{E}^{H_V^R}(\tilde{H}_u^\ell - H_V^R). \tag{3.21}$$

The last term can be rewritten as

$$\mathbb{E}_V^R(\tilde{H}_u^\ell - H_V^R) = \mathbb{E}_V^R(\tilde{H}_u^\ell - H_V^\ell) + \mathbb{E}_V^R(H_V^\ell - H_V^R). \tag{3.22}$$

Using that L_R^ℓ is positive definite, $\mathbf{L}_R^\ell \geqslant 0$, and by (3.8), the last term in (3.22) is bounded by

$$\mathbb{E}_V^R(H_V^\ell - H_V^R) = -N^2 \mathbb{E}_V^R \mathbf{L}_R^\ell + N \log(R/\ell) + N^2 K_R^\ell \leq N \log(R/\ell) + N^2 K_R^\ell.$$

To bound the first term in (3.22), by the mean-value theorem for continuous functions, we may average (3.21) over u in the square $[-b/2, b/2]^2$. By the definition of $\bar{Y}$ in (3.19), we have

$$\frac{1}{b^2} \int_{[-b/2,b/2]^2} \mathrm{d}u \, \mathbb{E}_V^R(\tilde{H}_u^\ell - H_V^\ell) = \mathbb{E}_V^R\Big[N \sum_j (\bar{Q}(z_j) - Q(z_j))\Big] + \mathbb{E}_V^R\Big[\sum_{i \neq j} (\bar{Y}(z_i,z_j) - Y^\ell(z_i - z_j))\Big]. \tag{3.23}$$

For the particles in the bulk, the term involving Q is bounded using (3.10) and $\ell \leqslant b$ by

$$N \sum_j (Q(z_j) - \bar{Q}(z_j)) \mathbb{1}_{z_j \in D} = \mathrm{O}(N^2 \ell^2 b)(\|\nabla \rho_V\|_\infty + \|\nabla^2 \rho_V\|_\infty). \tag{3.24}$$

For the particles in the boundary region, the difference of Q and $\tilde{Q}$ is

$$2N \int_D Y^\ell(z_j - w) \, \mu_V(\mathrm{d}w).$$

By the local density estimate, Remark A.4, the sum of the last line over the particles z_j in the boundary region is $\mathrm{O}(N^2\ell^2 b)$. Moreover, dividing the sum over $i \neq j$ for the pair interaction in (3.23) into the boundary and bulk particles, Lemma 3.10 below implies

$$\mathbb{E}_V^R \sum_{i \neq j} [\bar{Y}(z_i,z_j) - Y^\ell(z_i - z_j)] = N^\varepsilon \mathrm{O}(N^2 \ell^3 b^{-1}),$$

where we used that the contribution for the above sum where both z_i and z_j are outside $\bigcup D$ vanishes since then $\bar{Y}(z_i,z_j) = Y^\ell(z_i - z_j)$. This completes the proof. □

Lemma 3.10. *For any u,*

$$\mathbb{E}_V^R \sum_{i,j} \mathbb{1}_{z_i,z_j \in \bigcap D} [g(z_i - z_j) - Y^\ell(z_i - z_j)] = \mathrm{O}(N^\varepsilon N^2 \ell^3 b^{-1}) \|\rho_V\|_\infty,$$

$$\mathbb{E}_V^R \sum_{i,j} \mathbb{1}_{z_i \in B, z_j \in \bigcup D} [\tilde{Y}(z_i,z_j) - Y^\ell(z_i - z_j)] = \mathrm{O}(N^\varepsilon N^2 \ell^3) \|\rho_V\|_\infty.$$

Proof. We use the local density for the Yukawa gas, Theorem A.1 with Remark A.4, stating that balls of radius $r \gg N^{-1/2}$ contain $\mathrm{O}(Nr^2)$ particles with very high probability (provided that the distance to the boundary is at least $b' \gg N^{-1/4}$). In addition, we use that for $|z_i - z_j| \geqslant \ell N^{\varepsilon}$ we have $Y^\ell(z_i - z_j) \leqslant \mathrm{e}^{-cN^{\varepsilon}}$ so that any such contributions can be neglected. As a consequence,

$$\mathbb{E}_V^R \sum_{i,j} [g(z_i - z_j) - Y^\ell(z_i - z_j)] = \mathrm{O}(N^\varepsilon N(N\ell^2)(\ell/b))$$

since each of the at most N particles z_i interacts with $\mathrm{O}(N^\varepsilon N\ell^2)$ particles z_j, and the difference $g - Y^\ell$ is of order ℓ/b by Lemma 3.9. The boundary layer has distance at least b' to the boundary of the support of the equilibrium measure, so that the local density estimate can be applied. We have

$$\mathbb{E}_V^R \sum_{i,j} \mathbb{1}_{z_i \in B, z_j \in \bigcup D} [\tilde{Y}(z_i, z_j) - Y^\ell(z_i - z_j)] = \mathrm{O}(N^\varepsilon (Nb)(N\ell^2)(\ell/b)).$$

Since there are $\mathrm{O}(Nb)$ boundary particles z_i which by Theorem A.1 each interact with $\mathrm{O}(N^\varepsilon N\ell^2)$ particles, and from Lemma 3.9 we have that the difference of the interactions is of order ℓ/b. $\square$

To bound the boundary contribution, we will need the following estimate. For $z \in S_V \setminus D$, recall from (3.16) and the Euler–Lagrange equation (3.9) that

$$\tilde{Q}(z) = Q(z) + 2\int_D Y^\ell(z - w)\, \mu_V(\mathrm{d}w) = 2c_V - 2\int_B Y^\ell(z - w)\, \mu_V(\mathrm{d}w),$$

and define the constant

$$\begin{aligned} I_{Q,B} &= \int_B \tilde{Q}(z)\, \mu_V(\mathrm{d}z) + \iint_{B^2} Y^\ell(z - w)\, \mu_V(\mathrm{d}z)\, \mu_V(\mathrm{d}w) \\ &= 2c_V \mu_V(B) - \iint_{B^2} Y^\ell(z - w)\, \mu_V(\mathrm{d}z)\, \mu_V(\mathrm{d}w). \end{aligned} \tag{3.25}$$

Proposition 3.11. *For any u,*

$$\begin{aligned} &\frac{1}{\beta} \log \int_{(\mathbb{C}\setminus D)^{n_B}} \mathrm{e}^{-\beta N \sum_j \tilde{Q}(z_j) - \beta \sum_{j \neq k} Y^\ell(z_j - z_k)}\, m(\mathrm{d}\mathbf{z}) \\ &\qquad\qquad \leqslant -N^2 I_{Q,B} - 2c_V N(n_B - N\mu_V(B)) + \mathrm{O}(n_B \log N). \end{aligned} \tag{3.26}$$

Proof. Let

$$E(n_B) = \inf_{\int \omega = n_B} \left[N \int_{\mathbb{C}\setminus D} \tilde{Q}(z)\, \omega(\mathrm{d}z) + \iint_{(\mathbb{C}\setminus D)^2} Y^\ell(z - w)\, \omega(\mathrm{d}z)\, \omega(\mathrm{d}w) \right],$$

where ω is a positive measure of total mass n_B supported on $\mathbb{C} \setminus D$. As in the upper bound bound on the partition function in Proposition A.14, the left-hand side of (3.26) is bounded from above by

$$- E(n_B) + \mathrm{O}(n_B \log N). \tag{3.27}$$

(In particular, note that regularity of the equilibrium measure assumed in that proposition is only required for the lower bound for the partition function.) It thus suffices to show that

$$E(n_B) - N^2 I_{Q,B} \geqslant 2c_V N(n_B - N\mu_V(B)).$$

To do so, with $\tilde\omega = \omega - N\mu_V$ inside the infimum, we write

$$
\begin{aligned}
&E(n_B) - N^2 I_{Q,B}\\
&= \inf_{\int \omega = n_B} \left[N \int_{D^c} \tilde{Q}(z)\, \omega(\mathrm{d}z) + \iint_{(D^c)^2} Y^\ell(z-w)\, \omega(\mathrm{d}z)\, \omega(\mathrm{d}w) \right]\\
&\qquad - N^2 \int_{D^c} \tilde{Q}(z)\, \mu_V(\mathrm{d}z) - N^2 \iint_{(D^c)^2} Y^\ell(z-w)\, \mu_V(\mathrm{d}z)\, \mu_V(\mathrm{d}w)\\
&= \inf_{\int \omega = n_B} \left[N \int_{D^c} \tilde\omega(\mathrm{d}z) \left[\tilde{Q}(z) + 2\int_{D^c} Y^\ell(z-w)\, \mu_V(\mathrm{d}w) \right] + \int_{D^c} Y^\ell(z-w)\, \tilde\omega(\mathrm{d}z)\, \tilde\omega(\mathrm{d}w) \right].
\end{aligned}
$$

The last term on the right-hand side is nonnegative, and can therefore be dropped. By definition of $\tilde Q$ and the Euler–Lagrange equation (3.9), also

$$
\tilde{Q}(z) + 2\int_{D^c} Y^\ell(z-w)\, \mu_V(\mathrm{d}w) = Q(z) + 2\int Y^\ell(z-w)\, \mu_V(\mathrm{d}w) \geqslant 2c_V.
$$

Since the same relation holds with equality on the support of μ_V, therefore

$$
E(n_B) - N^2 I_{Q,B} \geqslant 2c_V N \int_{D^c} \tilde\omega(\mathrm{d}z) = 2c_V N (n_B - N\mu_V(B)).
$$

This completes the proof. □

Proof of Proposition 3.5. Summing over the possible particle profiles, we have

$$
\int \mathrm{e}^{-\beta \tilde{H}^\ell_u(\mathbf{z})}\, m(\mathrm{d}\mathbf{z}) = \sum_{\mathbf{n}} \binom{N}{\mathbf{n}} \int \mathrm{e}^{-\beta \tilde{H}^\ell_u(\mathbf{z})}\, \mathbb{1}_{\mathbf{n}(\mathbf{z}) = \mathbf{n}}\, m(\mathrm{d}\mathbf{z}),
$$

where $\mathbf{n}(\mathbf{z})$ is the particle profile of the configuration $\mathbf{z} \in \mathbb{C}^N$. By definition of $\tilde H$, for any u, the integral on the right-hand side factorizes as

$$
\left(\prod_{\alpha \subset D} \int_{\alpha^{n_\alpha}} \mathrm{e}^{-\beta \hat{H}_\alpha(\mathbf{z})}\, m(\mathrm{d}z) \right) \times \left(\int_{(\mathbb{C}\setminus D)^{n_B}} \mathrm{e}^{-\beta N \sum_j \tilde{Q}(z_j) - \beta \sum_{j\neq k} Y^\ell(z_j - z_k)}\, m(\mathrm{d}\mathbf{z}) \right).
$$

The claim now follows from Propositions 3.8 and 3.11. □

3.5. Lower bound of the partition function: set-up and embedding of the torus. To obtain a lower bound on the partition function, we first restrict the particle profile to $\bar{\mathbf{n}}$. For this, we define the indicator function

$$
\hat\chi(\mathbf{z}) = \mathbb{1}\Big(n_B(z) = \bar{n}_B\Big) \prod_\alpha \mathbb{1}\Big(n_\alpha(z) = \bar{n}_\alpha\Big) \prod_j \mathbb{1}\Big(z_j \in D \cup B\Big). \tag{3.28}
$$

where $\mathbf{n}(\mathbf{z})$ is the particle profile of the configuration $\mathbf{z} \in \mathbb{C}^N$. We then start with

$$
\frac{1}{\beta} \log \int \mathrm{e}^{-\beta H^R_V(\mathbf{z})}\, m(\mathrm{d}\mathbf{z}) \geq \frac{1}{\beta} \log \int \mathrm{e}^{-\beta H^R_V(\mathbf{z})}\, \hat\chi(\mathbf{z})\, m(\mathrm{d}\mathbf{z}). \tag{3.29}
$$

Next we break the permutation symmetry of the particles. We order the squares α arbitrarily as $\alpha_1, \alpha_2, \dots$ and write $\tilde\chi(\mathbf{z})$ for $\hat\chi(\mathbf{z})$ multiplied with the indicator function of the event in which the particles $z_1, \dots, z_{\bar{n}_{\alpha_1}}$ are in α_1, the particles $z_{\bar{n}_{\alpha_1}+1}, \dots, z_{\bar{n}_{\alpha_1}+\bar{n}_{\alpha_2}}$ are in α_2, and so on. Then

$$
\frac{1}{\beta} \log \int \mathrm{e}^{-\beta H^R_V(\mathbf{z})}\, \hat\chi(\mathbf{z})\, m(\mathrm{d}\mathbf{z}) = \frac{1}{\beta} \log \binom{N}{\mathbf{n}} + \frac{1}{\beta} \log \int \mathrm{e}^{-\beta H^R_V(\mathbf{z})}\, \tilde\chi(\mathbf{z})\, m(\mathrm{d}\mathbf{z}).
$$

Embedding of the torus. To each square α, we associate a torus $\mathbb{T}_\alpha$ of side length b and a map

$$\Psi_\alpha : \mathbb{T}_\alpha \to \alpha. \tag{3.30}$$

Thus the maps Ψ_α go in the opposite direction of the maps Φ_α in (2.24) that appeared in the proof of the upper bound. Moreover, we will now choose these maps Ψ_α randomly, independently for different squares with the same distribution, and also independent of the particle configurations; the random choice will be specified below. Given any choice of the maps Ψ_α, the quasi-free approximation can be expressed as follows. Let ω_α be the Yukawa gas on $\mathbb{T}_\alpha^{n_\alpha}$, with density

$$\frac{1}{Z_\alpha}\mathrm{e}^{-\beta\hat{H}_\alpha(\mathbf{u}^\alpha)}m(\mathrm{d}\mathbf{u}^\alpha). \tag{3.31}$$

For the boundary, we take ω_B to be the measure under which the particles are independently distributed according to the equilibrium measure, i.e., $\omega_B = \mu_V|_B^{\otimes n_B}$ on B and $\Psi_B : B \to B$ to be the identity map. With the fixed particle profile $\mathbf{n} = \bar{\mathbf{n}}$, the quasi-free approximation is the product measure $\boldsymbol{\omega} = \prod_\alpha \omega_\alpha$. Given the maps Ψ_α, define Ψ by

$$\Psi : \prod_\alpha \mathbb{T}_\alpha^{n_\alpha} \times B^{n_B} \to \mathbb{C}^N, \quad \Psi(\{\mathbf{u}\}) = (\{\Psi_\alpha \mathbf{u}^\alpha\}) \in \mathbb{C}^N. \tag{3.32}$$

In particular, $\Psi^*\boldsymbol{\omega} = \prod_\alpha \Psi_\alpha^* \boldsymbol{\omega}_\alpha$ is a measure on configurations of N particles in $\mathbb{C}$. Under the map Ψ_α with $|\mathrm{d}\Psi| = 1$, the measure ω_α transforms to

$$\frac{1}{Z_\alpha}\mathrm{e}^{-\beta\hat{H}_\alpha(\boldsymbol{\Psi}_\alpha^{-1}(z))}\prod_i \mathrm{d}\Psi_\alpha^{-1}(z_i) = \frac{1}{Z_\alpha}\mathrm{e}^{-\beta\hat{H}_\alpha(\boldsymbol{\Psi}_\alpha^{-1}(z))}\prod_i \mathrm{d}z_i.$$

This expression is analyzed in Section 3.6 below; before we specify the choice of the maps Ψ_α.

Choice of the maps Ψ_α. We now specify the random choice of the maps Ψ_α. Since these maps are independent for different squares and have the same distribution, it suffices to consider a single square. Thus we define maps $\Psi : \mathbb{T} \to [-1/2, 1/2)^2$ where $\mathbb{T}$ is the unit torus; the maps Ψ_α are then defined by recentering and rescaling.

The main reason to introduce *random* Ψ is to resolve the issue that the torus distance and Euclidean distance are incompatible. The range of the Yukawa interaction ℓ, appearing in the quasi-free gas, is small. On the other hand, we wish to use it to approximate the Coulomb energy which is long range. The Coulomb interaction will be pushed back to the torus; this creates discontinuities since the torus is periodic. The naive embedding of the square onto the torus used in Section 2.3 is discontinuous along a horizontal and a vertical line. (In that section, this is not an issue since we may assume from the start that the range of the Yukawa interaction is small by Lemma 2.3.) This discontinuity can be averaged out using the translational invariance of the torus, but the resulting interaction on the torus is still not smooth enough to apply the rigidity estimate. Therefore we now choose Ψ to involve a more sophisticated average than the simple average over the discontinuity lines so as to have a smooth interaction after pushing back the Coulomb interaction to the torus.

To define the maps Ψ, we define $[u]$ through

$$-\frac{1}{2} \leqslant [u] < \frac{1}{2},\ u - [u] \in \mathbb{Z} \quad \text{for } u \in \mathbb{R}, \qquad [z] = ([z_1],[z_2]) \in \mathbb{T} \quad \text{for } z \in \mathbb{C} \cong \mathbb{R}^2. \tag{3.33}$$

Then we define maps $\Phi_1, \Phi_2 : \mathbb{T} \to \mathbb{T}$ by

$$\Phi_1(z) = ([z_1 + m_1 s(z_2)], z_2), \quad \Phi_2(z) = (z_1, [z_2 + m_2 s(z_1)]), \tag{3.34}$$

where we will choose $s(z) = \sin(2\pi x)$ (or any smooth periodic function with similar oscillation). Let $\Phi = \Phi_1 \circ \Phi_2$. We choose m_1, m_2 as independent random variables with the distribution of tX with X a random variable with smooth and compactly supported density, $\mathbb{E}(X) = 0$, and $N^{-1/2} \ll t \ll 1$ is some mesoscopic scale. Eventually, we will choose $t = N^{-1/4}$.

Finally, let $\Psi_z = [\Phi(z)+(a_1, a_2)]$, where (a_1, a_2) is a random shift, with a_1 and a_2 independent and uniform on $[-1/2, 1/2)$. Note that Φ and Ψ are smooth function on the torus and they preserve volumes:

$$|\mathrm{d}\Phi| = |\mathrm{d}\Psi| = 1. \tag{3.35}$$

In the following statement, Y^γ is the Yukawa potential on the plane (2.1), but the expectation $\mathbb{E}^\gamma$ refers to the Yukawa gas with N particles and range γ on the unit torus. The expectation $\mathbb{E}^\Psi$ refers to the randomness of Ψ, i.e., the independent random variables m_1, m_2, a_1, a_2. As usually, we also denote $\hat\mu$ the empirical spectral measure and $\tilde\mu = \hat\mu - m$ where m is the uniform probability measure on $\mathbb{T}$. In the statement below and this section, it is understood that all double integrals are evaluated on $\{z \neq w\}$. The following proposition will be proved in Appendix C. The assumption $\gamma \gg N^{-1/4}$ below is technical: our proof of Proposition 3.12 gives non-trivial bounds for any mesoscopic scale γ. However the obtained bounds take a simpler form for $\gamma \gg t = N^{-1/4}$, which is the regime of interest to us.

Proposition 3.12. *Choose $t = N^{-1/4}$ in (3.34) and let $c, \varepsilon > 0$ be small constants. For any $N^{-1/4} \ll \gamma < N^{-c}$, the following three terms are of order $N^\varepsilon \mathrm{O}(\gamma^{-4} + N^{4/5}\gamma^{-2/5})$:*

$$\mathbb{E}^\Psi \mathbb{E}^\gamma N^2 \iint (Y^\gamma(\Psi_z - \Psi_w) - Y^\gamma(z - w))\, \hat\mu(\mathrm{d}z)\, \hat\mu(\mathrm{d}w),$$

$$\mathbb{E}^\Psi \mathbb{E}^\gamma N^2 \iint (Y^R(\Psi_z - \Psi_w) - Y^\gamma(\Psi_z - \Psi_w))\, \tilde\mu(\mathrm{d}z)\, \tilde\mu(\mathrm{d}w),$$

$$\mathbb{E}^\Psi \mathbb{E}^\gamma N^2 \iint (U^\gamma(z - w) - Y^\gamma(\Psi_z - \Psi_w))\, \tilde\mu(\mathrm{d}z)\, \tilde\mu(\mathrm{d}w).$$

Similar estimates hold for the Yukawa gas of range ℓ on a torus of width b provided that the measures $\hat\mu$ is understood as in (3.45) *and $\gamma = \ell/b$. If we use the standard convention to denote the total number of particles for the Yukawa gas on $\mathbb{C}$ by N, then the number of particles N appeared in the equations of this proposition should be replaced by Nb^2.*

3.6. Proof of Proposition 3.6. Defining $\hat H_\Psi(\mathbf{u}) = \sum_\alpha \hat H_\alpha \circ \Psi_\alpha^{-1}(\mathbf{u}^\alpha)$, by Jensen's inequality,

$$\frac{1}{\beta} \log \int \mathrm{e}^{-\beta H_V^R(\mathbf{z})} \tilde\chi(\mathbf{z})\, m(\mathrm{d}z) \geq \frac{1}{\beta} \log \int \mathrm{e}^{-\beta \hat H_\Psi(\mathbf{z})} \tilde\chi(\mathbf{z})\, m(\mathrm{d}z) + \mathbb{E}^{\Psi^* \boldsymbol{\omega}}(\hat H_\Psi - H_V^R). \tag{3.36}$$

Reversing the change of variables and averaging over the distribution of maps Ψ with $|\mathrm{d}\Psi| = 1$, whose expectation is denoted by $\mathbb{E}^\Psi$,

$$\frac{1}{\beta} \log \int \mathrm{e}^{-\beta H} \tilde\chi \geq \sum_\alpha \frac{1}{\beta} \log \int \mathrm{e}^{-\beta \hat H_\alpha(\mathbf{u}^\alpha)} \prod_i du_i^\alpha + \mathbb{E}^\Psi \mathbb{E}^{\boldsymbol{\omega}}(\hat H(\mathbf{u}) - H_V^R(\Psi \mathbf{u})), \tag{3.37}$$

where $\hat H(\mathbf{u}) = \sum_\alpha \hat H_\alpha(\mathbf{u}^\alpha)$. We abbreviate $\mathbb{E}^\Psi \mathbb{E}^{\boldsymbol{\omega}}$ by $\hat{\mathbb{E}}$. Then, in summary, we need to estimate

$$\hat{\mathbb{E}}(\hat H(\mathbf{u}) - H_Q^\ell(\Psi \mathbf{u})) + \hat{\mathbb{E}}(H_Q^\ell(\Psi \mathbf{u}) - H_V^R(\Psi \mathbf{u})). \tag{3.38}$$

These two terms are estimated in Lemmas 3.13 and 3.15 below. In preparation, we observe that, by construction, the expected empirical measure $\hat\mu$ under $\hat{\mathbb{E}}$ in each square α is uniform with total mass n_α/N (the ˆ on $\hat\mu$ stands for empirical measure and is not related to that on $\hat{\mathbb{E}}$):

$$N \hat{\mathbb{E}}(\hat\mu|_\alpha) = n_\alpha \mu_\alpha, \quad \text{where } \mu_\alpha(dz) = b^{-2} \mathbb{1}_{z \in \alpha}\, m(\mathrm{d}z). \tag{3.39}$$

Bound of the first term in (3.38). We write $\hat{H}_D = \sum_{\alpha\subset D}\hat{H}_\alpha$ and also decompose $H_Q^\ell(\mathbf{z})$ into bulk and boundary contributions as

$$H^\ell_{Q,D}(\mathbf{z}) = N\sum_j Q(z_j)\mathbb{1}_{z_j\in D} + \sum_{j\neq k} Y^\ell(z_j-z_k)\mathbb{1}_{z_j,z_k\in D}, \quad H^\ell_{Q,B}(\mathbf{z}) = H^\ell_Q(\mathbf{z}) - H^\ell_{Q,D}(\mathbf{z}). \tag{3.40}$$

Lemma 3.13. *Assume $\ell/b \gg (Nb^2)^{-1/4}$. Then there exists a $\tau>0$ such that*

$$\hat{\mathbb{E}}(\hat{H}_D(\mathbf{u}) - H^\ell_{Q,D}(\Psi\mathbf{u})) = E + \mathrm{O}(N^2(\ell^3+b^2\ell^2)(\|\rho\|+\|\nabla\rho\|)^2 + N^2\ell^4\|\rho\|\|\nabla^2\rho\|) \tag{3.41}$$

$$N^2 I_{B,Q} - \hat{\mathbb{E}}(H^\ell_{Q,B}(\mathbf{u})) = \mathrm{O}(N^2\ell^2 b)\|\nabla\rho\|_V + \mathrm{O}(\bar{n}_B\log N), \tag{3.42}$$

where $E = \mathrm{O}(N^{1-\tau} + N^\varepsilon b^2\ell^{-4})$. More precisely, $\mathrm{O}(N^{1-\tau})$ is $N^\varepsilon\mathrm{O}(N^{4/5}/\ell^{2/5})$.

The main error in (3.41) is the one with the factor $N^2(\ell^3+b^2\ell^2)$, which is of order b smaller than the main error term in the upper bound (3.18). The reason we gain an additional factor b here, roughly speaking, is due to the fact that the leading error from the left side of a square is cancelled by that from the right side provided that the densities of the two neighboring squares are the same. Since the density variation is of order b, the next order error carries an additional factor b. (A similar cancelation could be obtained also in the upper bound (3.18). Since this refined estimate is not needed in this paper, we chose not to present it for the sake of simplicity.)

Proof of (3.41). Estimating Q by (3.10) and $\bar{n}_\alpha$ by (3.14), the difference of the contributions of the external potential is

$$\begin{aligned}
&\Big|\hat{\mathbb{E}}\Big[N\sum_{\alpha\subset D}\sum_i (Q(z_i) - Q(\alpha)\mathbb{1}_{z_i\in\alpha})\Big]\Big| = \Big|N\sum_{\alpha\subset D} n_\alpha\int (Q(z)-Q(\alpha))\,\mu_\alpha(\mathrm{d}z)\Big|\\
&\leqslant 4\pi\ell^2 N\sum_{\alpha\subset D} n_\alpha\Big|\int(\rho_V(\alpha)-\rho_V(z))\,\mu_\alpha(\mathrm{d}z)\Big| + N\sum_{\alpha\subset D}\mathrm{O}(Nb^2\|\rho_V\|_\infty)\mathrm{O}(N^\varepsilon\ell^4\|\nabla^2\rho_V\|_\infty)\\
&\leqslant \mathrm{O}(N^2b^2\ell^2\|\nabla\rho_V\|_\infty^2) + \mathrm{O}(N^\varepsilon N^2\ell^4\|\rho_V\|_\infty\|\nabla^2\rho_V\|_\infty).
\end{aligned}$$

For the two-particle interactions, we will show that

$$\begin{aligned}
\sum_{\alpha,\beta\subset D}\hat{\mathbb{E}}\Big[\sum_{i\neq j}\mathbf{1}_{u_i\in\mathbb{T}_\alpha}\mathbf{1}_{u_j\in\mathbb{T}_\beta}(U^\ell_\alpha(u_i-u_j)\mathbb{1}_{\alpha=\beta} - Y^\ell(\Psi_\alpha(u_i)-\Psi_\beta(u_j)))\Big]&\\
= E + \mathrm{O}(N^2\ell^3)(\|\rho_V\|+\|\nabla\rho_V\|_\infty)^2,&
\end{aligned} \tag{3.43}$$

where

$$E = \sum_{\alpha\subset D} n_\alpha^2\hat{\mathbb{E}}\iint (U^\ell_\alpha(u-v) - Y^\ell(\Psi_\alpha(u)-\Psi_\alpha(v)))\,\tilde{\mu}_\alpha(\mathrm{d}v)\,\tilde{\mu}_\alpha(\mathrm{d}w), \tag{3.44}$$

and U^ℓ_α is the Yukawa potential on the torus $\mathbb{T}_\alpha$ and $\tilde{\mu}_\alpha = \hat{\mu}|_\alpha - \mu_\alpha$ where μ_α is the normalized uniform measure on $\mathbb{T}_\alpha$. Denoting $\hat{\mu}_\alpha = \hat{\mu}|_\alpha$, we have

$$\hat{\mu}_\alpha(z) = \hat{\mu}|_\alpha(z) = N^{-1}\sum_j\delta(z-z_j)\mathbb{1}(z\in\alpha) = n^{-1}\sum_j\delta_\alpha(z-z_j) \tag{3.45}$$

where δ_α is the delta function normalized w.r.t. the *probability* measure μ_α. By Proposition 3.12 (in particular the remark about rescaling, with N replaced by Nb^2) and the fact that there are $\mathrm{O}(b^{-2})$ many bulk squares α, the term E is of order

$$E = N^\varepsilon\mathrm{O}\Big(b^{-2}\left((Nb^2)^{4/5}/(\ell/b)^{2/5} + (\ell/b)^{-4}\right)\Big) = N^\varepsilon\mathrm{O}\Big(N^{4/5}/\ell^{2/5} + b^2\ell^{-4}\Big). \tag{3.46}$$

It remains to prove (3.43). We first note that the contribution of the nonadjacent squares on the left-hand side is bounded by $\mathrm{O}(\mathrm{e}^{-c\ell/b}) = \mathrm{O}(\mathrm{e}^{-N^{\varepsilon}})$. Denoting by $\alpha \sim \beta$ that the squares α and β are adjacent, we therefore have (up to an additive error $\mathrm{O}(\mathrm{e}^{-N^{\varepsilon}})$)

$$\sum_{\alpha,\beta\subset D} \hat{\mathbb{E}}\Big[\sum_{i\neq j} \mathbf{1}_{z_i\in\alpha}\mathbf{1}_{z_j\in\beta}(U^\ell(u_i,u_j)\mathbb{1}_{\alpha=\beta} - Y^\ell(\Psi(u_i)-\Psi(u_j)))\Big]$$
$$= \sum_{\alpha} n_\alpha^2 \hat{\mathbb{E}}\Big[\iint (U^\ell(u,v) - Y^\ell(\Psi(u)-\Psi(v)))\hat{\mu}_\alpha(\mathrm{d}u)\hat{\mu}_\alpha(\mathrm{d}v)\Big] - \sum_{\alpha\sim\beta} n_\alpha n_\beta \bar{Y}_{\alpha\beta}, \tag{3.47}$$

with

$$\bar{Y}_{\alpha\beta} = \iint_{\alpha\times\beta} Y^\ell(\Psi_\alpha(u)-\Psi_\beta(v))\,\mu_\alpha(\mathrm{d}u)\,\mu_\beta(\mathrm{d}v) = \iint_{\alpha\times\beta} Y^\ell(u-v)\,\mu_\alpha(\mathrm{d}u)\,\mu_\beta(\mathrm{d}v),$$

and where we used $\hat{\mathbb{E}}\hat{\mu}_\alpha(\mathrm{d}u) = \mu_\alpha(\mathrm{d}u)$ and that $\hat{\mu}_\alpha$ and $\hat{\mu}_\beta$ are independent under $\hat{\mathbb{E}}$ for $\alpha \neq \beta$, and that $|\mathrm{d}\Psi_\alpha| = |\mathrm{d}\Psi_\beta| = 1$. By Proposition 3.12 (with the remark about rescaling, with N replaced by Nb^2), and using that the number of squares is $\mathrm{O}(b^{-2})$, we may remove Ψ in the first term on the right-hand side of (3.47) with admissible error E as in (3.46).

To prove (3.43), for the squares $\alpha \subset D$ not touching the boundary, we use the cancelation (3.48) below, and for the squares touching the boundary instead the estimate (3.49). Finally, summing over α, using that there $\mathrm{O}(b^{-2})$ squares α not touching a boundary square and $\mathrm{O}(b^{-1})$ squares touching the boundary, it follows that

$$\sum_\alpha \left(n_\alpha^2 \iint_{\alpha^2} (U^\ell(u-v) - Y^\ell(u-v))\mu_\alpha(\mathrm{d}u)\mu_\alpha(\mathrm{d}v) - \sum_{\beta\sim\alpha} n_\alpha n_\beta \bar{Y}_{\alpha\beta}\right)$$
$$= \mathrm{O}(N^2\ell^3)(\|\rho_V\|_\infty + \|\nabla\rho_V\|_\infty)^2.$$

This proves (3.43). □

Lemma 3.14. *For any square α such that the neighboring squares are not touching the boundary,*

$$\bar{n}_\alpha^2 \iint_{\alpha^2} (U^\ell(u-v) - Y^\ell(u-v))\,\mu_\alpha(\mathrm{d}u)\,\mu_\alpha(\mathrm{d}v) - \sum_{\beta\sim\alpha} \bar{n}_\alpha \bar{n}_\beta \bar{Y}_{\alpha\beta}$$
$$= \mathrm{O}\big(N^2 b^2 \ell^3 \|\rho_V\|_\infty \|\nabla\rho_V\|_\infty\big) + \mathrm{O}(\mathrm{e}^{-N^{\varepsilon}}). \tag{3.48}$$

For the squares α touching the boundary, we still have

$$\bar{n}_\alpha^2 \iint_{\alpha^2} (U^\ell(u-v) - Y^\ell(u-v))\,\mu_\alpha(\mathrm{d}u)\,\mu_\alpha(\mathrm{d}v) - \sum_{\beta\sim\alpha} \bar{n}_\alpha \bar{n}_\beta \bar{Y}_{\alpha\beta} = \mathrm{O}(N^2 b\ell^3)\|\rho_V\|_\infty^2. \tag{3.49}$$

Proof. For any fixed square α of side length $b \gg l$, using that contributions for distances $\gg \ell$ are negligible, by unfolding the periodized interaction we have

$$\int_\alpha \int_\alpha U^\ell(u-v)\,m(\mathrm{d}u)\,m(\mathrm{d}v) = \int_\alpha \int_{\cup_{\beta\sim\alpha}\beta\cup\alpha} Y^\ell(z-w)\,m(\mathrm{d}u)\,m(\mathrm{d}v) + \mathrm{O}(\mathrm{e}^{-N^{\varepsilon}}),$$

and thus

$$\iint_{\alpha^2} (U^\ell(u-v) - Y^\ell(u-v))\,\mu_\alpha(\mathrm{d}u)\,\mu_\alpha(\mathrm{d}v) = \sum_{\beta\sim\alpha} \iint_{\alpha\times\beta} Y^\ell(z-w)\,\mu_\alpha(\mathrm{d}z)\,\mu_\beta(\mathrm{d}w) + \mathrm{O}(\mathrm{e}^{-N^{\varepsilon}}).$$

Therefore (3.48) equals

$$\bar{n}_\alpha \sum_{\beta\sim\alpha} \big(\bar{n}_\alpha - \bar{n}_\beta\big) \iint_{\alpha\times\beta} Y^\ell(z-w)\mu_\alpha(\mathrm{d}z)\mu_\beta(\mathrm{d}w) + \mathrm{O}(\mathrm{e}^{-N^\varepsilon}).$$

Note that

$$\iint_{\alpha\times\beta} Y^\ell(z-w)\mu_\alpha(\mathrm{d}z)\mu_\beta(\mathrm{d}w) = \mathrm{O}(b^{-2})\mathrm{O}(\ell b^{-1}) \sup_{z\in\alpha} \int Y^\ell(z-w)\, m(\mathrm{d}w) = \mathrm{O}(b^{-3}\ell^3).$$

Using $|\bar{n}_\alpha - \bar{n}_\beta| = \mathrm{O}(Nb^3)\|\nabla\rho_V\|_\infty$ and $\bar{n}_\alpha = \mathrm{O}(Nb^2)\|\rho_V\|_\infty$, the claim (3.48) follows.

For the boundary squares, we do not use any cancelation between the two terms in (3.48), but we still use the cancellation between U^ℓ and Y^ℓ. Analogously to the above, the two terms are each bounded by

$$\mathrm{O}(Nb^2)^2\mathrm{O}(b^{-3}\ell^3)\|\rho_V\|_\infty^2 = \mathrm{O}(N^2 b\ell^3)\|\rho_V\|_\infty^2.$$

This completes the proof. □

Proof of (3.42). By definition, we have

$$\hat{\mathbb{E}}[H^\ell_{Q,B}] = \hat{\mathbb{E}}\left[\sum_{i\neq j} Y^\ell(z_i - z_j)\mathbb{1}_{z_i,z_j\in B} + N\sum_{z_i\in B} Q(z_j) + 2\sum_{z_i\in D}\sum_{z_j\in B} Y^\ell(z_i-z_j)\right].$$

Moreover, by definition of the expectation $\hat{\mathbb{E}}$, the particles in B are distributed independently according to the equilibrium measure μ_V. If the particles in D were also distributed independently according to the equilibrium measure, the above right-hand side would be $N^2 I_{Q,B} + \mathrm{O}(\bar{n}_B \log N)$, with the error term $\mathrm{O}(\bar{n}_B\log m(B)) = \mathrm{O}(\bar{n}_B\log N)$ resulting from the inclusion of the diagonal $i=j$ in the first sum. In reality, the particles in D are distributed according to the periodic Yukawa gas in the squares α; under this measure the expected empirical measure is uniform on the squares α with constant density $\bar{n}_\alpha/N$; we may replace this constant density in the bulk squares by the density of the equilibrium measure with an error $\mathrm{O}(Nn_B\ell^2 bb')\|\nabla\rho_V\|_\infty = \mathrm{O}(Nn_B\ell^2 b)\|\nabla\rho_V\|_\infty$. In summary, we have

$$\hat{\mathbb{E}}[H^\ell_{Q,B}] = N^2 I_{Q,B} + \mathrm{O}(N^2\ell^2 b)\|\nabla\rho_V\|_\infty + \mathrm{O}(\bar{n}_B\log N)$$

as claimed. □

Bound of second term in (3.38). Recall the decomposition $H^\ell_Q = H^\ell_{Q,D} + H^\ell_{Q,B}$ from (3.40), and decompose H^R_V analogously. The following lemma is the key difficulty in the lower bound; it is in this term we need to use the randomness of the map Ψ in an essential way.

Lemma 3.15. *Assume $\ell/b \gg (Nb^2)^{-1/4}$. Then there exists a $\tau > 0$ such that*

$$\hat{\mathbb{E}}(H^\ell_{Q,D}(\Psi\mathbf{u}) - H^R_{V,D}(\Psi\mathbf{u})) - N\log(R/\ell) - N^2 K^\ell_R = N^\varepsilon \mathrm{O}(N^{1-\tau} + b^2\ell^{-4}) + \mathrm{O}(N^2 b^4)\|\nabla\rho\|_2^2 \tag{3.50}$$

$$\hat{\mathbb{E}}(H^\ell_{Q,B}(\Psi\mathbf{u}) - H^R_{V,B}(\Psi\mathbf{u})) = \mathrm{O}((\log N)b^{-2} + n_B\log N). \tag{3.51}$$

More precisely, $\mathrm{O}(N^{1-\tau})$ *is* $N^\varepsilon\mathrm{O}(N^{4/5}/\ell^{2/5})$.

Proof of (3.50). We start from (3.8), which states

$$\hat{\mathbb{E}}(H_Q^\ell \circ \Psi - H_V^R \circ \Psi) - N\log(R/\ell) - N^2 K_R^\ell = -N^2\hat{\mathbb{E}}(\mathbf{L}_R^\ell \circ \Psi) = -N^2\hat{\mathbb{E}}\sum_{\alpha,\beta}(\Omega_{\alpha\beta}\circ\Psi), \quad (3.52)$$

where

$$\Omega_{\alpha\beta}(\mathbf{z}) = \int_{v\in\alpha}\int_{w\in\beta} L_R^\ell(v-w)\,\tilde{\mu}_V^{\mathbf{z}}(\mathrm{d}v)\,\tilde{\mu}_V^{\mathbf{z}}(\mathrm{d}w),$$

and we have made the dependence of $\tilde{\mu}$ on $\mathbf{z}\in\mathbb{C}^N$ through the empirical measure $\hat{\mu}$ explicit. For $H_{Q,D}^\ell$ we restrict to $\alpha,\beta\subset D$. We rewrite $\Omega_{\alpha\beta}\circ\Psi$ as

$$\Omega_{\alpha\beta}(\Psi(\mathbf{u})) = \int_{\mathbb{T}_\alpha}\int_{\mathbb{T}_\beta} L_R^\ell(\Psi(v)-\Psi(w))\,(\hat{\mu}^{\mathbf{u}}(\mathrm{d}v)-\rho_V(\Psi(v))m(dv))\,(\hat{\mu}^{\mathbf{u}}(\mathrm{d}w)-\rho_V(\Psi(w))m(dw)).$$

For each bulk square $\alpha\subset D$, in particular, we have the important relation

$$N(\hat{\mu}^{\mathbf{u}}(\mathrm{d}v)-\rho_V(\Psi(v))m(dv))|_\alpha = n_\alpha\tilde{\mu}_\alpha(\mathrm{d}v) - N[\rho_V(\Psi(v))-\rho_V(\alpha)]\,m(\mathrm{d}v). \quad (3.53)$$

Since under $\hat{\mathbb{E}}$ the density on each square is a constant and since Ψ preserves the density, for any deterministic function g on the square indexed by α,

$$\hat{\mathbb{E}}\int g(\Psi(v))\,\tilde{\mu}_\alpha(\mathrm{d}v) = 0. \quad (3.54)$$

Expanding $\Omega_{\alpha\beta}$ using (3.53), and using that the cross-term vanishes in expectation by (3.54),

$$\begin{aligned} N^2\hat{\mathbb{E}}\Omega_{\alpha\beta} &= n_\alpha n_\beta\hat{\mathbb{E}}\iint_{z\in\alpha,w\in\beta} L_R^\ell(\Psi(z)-\Psi(w))\,\tilde{\mu}_\alpha(\mathrm{d}z)\,\tilde{\mu}_\beta(\mathrm{d}w) \\ &+ N^2\hat{\mathbb{E}}\iint_{z\in\alpha,w\in\beta} L_R^\ell(\Psi(z)-\Psi(w))[\rho_V(\Psi(z))-\rho_V(\alpha)]\,m(\mathrm{d}z)[\rho_V(\Psi(w))-\rho_V(\beta)]\,m(\mathrm{d}w). \end{aligned} \quad (3.55)$$

For the first term on the right-hand side of (3.55), notice that it vanishes for $\alpha\neq\beta$, using (3.54) and that particles in different squares are independently distributed. By Proposition 3.12 (in particular the remark about rescaling, with N replaced by Nb^2), we have

$$\begin{aligned} \sum_{\alpha\subset D} n_\alpha^2\hat{\mathbb{E}}\iint_{z\in\alpha,w\in\alpha} L_R^\ell(\Psi(z)-\Psi(w))\,\tilde{\mu}_\alpha(\mathrm{d}z)\,\tilde{\mu}_\alpha(\mathrm{d}w) &= N^\varepsilon b^{-2}\mathrm{O}((Nb^2)^{4/5}/(\ell/b)^{2/5}+b^4\ell^{-4}) \\ &= N^\varepsilon\mathrm{O}(N^{4/5}/\ell^{2/5}+b^2\ell^{-4}). \end{aligned}$$

For the second term on the right-hand side of (3.55), we can remove the Ψ, by changing variables using $|\mathrm{d}\Psi_\alpha|=1$. In Lemma 3.16 below, we show that its sum over α,β is smaller than the claimed error term, completing the proof of (3.50). □

In the following lemma, the naive size of the left-hand side is N^2b^2. We gain an extra factor b for each integration variable and thus obtain the resulting stronger estimate.

Lemma 3.16.

$$\begin{aligned} N^2\sum_{\alpha,\beta\subset D}\hat{\mathbb{E}}\iint_{\alpha\times\beta} L_R^\ell(z-w)[\rho_V(z)-\rho_V(\alpha)]\,m(\mathrm{d}z)[\rho_V(w)-\rho_V(\beta)]\,m(\mathrm{d}w) \\ = N^\varepsilon\mathrm{O}(N^2b^4)(\|\nabla\rho\|+\|\nabla^2\rho\|+\|\nabla^3\rho\|)^2. \end{aligned} \quad (3.56)$$

Proof. We first consider the diagonal terms $\alpha = \beta$ on the left-hand side of (3.56). The contribution of each such term is $\mathrm{O}(\log N)^2 N^2 b^6 \|\nabla\rho_V\|_\infty^2$, where a factor $b^2 \log N$ each is from the integrations of z and w and a factor $b^2\|\nabla\rho_V\|_\infty^2$ is from the size of $[\rho_V(z)-\rho(\alpha)][\rho_V(w)-\rho(\beta)]$. Since there are $\mathrm{O}(b^{-2})$ many bulk squares, this bounds the sum over the terms $\alpha=\beta$ as claimed.

Next we consider the off-diagonal terms $\alpha \neq \beta$. We use a Taylor expansion to find that the sum of these terms is bounded by

$$N^2 \sum_{\alpha\neq\beta} \iint_{\alpha\times\beta} \Big[\nabla\rho(\alpha)(z-\alpha) + \nabla^2\rho(\alpha)(z-\alpha)^2\Big]\Big[\nabla\rho(\beta)(w-\beta)+\nabla^2\rho(\beta)(w-\beta)^2\Big]$$
$$\times \Big[L^\ell(\alpha-\beta) + \nabla L^\ell(\alpha-\beta)(z-\alpha-w+\beta) + \nabla^2 L^\ell(\alpha-\beta)(z-\alpha-w+\beta)^2\Big]\, m(\mathrm{d}z)\, m(\mathrm{d}w),$$

where we have neglected the remainder term, which is bounded similarly without using symmetry and produces the error terms depending on $\|\nabla^3\rho\|$. By symmetry, the odd terms in $(z-\alpha)$ and $(w-\beta)$ do not contribute. The leading terms are therefore the quartic terms. These terms are bounded by

$$N^2 b^4 (\|\nabla\rho\| + \|\nabla^2\rho\|)^2.$$

The factor b^4 comes from $b^{-4}b^4b^4$ with the factor b^{-4} coming from the summation over squares; the b^4 factor coming from the volume of the integration of z and w, and the last b^4 factor comes from the size of products of $(z-\alpha)$ and $(w-\beta)$ in the formula. This concludes the proof. □

Proof of (3.51). We now bound $\hat{\mathbb{E}}\Omega_{\alpha\beta}$ for $\beta = B$ and $\alpha \subset D$. Since $\tilde\mu_V = \hat\mu - \mu_V$ and $\hat{\mathbb{E}}\hat\mu|_\alpha$ is the uniform measure on α with total mass $\bar n_\alpha/N$, we have

$$\hat{\mathbb{E}}(\tilde\mu_V|_\alpha(\mathrm{d}z)) = \left(\frac{\bar n_\alpha}{Nb^2\rho_V(z)} - 1\right)\mu_V|_\alpha(\mathrm{d}z),$$
$$\hat{\mathbb{E}}(\tilde\mu_V|_B(\mathrm{d}z)) = \left(\frac{\bar n_B}{N\mu_V(B)} - 1\right)\mu_V|_B(\mathrm{d}z).$$

Since $\hat\mu|_B$ and the $\hat\mu|_\alpha$ are independent under $\hat{\mathbb{E}}$, and since the number of squares α is $\mathrm{O}(b^{-2})$, therefore

$$\begin{aligned}
&N^2\sum_\alpha \hat{\mathbb{E}}\int_{z\in B}\int_{w\in\alpha} L_R^\ell(\Psi_B(z)-\Psi_\alpha(w))\,\tilde\mu_V(\mathrm{d}z)\,\tilde\mu_V(\mathrm{d}w)\\
&= \left(\frac{\bar n_B}{\mu_V(B)} - N\right)\sum_\alpha \int_{z\in B}\int_{w\in\alpha} L_R^\ell(z-w)\left(\frac{\bar n_\alpha}{b^2\rho_V(z)} - N\right)\mu_V(\mathrm{d}z)\,\mu_V(\mathrm{d}w)\\
&= \mathrm{O}\left((\log N)\big|\bar n_B - N\mu_V(B)\big|\sum_\alpha\big|\bar n_\alpha - N\mu_V(\alpha)\big|\right) = \mathrm{O}\left((\log N)b^{-2}\right).
\end{aligned}$$

Similarly, for $\alpha=\beta=B$, we have

$$\begin{aligned}
&N^2\mathbb{E}^\omega\int_{z\in B}\int_{w\in B} L_R^\ell(z-w)\,\mathbb{1}_{z\neq w}\,\tilde\mu(\mathrm{d}z)\,\tilde\mu(\mathrm{d}w)\\
&= \left(\frac{\bar n_B}{\mu_V(B)} - N\right)^2\iint_{B\times B} L_R^\ell(z-w)\,\mu_V(\mathrm{d}z)\,\mu_V(\mathrm{d}w)\\
&= \left(\frac{\bar n_B}{\mu_V(B)} - N\right)^2 \mathrm{O}(\log N)\mu_V(B)^2 = \mathrm{O}(\bar n_B\log N).
\end{aligned}$$

This completes the proof. □

Remark 3.17. *The term E in* (3.44) *can be alternatively be bounded by*

$$E = \mathrm{O}(N^2\ell^3 b^{-1}). \tag{3.57}$$

To see this, we can use (3.58), (3.59) *below, with N replaced by Nb^2. Using that there are b^{-2} many squares α implies the bound $b^{-2}(Nb^2)^2(\ell/b)^3 b^{-1} = N^2\ell^3 b^{-1}$. Notice that a similar argument was already used in the proof of the upper bound in* (3.18) *where a similar term can be found.*

Lemma 3.18. *Let $\mathbb{T}_\alpha$ be a torus of side length $b \gg \ell$. Then for any translation invariant random points z and w in $\mathbb{T}_\alpha$,*

$$\mathbb{E}\left(U^\ell_\alpha(z,w) - Y^\ell(z-w)\right) = \mathbb{E}\left(\frac{b|x| + b|y| - |xy|}{b^2} Y^\ell(d_\alpha(z,w))\right) + \mathrm{O}(\mathrm{e}^{-cb/\ell}), \tag{3.58}$$

where $d_\alpha(z,w)$ is the periodic distance and $z - w = (x,y)$ (with the difference on the torus).

Proof. The proof uses the same reasoning as in Lemma 3.9. Namely, the difference $U^\ell_\alpha(z,w) - Y^\ell(z-w)$ is negligible unless z and w have periodic distance order ℓ and Euclidean distance order b, i.e., z and w are on the opposite sides of the torus α. In this case the second term is negligible, i.e., $\mathrm{O}(\mathrm{e}^{-cb/\ell})$, and the first term is $Y^\ell(d_\alpha(z,w)) + \mathrm{O}(\mathrm{e}^{-cb/\ell})$. By translation invariance we may average over the position of the pair (z,w) with respect to the boundary; the prefactor inside the expectation on the right-hand side of (3.58) is the probability that z and w fall on opposite sides of the torus when the center is chosen uniformly randomly. □

Lemma 3.19. *Let $\mathbb{E}^\ell$ be the expectation of the Yukawa gas of range $N^{-1/2} \ll \ell \ll 1$ with N particles on the unit torus. Then*

$$N^2\mathbb{E}^\ell \iint (|x| + |y| - |xy|) U^\ell(z-w)\, \tilde{\mu}(\mathrm{d}z)\, \tilde{\mu}(\mathrm{d}w) = N^\varepsilon \mathrm{O}(N^2\ell^3). \tag{3.59}$$

Proof. It is sufficient to show that

$$N^2\mathbb{E}^\ell \iint |z-w|\, U^\ell(z-w)\, \hat{\mu}(\mathrm{d}z)\, \hat{\mu}(\mathrm{d}w) = N^\varepsilon \mathrm{O}(N^2\ell^3).$$

The estimate follows from the local density for $\hat{\mu}$ stated in Theorem A.2. More precisely, diving the unit torus into squares of length $b = N^\varepsilon \ell$, with very high probability, each square contains $\mathrm{O}(Nb^2)$ particles. Denoting the squares by α and β, the left-hand side of (3.59) is bounded by

$$N^2\mathbb{E}^\ell \sum_{\alpha,\beta} \iint_{\alpha\times\beta} |z-w|\, U^\ell(z-w)\, \hat{\mu}(\mathrm{d}z)\, \hat{\mu}(\mathrm{d}w). \tag{3.60}$$

Using the exponential decay of $U^\ell(z-w)$, up to an error of order $\mathrm{O}(\mathrm{e}^{-cN^\varepsilon})$, only the neighboring or equal pairs of squares α, β contribute to this sum. For each such pair, the contribution is $\mathrm{O}(N^2 b^5)$ with two factors of b^2 from the integrals over z and w and one from the factor $|z-w|$. Summing over the $\mathrm{O}(b^{-2})$ terms and using that $b = N^\varepsilon \ell$ the estimate (3.59) follows. □

3.7. Estimate for quasi-free free energy. Analogously to (2.33), we define

$$h_\alpha(\mathbf{n}) = 2\pi\gamma^2(n_\alpha - \bar{n}_\alpha)^2 - n_\alpha\zeta - \frac{1}{2}n_\alpha \log n_\alpha - \Big(\frac{1}{2} - \frac{1}{\beta}\Big)n_\alpha \log b^{-2}. \tag{3.61}$$

Then, similarly to Lemma 2.8, we have the following estimate for $F(\mathbf{n})$ defined in (3.11).

Lemma 3.20. *There exists $\tau > 0$ such that*

$$F(\mathbf{n}) + N\log\ell - N^2K_R^\ell - N^2I_V^R = \frac{1}{\beta}\log\binom{N}{\mathbf{n}} - \sum_{\alpha\subset D} h_\alpha(\mathbf{n}) + \mathrm{O}(N^2\ell^2 b)(1+\|\rho_V\|_\infty)^2 + \mathrm{O}(N^{1-\tau}). \tag{3.62}$$

More precisely, $\mathrm{O}(N^{1-\tau})$ is $N^\varepsilon\mathrm{O}(N^{7/8}/b^{1/4} + \ell^{-2})$.

Proof. Recall that

$$F(\mathbf{n}) = \frac{1}{\beta}\log\binom{N}{\mathbf{n}} - \sum_\alpha T_\alpha(n_\alpha) - 2c_V N n_B, \quad T_\alpha(n_\alpha) := -\frac{1}{\beta}\log\int_{\mathbb{T}_\alpha^{n_\alpha}} \mathrm{e}^{-\beta\hat{H}_\alpha(\mathbf{z})}\, m(\mathrm{d}\mathbf{z}).$$

It suffices to show that

$$\begin{aligned}\sum_{\alpha\subset D} T_\alpha(n_\alpha) + N^2 I_{Q,B} + 2c_V N(n_B - N\mu_V(B)) - N^2K_R^\ell \\ = N^2 I_V^R - N\log\ell + \sum_{\alpha\subset D} h_\alpha(\mathbf{n}) + \mathrm{O}(N^2\ell^2 b) + \mathrm{O}(N^{1-\tau}),\end{aligned}$$

which we now prove. By definition of T_α,

$$\begin{aligned}T_\alpha(n_\alpha) &= Nn_\alpha Q(\alpha) + 2\pi\gamma^2 n_\alpha^2 - n_\alpha\log\ell - \xi_b^{(\gamma)}(n_\alpha)\\ &= Nn_\alpha Q(\alpha) + 2\pi\gamma^2 n_\alpha^2 - n_\alpha\log\ell - n_\alpha\zeta^{(\gamma)}(n_\alpha) - \frac{1}{2}n_\alpha\log n_\alpha - \Big(\frac{1}{2} - \frac{1}{\beta}\Big)n_\alpha\log b^{-2}.\end{aligned}$$

By Theorem 2.1, $n_\alpha\zeta^{(\gamma)}(n_\alpha) = n_\alpha\zeta + N^\varepsilon\mathrm{O}(n_\alpha^{7/8} + 1/\gamma^2)$ so that $\sum_\alpha n_\alpha\zeta^{(\gamma)}(n_\alpha) = N\zeta + b^{-2}N^\varepsilon\mathrm{O}((Nb^2)^{7/8} + (\ell/b)^{-2}) = N\zeta + N^\varepsilon\mathrm{O}(N^{7/8}/b^{1/4} + \ell^{-2})$. Therefore

$$\begin{aligned}\sum_\alpha T_\alpha(n_\alpha) = \sum_\alpha\Big(Nn_\alpha Q(\alpha) + 2\pi\gamma^2 n_\alpha^2 - \frac{1}{2}n_\alpha\log n_\alpha - \Big(\frac{1}{2} - \frac{1}{\beta}\Big)n_\alpha\log b^{-2}\Big)\\ - N\log\ell - N\zeta + N^\varepsilon\mathrm{O}(N^{7/8}/b^{1/4} + \ell^{-2}).\end{aligned}$$

By definition of $h_\alpha(\mathbf{n})$ in (3.61) and since

$$\sum_\alpha 2\pi\gamma^2 n_\alpha^2 = 2\pi\gamma^2\sum_\alpha(n_\alpha - \bar{n}_\alpha)^2 + 4\pi\gamma^2\sum_\alpha n_\alpha\bar{n}_\alpha - 2\pi\gamma^2\sum_\alpha\bar{n}_\alpha^2,$$

we obtain

$$\sum_\alpha T_\alpha(n_\alpha) - \sum_\alpha h_\alpha(n_\alpha) + N\log\ell = \sum_\alpha\big(Nn_\alpha Q(\alpha) + 4\pi\gamma^2 n_\alpha\bar{n}_\alpha - 2\pi\gamma^2\bar{n}_\alpha^2\big) + N^\varepsilon\mathrm{O}(N^{7/8}/b^{1/4} + \ell^{-2}).$$

We now compute the right-hand side of the last equation. Using that $\gamma = \ell/b$, that $2\pi\ell^2 = \int Y^\ell(z)\, m(\mathrm{d}z) = \int_\alpha Y^\ell(z)\, m(\mathrm{d}z) + \mathrm{O}(\mathrm{e}^{-N^\varepsilon})$, and that $\bar n_\alpha = Nb^2\rho_V(z) + \mathrm{O}(Nb^3)\|\nabla\rho_V\|_\infty = N\int_\alpha \rho_V(w)\, m(\mathrm{d}w) + \mathrm{O}(Nb^3)\|\nabla\rho_V\|_\infty$ for any $z \in \alpha$, we obtain

$$\begin{aligned} 2\pi\gamma^2 \sum_\alpha \bar n_\alpha^2 &= N^2 \sum_\alpha \iint_{D\times\alpha} Y^\ell(z-w)\, \rho_V(z)\, \rho_V(w)\, m(\mathrm{d}z)\, m(\mathrm{d}w) + \mathrm{O}(N^2\ell^2 b)\|\rho_V\|_\infty \|\nabla\rho_V\|_\infty \\ &= N^2 \iint_{D\times D} Y^\ell(z-w)\, \mu_V(\mathrm{d}z)\, \mu_V(\mathrm{d}w) + \mathrm{O}(N^2\ell^2 b)\|\rho_V\|_\infty \|\nabla\rho_V\|_\infty. \end{aligned} \tag{3.63}$$

Analogously, we obtain

$$4\pi\gamma^2 \bar n_\alpha = 2N \int Y^\ell(\alpha - z)\rho_V(z)\, m(\mathrm{d}z) + \mathrm{O}(N\ell^2 b)\|\nabla\rho_V\|_\infty.$$

It follows that

$$\begin{aligned} \sum_\alpha [N n_\alpha Q(\alpha) + 4\pi\gamma^2 n_\alpha \bar n_\alpha] &= N \sum_\alpha n_\alpha \left[Q(\alpha) + 2\int Y^\ell(\alpha - z)\, \mu_V(\mathrm{d}z) \right] + \mathrm{O}(N^2\ell^2 b)\|\nabla\rho_V\|_\infty \\ &\qquad + 2N \sum_\alpha n_\alpha \int Y^\ell(\alpha - z)\, [\rho_V(z)\, m(\mathrm{d}z) - \mu_V(\mathrm{d}z)] \\ &= 2c_V N(N - n_B) + \mathrm{O}(N^2\ell^2 b)\|\nabla\rho_V\|_\infty \\ &\qquad - 2N^2 \iint_{D\times B} Y^\ell(z-w)\, \mu_V(\mathrm{d}z)\, \mu_V(\mathrm{d}w) + \mathrm{O}(N^2\ell^3 \|\rho_V\|_\infty^2), \end{aligned} \tag{3.64}$$

where the second equality follows from the Euler–Lagrange equation (3.9) and $\sum_\alpha n_\alpha = N - n_B$, and using that in the computation of $\iint_{D\times B} Y^\ell(z-w)\, \mu_V(\mathrm{d}z)\, \mu_V(\mathrm{d}w)$, the contribution of the absolutely continuous part of μ_V in B is of order $N^2\ell^2\|\rho_V\|_\infty^2$. Using also that $I_{Q,B} - 2c_V\mu_V(B) = -\iint_{B\times B} Y^\ell(z-w)\, \mu_V(\mathrm{d}z)\, \mu_V(\mathrm{d}w)$ by (3.25), in summary, we have proved

$$\begin{aligned} &\sum_\alpha T_\alpha(n_\alpha) + N^2 I_{Q,B} + 2c_V N(n_B - N\mu_V(B)) - \sum_\alpha h_\alpha(n_\alpha) + N\log\ell \\ &\qquad = 2c_V N^2 - N^2 \iint_{\mathbb{C}^2} Y^\ell(z-w)\, \mu_V(\mathrm{d}z)\, \mu_V(\mathrm{d}w) + \mathrm{O}(N^2\ell^2 b)(\|\rho_V\|_\infty + \|\nabla\rho_V\|_\infty)^2. \end{aligned}$$

The claim (3.62) now follows from the Euler–Lagrange equation (3.9), which implies

$$\begin{aligned} 2c_V &= 2\iint Y^R(z-w)\, \mu_V(\mathrm{d}z)\, \mu_V(\mathrm{d}w) + \int V(z)\, \mu_V(\mathrm{d}z) \\ &= \iint Y^R(z-w)\, \mu_V(\mathrm{d}z)\, \mu_V(\mathrm{d}w) + I_V^R = \iint Y^\ell(z-w)\, \mu_V(\mathrm{d}z)\, \mu_V(\mathrm{d}w) + K_R^\ell + I_V^R. \end{aligned}$$

This completes the proof. ∎

We need the following bound showing that in the sum over $\mathbf{n}$ the dominant term is $\mathbf{n} = \bar{\mathbf{n}}$.

Lemma 3.21. *Assume that $|\mathcal{E}_\alpha(n) - \mathcal{E}_\alpha(m)| \leqslant |n-m|(n+m)^\varepsilon$ and define*

$$\mathcal{E}(\mathbf{n}) = \sum_{\alpha\subset D} \Big[-2\pi\gamma^2(n_\alpha - \bar n_\alpha)^2 + \mathcal{E}_\alpha(n_\alpha) \Big]. \tag{3.65}$$

Then

$$\frac{1}{\beta}\log\sum_{\mathbf{n}} e^{\beta\mathcal{E}(\mathbf{n})+\beta O(n_B\log N)} \leqslant \mathcal{E}(\bar{\mathbf{n}}) + N^{\varepsilon} O(Nb' + \ell^{-2}\|\rho_V\|_\infty). \tag{3.66}$$

Proof. By definition,

$$\begin{aligned}\frac{1}{\beta}\log\sum_{\mathbf{n}} e^{\beta\mathcal{E}(\mathbf{n})+\beta O(n_B\log N)} - \mathcal{E}(\bar{\mathbf{n}}) &= \frac{1}{\beta}\log\sum_{\mathbf{n}} e^{\beta(\mathcal{E}(\mathbf{n})-\mathcal{E}(\bar{\mathbf{n}}))+\beta O(n_B\log N)}\\ &= \frac{1}{\beta}\log\sum_{\mathbf{n}}\exp\left[\sum_\alpha \beta\Big[-2\pi\gamma^2(n_\alpha-\bar n_\alpha)^2 + (\mathcal{E}_\alpha(n_\alpha)-\mathcal{E}_\alpha(\bar n_\alpha)\Big] + O(\beta n_B\log N)\right].\end{aligned} \tag{3.67}$$

By the constraint $N=\sum_\alpha n_\alpha = n_B + \sum_{\alpha\subset D} n_\alpha$, we can add the factor

$$\begin{aligned}\mathbb{1}\Big(n_B-\bar n_B = \sum_{\alpha\subset D}(\bar n_\alpha - n_\alpha)\Big) &\leqslant \mathbb{1}\Big(|n_B-\bar n_B| \leqslant \sum_{\alpha\subset D}|\bar n_\alpha - n_\alpha|\Big)\\ &\leqslant \exp\left[-\frac{\beta 2\pi\gamma^2}{2\#\{\alpha\subset D\}}(n_B-\bar n_B)^2 + \frac{\beta2\pi\gamma^2}{2}\sum_{\alpha\subset D}(\bar n_\alpha - n_\alpha)^2\right],\end{aligned}$$

where we used that $\mathbb{1}(a\leqslant b)\leqslant e^{-ca^2+cb^2}$ and $(\sum_\alpha x_\alpha)^2 \leqslant \#\{\alpha\subset D\}\sum_\alpha x_\alpha^2$ where $\#\{\alpha\subset D\} = O(b^{-2})$ is the number of squares. Thus, at the cost of replacing $2\pi\gamma^2$ by $\pi\gamma^2$ in (3.67), we can add the factor

$$\exp\left[-c\beta b^2\gamma^2(n_B-\bar n_B)^2\right] = \exp\left[-\beta c\ell^2(n_B-\bar n_B)^2\right].$$

With this preparation, to get an upper bound, we now drop the constraint $\sum_\alpha n_\alpha = N$ on $\mathbf{n}$, and sum each n_α independently. For the bulk squares, we use $|\mathcal{E}_\alpha(n)-\mathcal{E}_\alpha(m)| \le |n-m|(n+m)^\varepsilon$ and the elementary inequality that for any positive fixed number $c>0$ and any integer $m\geq 0$,

$$\sum_{n=0}^{\infty}\exp\left[|n-m|(n+m)^\varepsilon - c\gamma^2(n-m)^2\right] \le C\gamma^{-1}(m+\gamma^{-2})^{2\varepsilon} e^{C\gamma^{-2}(m+\gamma^{-2})^{2\varepsilon}}.$$

For the boundary layer B, we similarly use

$$\sum_{n=0}^{\infty}\exp\left[n\log N - c\ell^2(n-m)^2\right] \leqslant C\ell^{-1}e^{C(m+\ell^{-2})(\log N)^2}.$$

In summary, using $\bar n_\alpha = O(Nb^2)\|\rho_V\|_\infty$ for $\alpha\subset D$ and $\bar n_B = O(Nb')$, the left-hand side of (3.67) is bounded by

$$C(\log N)\sum_{\alpha\subset D}\gamma^{-2}(\bar n_\alpha+\gamma^{-2})^{2\varepsilon}\|\rho_V\|_\infty + C(\log N)^2(\bar n_B+\ell^{-2}) \le CN^{2\varepsilon}\ell^{-2}\|\rho_V\|_\infty + CNb'(\log N)^2.$$

This completes the proof of the lemma. □

3.8. Existence of free energy of Yukawa gas: proof of Theorem 3.2. The proof of Theorem 3.2 below is analogous to that of Proposition 2.10.

Proof of Theorem 3.2. We apply Propositions 3.5, 3.6. First of all, any choice $N^{-1/4} \ll b' \ll 1$ is admissible since the error terms involving b' are $N^\varepsilon O(Nb')$ using that $n_B = O(Nb')$. In the situation of Theorem 1.1, this error term is smaller than the claimed error term. In the situation of Remark 3.1, it is $N^{\varepsilon}\mathrm{O}(N^{1-a})$, as claimed in the remark. Moreover, in Propositions 3.5, 3.6, the range parameter ℓ is not required to be the same in the upper and lower bound. We denote the value of ℓ by ℓ_+ for the upper bound and by ℓ_- for the lower bound.

We first consider the case $N^{-1/2+\sigma} \leqslant R \leqslant 1$. Take $b = N^{-1/2+\sigma/10}$. For $\ell_+ = N^{-1/2+\sigma/100}$, the error terms in (3.12) are bounded by $N^{1-\sigma/1000}$. For $\ell_- = N^{-1/2+9\sigma/100}$, the error terms in (3.13) are also bounded by $N^{1-\sigma/1000}$ (we used $\bar{n}_B = \mathrm{O}(Nb')$).

With Lemma 3.20, for some $\kappa = \kappa(\sigma) > 0$ we therefore obtain

$$\frac{1}{\beta}\log\int \mathrm{e}^{-\beta H_V^R(\mathbf{z})} m(\mathrm{d}\mathbf{z}) \geqslant -N^2 I_V^R + N\log R + \frac{1}{\beta}\log\binom{N}{\bar{\mathbf{n}}}\mathrm{e}^{-\beta\sum_\alpha h_\alpha(\bar{\mathbf{n}})} - \mathrm{O}(N^{1-\kappa}), \tag{3.68}$$

$$\frac{1}{\beta}\log\int \mathrm{e}^{-\beta H_V^R(\mathbf{z})} m(\mathrm{d}\mathbf{z}) \leqslant -N^2 I_V^R + N\log R + \frac{1}{\beta}\log\sum_{\mathbf{n}}\binom{N}{\mathbf{n}}\mathrm{e}^{-\beta\sum_\alpha h_\alpha(\mathbf{n})} + \mathrm{O}(N^{1-\kappa}). \tag{3.69}$$

By Stirling's formula as in (2.39), (3.61), and the definition of $\mathcal{E}$ in (3.65) with $\mathcal{E}_1(n) = (\frac{1}{2} - \frac{1}{\beta})n\log(nb^{-2})$, we can rewrite (3.68), (3.69) as

$$\frac{1}{\beta}\log\int \mathrm{e}^{-\beta H_V^R(\mathbf{z})}\, m(\mathrm{d}\mathbf{z}) + N^2 I_V^R \geqslant \mathcal{E}(\bar{\mathbf{n}}) + \zeta + N\log R + \frac{1}{\beta}N\log N + \mathrm{O}(N^{1-\kappa}), \tag{3.70}$$

$$\frac{1}{\beta}\log\int \mathrm{e}^{-\beta H_V^R(\mathbf{z})}\, m(\mathrm{d}\mathbf{z}) + N^2 I_V^R \leqslant \frac{1}{\beta}\log\sum_{\mathbf{n}}\mathrm{e}^{\beta\mathcal{E}(\mathbf{n})} + \zeta + N\log R + \frac{1}{\beta}N\log N + \mathrm{O}(N^{1-\kappa}). \tag{3.71}$$

By Lemma 3.21, we can replace the sum over $\mathbf{n}$ in (3.71) by the dominant term $\bar{\mathbf{n}}$ with error smaller than $\mathrm{O}(N^{1-\kappa})$. By a Riemann sum approximation using that ρ_V is C^1 in D,

$$\begin{aligned}\mathcal{E}(\bar{\mathbf{n}}) &= \left(\frac{1}{2}-\frac{1}{\beta}\right)\sum_\alpha \bar{n}_\alpha\log(\bar{n}_\alpha b^{-2})\\ &= \left(\frac{1}{2}-\frac{1}{\beta}\right)N\int \rho_V(z)\log\rho_V(z)\, m(\mathrm{d}z) + \left(\frac{1}{2}-\frac{1}{\beta}\right)N\log N + \mathrm{O}(N(b+b'))\|\rho_V\|_{C^1}.\end{aligned}$$

This completes the proof of (3.4) when $N^{-1/2+\sigma} \leqslant R \leqslant 1$.

When $R \geqslant 1$, we consider all error terms in details. In (3.12), the error is

$$\mathrm{O}(N^{\varepsilon})\left(N^2\ell_+^3 b^{-1} + N^2\ell_+^2 b\right)$$

while in (3.13) it is of order

$$\mathrm{O}(N^{\varepsilon})\left(b^2\ell_-^{-4} + N^2 b^4 + N^2\ell_-^2 b + Nb + N^{4/5}/\ell_-{}^{2/5}\right).$$

Lemma 3.20 gives analogues of (3.68) and (3.69) with an additional error term

$$\mathrm{O}(N^{\varepsilon})\left(N^{7/8}/b^{1/4} + 1/\ell_-^2 + 1/\ell_+^2\right).$$

The above three bounds yield the following possible parameters at the optimum: $b = N^{-1/3}$, $\ell_+ = N^{-23/48}$, $\ell_- = N^{-7/18}$. The common error then becomes $\mathrm{O}(N^{23/24+\varepsilon})$ for arbitrarily small $\varepsilon > 0$. Note that these choices of parameters satisfy the hypothesis $\ell_-/b \gg (Nb^2)^{-1/4}$ necessary to apply Proposition 3.12 (remember $t = N^{-1/4}$ and N is substituted by Nb^2 in our setting). The rest of the proof is unchanged. □

3.9. Existence of free energy of Coulomb gas: proof of Theorem 1.1. We now take $R \to \infty$ to deduce Theorem 1.1 from Theorem 3.2.

Proof of Theorem 1.1. The equilibrium measure μ_V of the Coulomb gas in Theorem 1.1 is characterized by the Euler–Lagrange equation

$$U^{\mu_V} + \frac{1}{2} V = c_V \tag{3.72}$$

in its support S_V and inequality in all of $\mathbb{C}$. Define the potential V_R via the equation

$$V_R(z) = V(z) + 2 \int \left(\log \frac{1}{|z-w|} - Y^R(z-w) + Y_0 + \log R \right) \mu_V(\mathrm{d}w). \tag{3.73}$$

Explicitly, one can check that in S_V,

$$U_R^{\mu_V} + \frac{1}{2} V_R = c_V^R, \qquad c_V^R = c_V + Y_0 + \log R, \tag{3.74}$$

holds and with the inequality $\geq c_V^R$ outside the support of S_V. Thus μ_V is also the equilibrium measure with respect to the Yukawa interaction and external potential V_R. Moreover, by (2.2),

$$\begin{aligned} I_{V_R}^R &= \int U_R^{\mu_V}(z) \mu_V(\mathrm{d}z) + \int V_R(z)\, \mu_V(\mathrm{d}z) \\ &= \int U_R^{\mu_V}(z) \mu_V(\mathrm{d}z) + \int V(z)\, \mu_V(\mathrm{d}z) + 2 \int (U^{\mu_V}(z) - U_R^{\mu_V}(z) + Y_0 + \log R) \\ &= I_V^{\mathcal{C}} + (Y_0 + \log R) + \mathrm{O}(\frac{1}{R}). \end{aligned} \tag{3.75}$$

Thus we have

$$\frac{1}{\beta} \log \int \mathrm{e}^{-\beta H_{V_R}^{Y^R}}\, m(\mathrm{d}\mathbf{z}) = \frac{1}{\beta} \log \int \mathrm{e}^{-\beta H_V^{\mathcal{C}}}\, m(\mathrm{d}\mathbf{z}) - N(N-1)(Y_0 + \log R) + \mathrm{O}\left(\frac{N^2}{R} \right).$$

Moreover, (2.2) and an analogous estimate for derivatives of (2.1) imply

$$\max_{k \leqslant 5} \|\nabla^k (V_R - V)\|_\infty = \mathrm{O}\left(\frac{1}{R} \right). \tag{3.76}$$

Thus, we may apply Theorem 3.2 with V replaced by V_R and with $R \gg N$, and Theorem 1.1 then follows with $\zeta_\beta^{\mathcal{C}} = \zeta - Y_0$. □

4 Proof of Theorem 1.2: central limit theorem

In this section, we prove Theorem 1.2. We use Theorem 1.1, the loop equation, and the extension of the local density estimate of [6] to the Coulomb gas with an additional small interaction given by a local angle term (Appendix A).

4.1. CLT for macroscopic test functions. We first prove Theorem 1.2 for macroscopic test functions f. For this, we first prove that a version of Theorem 1.2 holds up to certain random shift, the local angle term $\hat{A}^f_V$ defined by

$$\hat{A}^f_V = \frac{N}{2}\,\mathrm{Re}\iint_{z\neq w} \frac{h(z)-h(w)}{z-w}\mathrm{e}^{-\frac{|z-w|^2}{2\theta^2}}\,\tilde{\mu}_V(\mathrm{d}z)\,\tilde{\mu}_V(\mathrm{d}w), \qquad h(z)=\frac{\bar\partial f(z)}{\partial\bar\partial V(z)}, \tag{4.1}$$

where $\theta = N^{-1/2+\sigma}$. Note the integrand is singular at $z=w$ since

$$\frac{h(z)-h(w)}{z-w} = \partial h(z) + \bar\partial h(z)\frac{\bar z-\bar w}{z-w} + \mathrm{O}(|z-w|).$$

We recall the definitions of X^f_V and Y^f_V from (1.11) and (1.12), as well as the norms from (1.10), and we write $\|f\|_k$ for $\|f\|_{k,b}$ with $b=1$.

In the proof of [6, Theorem 1.2], more precisely in [6, Lemma 7.5], we showed that (4.1) is bounded by $\mathrm{O}(N^\varepsilon)$ with very high probability. Assuming this term was $\ll 1$ instead of $\mathrm{O}(N^\varepsilon)$, a small modification of the argument in [6, Section 7] would already imply Theorem 1.2. A similar strategy was used in [4, 5], where a version of (4.1) was shown to be approximately equal to $-\frac{1}{2}Y^f_V$ for $\beta=1$, by using the exactly known correlation kernel for the microscopic correlation functions in this integrable case. Our strategy now is to first prove a version of Theorem 1.2 in which the contribution of the angle term (4.1) has been removed (in Proposition 4.1 below), and then subsequently, by combining this argument with Theorem 1.1, prove that the angle term (4.1) is in fact negligible up to the constant $-\frac{1}{2}Y^f_V$ (in Proposition 4.2).

Proposition 4.1. *Suppose that V satisfies conditions* (1.4) *and* (1.9), *or more generally the conditions stated in Remark 3.1. Then for any small σ, the following holds. For any function f satisfying the same assumptions as in Theorem 1.2 (in particular the support of f has distance of order* 1 *to ∂S_V), for small ε and $tb^{-2}N^{2\sigma}+tb^{-2}\|f\|_{4,b}\ll 1$, we have for any $0\le|u|\le \mathrm{O}(t)$*

$$\begin{aligned}\frac{1}{t\beta N}\log\mathbb{E}_V\mathrm{e}^{-\beta N t(X^f_V-\hat{A}^f_{V+uf})} &= \frac{tN}{8\pi}\int|\nabla f(z)|^2\,m(\mathrm{d}z) - \frac{1}{\beta}Y^f_V\\ &\quad + \mathrm{O}(N^{-1/2+\varepsilon}b^{-1}+N^{-\sigma+\varepsilon})\|f\|_{3,b} + \mathrm{O}(tN^{2\sigma+\varepsilon}b^{-2})\|f\|^2_{4,b}.\end{aligned} \tag{4.2}$$

Proposition 4.2. *There exists $\kappa>0$ such that if $\sigma=\kappa/6$ and $0\le|u|,t\leqslant N^{-2\kappa/3}$,*

$$\frac{1}{t\beta N}\log\mathbb{E}_V\mathrm{e}^{\beta N t\hat{A}^f_{V+uf}} = -\frac{1}{2}Y^f_V + \mathrm{O}(N^{-\kappa/3})(1+\|f\|_5)^2. \tag{4.3}$$

The above two propositions will be proved in Sections 4.2, 4.5 below. Proposition 4.1 without the angle term $\hat{A}^f_{V+uf}$ would imply a CLT for X^f_V. This angle term is controlled in Proposition 4.2. By combining them, we next complete the proof of Theorem 1.2 for macroscopic test functions. For mesoscopic test functions, a similar argument applies after conditioning (see Section 4.6).

Proof of Theorem 1.2 for macroscopic test functions. By assumption, f is a macroscopic test function with $\|f\|_5$ bounded. Let σ and κ be as in Proposition 4.2. Then, with $\lambda=Nt$ in the identity

$$\frac{1}{t\beta N}\log\mathbb{E}_V\mathrm{e}^{-\beta N t X^f_V} = \frac{1}{t\beta N}\left(\log\mathbb{E}_V\mathrm{e}^{-\beta N t(X^f_V-\hat{A}^f_V)} - \log\mathbb{E}_{V+tf}\mathrm{e}^{\beta N t\hat{A}^f_V}\right), \tag{4.4}$$

the claim follows from using the estimates (4.2), (4.3) for the two terms on the right-hand side of (4.4), and replacing κ by 3κ. □

4.2. Loop equation with angle term. We start the proof with an integration by parts formula. Consider a smooth bounded function $v : \mathbb{C} \to \mathbb{C}$, and G smooth, defined on $z_1 \neq z_2$ such that $G(z_1, z_2) = G(z_2, z_1)$, and

$$\limsup_{|z_2| \to \infty} (|G(z_1, z_2)| / \log |z_2|) \leqslant 1. \tag{4.5}$$

for any fixed z_1. For any $\mathbf{z} \in \mathbb{C}^N$ we define

$$W_V^{G,v}(\mathbf{z}) = -\sum_{j \neq k} (v(z_j) - v(z_k)) \partial_{z_j} G(z_j, z_k) + \frac{1}{\beta} \sum_j \partial_j v(z_j) - N \sum_j v(z_j) \partial V(z_j). \tag{4.6}$$

The following elementary lemma is often referred to as Ward identity or loop equation. For example, it was used in [5] to study fluctuations of the empirical measure when $\beta = 1$, and in [6] to prove rigidity for all $\beta > 0$, with in both cases the interaction G being the Coulomb potential $\mathcal{C}$. Its relation to Conformal Field Theory is discussed in [**?**]. In this work we need a perturbation G of the Coulomb interaction by the local angle term.

Lemma 4.3. *Under the above assumptions, we have*

$$\mathbb{E}_V^G \left(W_V^{G,v} \right) = \frac{1}{2} \mathbb{E}_V^G \Big(\sum_{j \neq k} (v(z_j) + v(z_k)) (\partial_{z_k} + \partial_{z_j}) G(z_j, z_k) \Big),$$

where the expectation is with respect to $P_{N,V}^G$ defined in (1.3).

Proof. The proof is a classical simple integration by parts: for any $j \in [\![1, N]\!]$, we have

$$\mathbb{E} \left(\partial_{z_j} v(z_j) \right) = \beta \mathbb{E} \left(v(z_j) \partial_{z_j} H(\mathbf{z}) \right),$$

where both terms are absolutely summable and the boundary terms vanishes because (i) with probability 1, no two z_i's have the same real or imaginary part, (ii) v is bounded, G satisfies the growth condition (4.5), V satisfies the growth condition (1.4). Summation of the above equation over all $j \in [\![1, N]\!]$ therefore gives

$$\frac{1}{\beta N} \sum_{j=1}^N \mathbb{E}(\partial_{z_j} v(z_j)) = \mathbb{E} \Big(\sum_{j=1}^N v(z_j) \Big(\partial_{z_j} V(z_j) + \sum_{k \neq j} (\partial z_j G(z_j, z_k) + \partial z_j G(z_k, z_j)) \Big) \Big)$$

$$= \mathbb{E} \Big(\sum_{j=1}^N v(z_j) \Big(\partial_{z_j} V(z_j) + \sum_{k \neq j} (\partial z_j - \partial z_k) G(z_j, z_k) \Big) \Big) + \mathbb{E} \Big(\sum_{j=1} \sum_{k \neq j} v(z_j) (\partial z_j + \partial z_k) G(z_j, z_k) \Big)$$

Using $G(z_j, z_k) = G(z_k, z_j)$, we can continue the equation with

$$= \mathbb{E} \Big(\sum_{j=1}^N v(z_j) \partial_{z_j} V(z_j) + \frac{1}{2} \sum_{j \neq k} \Big(v(z_j)(\partial z_j - \partial z_k) G(z_j, z_k) + v(z_k)(\partial z_k - \partial z_j) G(z_k, z_j) \Big) \Big)$$
$$+ \frac{1}{2} \mathbb{E} \Big(\sum_{j \neq k} (v(z_j) + v(z_k)) (\partial z_j + \partial z_k) G(z_j, z_k) \Big)$$
$$= \mathbb{E} \Big(\sum_{j=1}^N v(z_j) \partial_{z_j} V(z_j) + \frac{1}{2} \sum_{j \neq k} \Big(v(z_j) - v(z_k) \Big) (\partial z_j - \partial z_k) G(z_j, z_k) \Big)$$
$$+ \frac{1}{2} \mathbb{E} \Big(\sum_{j \neq k} (v(z_j) + v(z_k)) (\partial z_j + \partial z_k) G(z_j, z_k) \Big).$$

This concludes the proof. ☐

Before considering the interaction G with additional angle term, we temporarily restrict our attention to the Coulomb case, where $\partial_{z_j}\mathcal{C}(z_j-z_k)=-\frac{1}{2}(z_j-z_k)^{-1}$.

Lemma 4.4. *For any $f:\mathbb{C}\to\mathbb{R}$ of class $\mathscr{C}^2$, supported on S_V, and $\mathbf{z}\in\mathbb{C}^N$, we have*

$$X_V^f=-\frac{1}{N}W_V^h(\mathbf{z})+\frac{1}{N\beta}\sum_k\partial h(z_k)+\frac{N}{2}\iint_{z\neq w}\frac{h(z)-h(w)}{z-w}\,\tilde{\mu}_V(\mathrm{d}z)\,\tilde{\mu}_V(\mathrm{d}w), \tag{4.7}$$

where h is is defined in (4.1) *depending on f and V, X_V^f as in* (1.11)*, and $\tilde{\mu}_V=\hat{\mu}-\mu_V$, and we used the notation $W_V^h=W_V^{\mathcal{C},h}$.*

Proof. First remember the following two identities:

$$\int\frac{\mu_V(\mathrm{d}w)}{z-w}=\partial V(z),\quad f(z)=\frac{1}{\pi}\int\frac{\bar{\partial}f(w)}{z-w}m(\mathrm{d}w). \tag{4.8}$$

The first equation holds for $z\in S_V$ and is obtained by the Euler-Lagrange equation, the second equation is a simple integration by parts. We therefore can write

$$\begin{aligned}
X_V^f&=\sum_j\int\frac{h(w)}{z_j-w}\mu_V(\mathrm{d}w)-N\iint\frac{h(w)}{z-w}\mu_V(\mathrm{d}w)\mu_V(\mathrm{d}z)\\
&=N\iint\frac{h(w)-h(z)}{z-w}\hat{\mu}_V(\mathrm{d}z)\mu_V(\mathrm{d}w)+\sum_j h(z_j)\partial V(z_j)-\frac{N}{2}\iint\frac{h(w)-h(z)}{z-w}\mu_V(\mathrm{d}w)\mu_V(\mathrm{d}z)\\
&=-\frac{1}{2N}\sum_{j\neq k}\frac{h(z_j)-h(z_k)}{z_j-z_k}+\sum_j h(z_j)\partial V(z_j)+\frac{N}{2}\iint_{z\neq w}\frac{h(z)-h(w)}{z-w}\tilde{\mu}_V(\mathrm{d}z)\tilde{\mu}_V(\mathrm{d}w),
\end{aligned}$$

which is equivalent to (4.7). In the first equation we used (1.6) and (4.8), and in the second equation we used (4.8). □

We now decompose the last term in (4.7) into a sum of the long-range and short-range terms. For this purpose, let $\varphi(z)=\mathrm{e}^{-|z|^2}$ and, given a mesoscopic scale $\theta=N^{-\frac{1}{2}+\sigma}$, we define

$$\begin{aligned}
\Phi(z-w,r)&=\frac{2}{\pi}\int\varphi\left(\frac{|z-\xi|}{r}\right)\varphi\left(\frac{|\xi-w|}{r}\right)\mathrm{d}m(\xi)=r^2\mathrm{e}^{-\frac{|z-w|^2}{2r^2}},\\
\Phi_\theta^-(z-w)&=\int_0^\theta\Phi(z-w,r)\frac{\mathrm{d}r}{r^5}=\frac{\mathrm{e}^{-\frac{|z-w|^2}{2\theta^2}}}{|z-w|^2},\\
\Phi_\theta^+(z-w)&=\int_\theta^\infty\Phi(z-w,r)\frac{\mathrm{d}r}{r^5}=\frac{1-\mathrm{e}^{-\frac{|z-w|^2}{2\theta^2}}}{|z-w|^2},
\end{aligned} \tag{4.9}$$

$$\Psi_h^\pm(z,w)=\Phi_\theta^\pm(z-w)(\bar{z}-\bar{w})(h(z)-h(w)),\quad \Psi^h(z,w)=\Psi_h^+(z,w)+\Psi_h^-(z,w). \tag{4.10}$$

As in the proof of [6, Lemma 7.5] (see also [20]), we have decomposed the last term in (4.7) into a relatively long range part and, essentially, a local angle term:

$$\frac{N}{2}\iint_{z\neq w}\frac{h(z)-h(w)}{z-w}\tilde{\mu}_V(\mathrm{d}z)\,\tilde{\mu}_V(\mathrm{d}w)=A_V^{h,+}+A_V^{h,-},$$

where

$$A_V^{h,+} = \frac{N}{2} \iint_{z \neq w} \Psi_h^+(z,w)\, \tilde{\mu}_V(\mathrm{d}z)\, \tilde{\mu}_V(\mathrm{d}w), \tag{4.11}$$

$$A_V^{h,-} = \frac{N}{2} \iint_{z \neq w} \Psi_h^-(z,w)\, \tilde{\mu}_V(\mathrm{d}z)\, \tilde{\mu}_V(\mathrm{d}w). \tag{4.12}$$

Note that, in the above decomposition, we could have considered any fixed non-negative function $\varphi \in C^\infty(\mathbb{C})$ with compact support or fast decay at infinity, as in [6, Lemma 7.5]. We here chose the Gaussian scale function for the sake of concreteness and some convenient simplifications. Compared with [6], we also write the mesoscopic scale as θ rather than $N^{-1/2}\theta$.

4.3. Coulomb gas with angle perturbation. We now define the perturbed Coulomb gas. The Coulomb gas, exponentially tilted by the real-part of the local angle term, is defined to have pair interaction and potential given by

$$G_t = \mathcal{C} - \frac{t}{2} \operatorname{Re} \Psi_h^-, \quad V_t = V + tf + tF, \quad F = \operatorname{Re} \int \Psi_h^-(\cdot, w)\, \mu_V(\mathrm{d}w), \tag{4.13}$$

where $h = \frac{\bar{\partial} f}{\partial \bar{\partial} V}$ coinciding with h_0 defined in (4.16) below. We also include a t-dependent constant in the perturbed Hamiltonian and define

$$H_t := H_{V_t}^{G_t} - \frac{t}{2} N^2 \operatorname{Re} \iint \Psi_h^-(z,w)\, \mu_V(\mathrm{d}z)\, \mu_V(\mathrm{d}w) = H_{V+tf}^{\mathcal{C}} - Nt\hat{A}_V^f. \tag{4.14}$$

For the proof of Proposition 4.1, we require the following local density estimate for this interaction. It is a minor modification of [6, Theorem 1.1], and is proved in Theorem A.3.

Proposition 4.5. *Consider the Coulomb gas with Hamiltonian (4.14), with $V, f \in \mathscr{C}^2$ and $tN^{2\sigma} \leqslant 1$ and $\|\nabla h\|_\infty \leqslant 1$ and $t \in [0,1]$. For all $s \in (0, \frac{1}{2})$, for all f supported in ball of radius $b = N^{-s}$ contained in S_V with distance of order 1 to the boundary, we have the local density estimate*

$$X_f^{V_t} \prec \sqrt{Nb^2} \|f\|_{2,b} \tag{4.15}$$

with respect to the measure $P_{V_t}^{G_t}$. In particular, for any ball as above, the number of particles in that ball is bounded with high probability by $\mathrm{O}(Nb^2)$.

For $0 \leq t \ll 1$, we define

$$h_t(z) = \frac{\bar{\partial} f(z)}{\partial \bar{\partial}(V(z) + tf(z))}, \qquad h = h_0. \tag{4.16}$$

In the next lemma, we collect some elementary estimates for h_t and F_t.

Lemma 4.6. *Assume that the support of f has distance $\gg N^{-1/2+\sigma}$ to ∂S_V, and that*

$$tb^{-2} \|f\|_{4,b} \ll 1. \tag{4.17}$$

Then the following estimates hold:

$$\|h_t\|_{k,b} \leq b^{-1} \|f\|_{k+1,b} \big[1 + tb^{-2} \|f\|_{k+2,b}\big], \tag{4.18}$$

$$tF(z) = \mathrm{O}(N^{-1+2\sigma}) tb^{-2} \|f\|_{2,b}, \tag{4.19}$$

$$t\Delta F(z) = \mathrm{O}(N^{-1/2+\sigma}) tb^{-3} \|f\|_{4,b}. \tag{4.20}$$

Proof. Using that $t\|\Delta f\|_\infty \ll 1$ and (4.17), we have

$$\|\nabla h_t\|_\infty \le \frac{\|\nabla\bar\partial f\|_\infty}{\|\partial\bar\partial(V+tf)\|_\infty} + \frac{\|\bar\partial f \nabla(\partial\bar\partial(V+tf))\|_\infty}{\|\partial\bar\partial(V+tf)\|_\infty^2} \le b^{-2}\|f\|_{2,b}\big[1+tb^{-2}\|f\|_{3,b}\big].$$

Similar estimates hold for higher derivatives and we get in general (4.18). We can bound tF by

$$tF(z) = t\int \frac{h(z)-h(w)}{z-w}\mathrm{e}^{-\frac{|z-w|^2}{2\theta^2}}\,\mu_V(\mathrm{d}w) = \mathrm{O}(N^{-1+2\sigma})t\|\nabla h\|_\infty = \mathrm{O}(N^{-1+2\sigma})tb^{-2}\|f\|_{2,b},$$

which is a small correction to $V+tf$. Similarly, we have

$$\begin{aligned} t\Delta F(z) = t\Delta h(z)\int \frac{\mathrm{e}^{-\frac{|z-w|^2}{2\theta^2}}}{z-w}\mu_V(\mathrm{d}w) - 2t\nabla h(z)\int\left(\nabla_w\frac{\mathrm{e}^{-\frac{|z-w|^2}{2\theta^2}}}{z-w}\right)\mu_V(\mathrm{d}w)\\ + th(z)\int\left(\Delta_w\frac{\mathrm{e}^{-\frac{|z-w|^2}{2\theta^2}}}{z-w}\right)\mu_V(\mathrm{d}w) = \mathrm{O}(N^{-1/2+\sigma+\varepsilon}), \end{aligned}$$

where for the last estimate we integrated w by parts to avoid the singularity. □

By using the local law of Proposition 4.5 in the loop equation, similarly as in [6, Section 7], we obtain the following estimate.

Lemma 4.7. *Suppose that the assumption* (4.17) *holds. Then for any* $0\le|u|\le\mathrm{O}(t)$

$$\begin{aligned} \frac{1}{t\beta N}\log\mathbb{E}_V\mathrm{e}^{-t\beta N(X_V^f-\hat A^f_{V+uf})} &= \frac{tN}{8\pi}\int|\nabla f(z)|^2\,m(\mathrm{d}z) - \frac{1}{\beta}Y_V^f + \frac{1}{t}\operatorname{Re}\int_0^t \mathbb{E}_{V_s}^{G_s}\left(A_{V_s}^{h_s,+}\right)\mathrm{d}s\\ &\quad + \mathrm{O}(N^{-1/2+\varepsilon}b^{-1})\|f\|_{3,b} + \mathrm{O}(tN^{2\sigma+\varepsilon}b^{-2})\|f\|_{4,b}^2, \qquad (4.21)\\ \frac{1}{t\beta N}\log\mathbb{E}_V\mathrm{e}^{-t\beta NX_V^f} &= \frac{tN}{8\pi}\int|\nabla f(z)|^2\,m(\mathrm{d}z) - \frac{1}{\beta}Y_V^f + \frac{1}{t}\operatorname{Re}\int_0^t \mathbb{E}_{V+sf}^{\mathcal{C}}\left(A_{V+sf}^{h_s,-}+A_{V+sf}^{h_s,+}\right)\mathrm{d}s\\ &\quad + \mathrm{O}(N^{-1/2+\varepsilon}b^{-1})\|f\|_{3,b} + \mathrm{O}(tN^{2\sigma+\varepsilon}b^{-2})\|f\|_{4,b}^2. \qquad (4.22) \end{aligned}$$

Proof. We focus on (4.21); the second bound (4.22) can be proved in a similar way. Note that the expectation on the right-hand side of (4.22) is with respect to the standard Coulomb gas without local angle term, and that the terms $A_{V+sf}^{h_s,\pm}$ are with respect to the external potential $V+sf$. The following proof is written for $u=0$ for the simplicity of notations; we will remark on the modification needed for the general case in the proof.

We denote by Z_t the partition function corresponding to the Hamiltonian (4.14). Then the left-hand side of (4.21) can be written as

$$\frac{1}{t\beta N}(\log Z_t - \log Z_0) + N\int f\,\mathrm{d}\mu_V = \frac{1}{t}\int_0^t\left[\partial_s\frac{1}{\beta N}\log Z_s + N\int f\,\mathrm{d}\mu_V\right]\mathrm{d}s.$$

Using the definition (4.14) of G_t, we get

$$\partial_t\frac{1}{\beta N}\log Z_t + N\int f\,\mathrm{d}\mu_V = N\int f(\mathrm{d}\mu_V-\mathrm{d}\mu_{V_t}) + \operatorname{Re}\mathbb{E}_{V_t}^{G_t}\Big(-X_{V_t}^f+\hat A_V^f\Big).$$

As $t \ll 1$ and Δf is bounded and supported in S_V, the supports S_V and S_{V_t} coincide. Together with the explicit formula for the equilibrium measure (1.6) and with (4.19), we have

$$N\int f(\mathrm{d}\mu_V - \mathrm{d}\mu_{V_t}) = \frac{Nt}{4\pi}\int |\nabla f|^2\,\mathrm{d}m + \frac{N}{4\pi}\int |ft\Delta F|\,\mathrm{d}m$$
$$= \frac{Nt}{4\pi}\int |\nabla f|^2\,\mathrm{d}m + \mathrm{O}(tb^{-2}N^{2\sigma})\|f\|_{2,b}^2,$$

where we have integrating by parts twice in getting the last inequality and also used that the support of the integrand has area $\mathrm{O}(b^2)$. Using (4.7) with the choice V_t for the external potential (and the unperturbed Coulomb pair interaction), we have

$$\mathbb{E}_{V_t}^{G_t}\Big(-X_{V_t}^f + A_V^{h,-}\Big) = \mathbb{E}_{V_t}^{G_t}\Big(\frac{1}{N}W_{V_t}^{h_t} - \frac{1}{N\beta}\sum_k \partial h_t(z_k) - A_{V_t}^{h_t,+} - A_{V_t}^{h_t,-} + A_V^{h,-}\Big). \tag{4.23}$$

The perturbed interaction G_t satisfies $G_t(z_j, z_k) = G_t(z_k, z_j)$ and the growth assumption (4.5), so Lemma 4.3 applies. Together with the definition of G_t and recalling $W_V^h = W_V^{\mathcal{C},h}$, we have

$$\mathbb{E}_{V_t}^{G_t}\Big(W_{V_t}^{h_t} + t\sum_{j\neq k}(h_t(z_j) - h_t(z_k))\partial_{z_j}\operatorname{Re}\Psi_{h_t}^-(z_j,z_k)\Big) = \mathbb{E}_{V_t}^{G_t}\Big(W_{V_t}^{G_t,h_t}\Big)$$
$$= \frac{1}{2}\mathbb{E}_{V_t}^{G_t}\Big(\sum_{j\neq k}(h_t(z_j) + h_t(z_k))(\partial_{z_k} + \partial_{z_j})G_t(z_j,z_k)\Big). \tag{4.24}$$

In summary, equations (4.23) and (4.24) give

$$\partial_t \frac{1}{\beta N}\log Z_t + N\int f\,\mathrm{d}\mu_V = \frac{Nt}{4\pi}\int |\nabla f|^2\,\mathrm{d}m + \operatorname{Re}\mathbb{E}_{V_t}^{G_t}\Big(-\frac{1}{N\beta}\sum_k \partial h_t(z_k) - A_{V_t}^{h_t,+} - A_{V_t}^{h_t,-} + A_V^{h,-}$$
$$+ \frac{1}{2N}\sum_{j\neq k}(h_t(z_j) + h_t(z_k))(\partial_{z_k} + \partial_{z_j})G_t(z_j,z_k)\Big) + \mathrm{O}(tb^{-2}N^{2\sigma})\|f\|_{2,b}^2. \tag{4.25}$$

We now evaluate all terms in the above expectation. The difference $A_V^{h_t,-} - A_V^{h,-}$ is bounded in Lemma 4.8 below. For the general cases with $u \neq 0$, $A_V^{h,-}$ should be replaced by $A_{V+uf}^{h,-}$. Notice that Lemma 4.8 is valid for all $0 \le |u| \le \mathrm{O}(t)$.

The other terms are bounded as follows. By (4.15),

$$\operatorname{Re}\mathbb{E}_{V_t}^{G_t}\Big(-\frac{1}{N\beta}\sum_k \partial h_t(z_k)\Big) = -\frac{1}{\beta}\operatorname{Re}\int \partial h_t\,\mathrm{d}\mu_{V_t} + \mathrm{O}(N^{-1/2+\varepsilon}b)\|\nabla h_t\|_{2,b}. \tag{4.26}$$

To compute the main term on the right-hand side, recall that $V_t = (V+tf)+tF$. By integration by parts and the explicit formula for the equilibrium density,

$$-\frac{1}{\beta}\operatorname{Re}\int \partial h_t\,\mathrm{d}\mu_{V_t} = -\frac{1}{4\pi\beta}\operatorname{Re}\int \partial\Big(\frac{\bar\partial f}{\partial\bar\partial(V+tf)}\Big)\Delta(V+tf)\,\mathrm{d}m + \mathrm{O}\Big(t\int \partial h_t\Delta F\,\mathrm{d}m\Big)$$
$$= -\frac{1}{4\pi\beta}\int \Delta f \log\Delta(V+tf)\,\mathrm{d}m + \mathrm{O}\Big(t\int \partial h_t\Delta F\,\mathrm{d}m\Big)$$
$$= -\frac{1}{\beta}Y_V^f + \mathrm{O}\Big(t\int |\Delta f|^2\,\mathrm{d}m\Big) + \mathrm{O}(tb^{-2}N^{2\sigma})\|f\|_{3,b}^2 \tag{4.27}$$
$$= -\frac{1}{\beta}Y_V^f + \mathrm{O}(tb^{-2}N^{2\sigma})\|f\|_{3,b}^2.$$

Finally, differentiation using (4.9) gives

$$\Big|\frac{t}{N}\sum_{j\neq k}(h_t(z_j)-h_t(z_k))\partial_{z_j}\operatorname{Re}\Psi^-_{h_t}(z_j,z_k)\Big|\leqslant C\frac{t}{N}\|\nabla h_t\|_\infty^2\sum_{j\neq k: z_j\in\Omega}\mathrm{e}^{-\frac{|z_j-z_k|^2}{2\theta^2}}\Big(1+\frac{|z_j-z_k|^2}{\theta^2}\Big)+\mathrm{e}^{-N^\varepsilon},$$

where Ω is the $N^\varepsilon\theta$-neighborhood of the support of h. Using the local density, implied by (4.15) and under the assumption (4.17), we therefore have

$$\operatorname{Re}\mathbb{E}^{G_t}_{V_t}\Big(\frac{t}{N}\sum_{j\neq k}(h_t(z_j)-h_t(z_k))\partial_{z_j}\operatorname{Re}\Psi^-_{h_t}(z_j,z_k)\Big)=\mathrm{O}\Big(tN^{2\sigma+\varepsilon}b^2\|\nabla h_t\|_\infty^2\Big)=\mathrm{O}\Big(tb^{-2}N^{2\sigma+\varepsilon}\|f\|_{2,b}^2\Big).$$

Similarly, (4.9) yields

$$\begin{aligned}
&\frac{1}{N}\sum_{j\neq k}(h_t(z_j)+h_t(z_k))(\partial_{z_j}+\partial_{z_k})G_t(z_j,z_k)\\
&=\frac{t}{N}\sum_{j\neq k}(h_s(z_j)+h_s(z_k))\frac{\partial h_t(z_j)-\partial h_t(z_k)}{z_j-z_j}\mathrm{e}^{-\frac{|z_k-z_j|^2}{2\theta^2}}\\
&=\mathrm{O}\Big(\frac{t}{N}\|h_t\|_\infty\|\nabla^2h_t\|_\infty\sum_{j\neq k: z_j\in\Omega}\mathrm{e}^{-\frac{|z_j-z_k|^2}{2\theta^2}}\Big)+\mathrm{O}(\mathrm{e}^{-N^\varepsilon}).\\
&=\mathrm{O}\Big(\frac{t}{N}b^{-4}\|f\|_{3,b}^2\sum_{j\neq k: z_j\in\Omega}\mathrm{e}^{-\frac{|z_j-z_k|^2}{2\theta^2}}\Big)+\mathrm{O}(\mathrm{e}^{-N^\varepsilon}).
\end{aligned}$$

The local density estimate (4.15) then again gives

$$\operatorname{Re}\mathbb{E}^{G_t}_{V_t}\Big(\frac{1}{N}\sum_{j\neq k}(h_t(z_j)+h_t(z_k))(\partial_{z_j}+\partial_{z_k})G_t(z_j,z_k)\Big)=\mathrm{O}\Big(tb^{-2}N^{2\sigma+\varepsilon}\|f\|_{3,b}^2\Big).\tag{4.28}$$

Collecting the error terms and using (4.29) and $b\geq\theta$, we get the error terms

$$N^{-1/2+\varepsilon}\|f\|_{3,b}+tN^\varepsilon\Big[b^{-2}N^{2\sigma}+N^{-1/2+3\sigma}b^{-3}\Big]\|f\|_{b,4}^2\leq N^{-1/2+\varepsilon}\|f\|_{3,b}+tN^{\varepsilon+2\sigma}b^{-2}\|f\|_{4,b}^2.$$

This concludes the proof. □

Lemma 4.8. *Recall h_t is defined in* (4.16)*. For any $0\leq|u|\leq\mathrm{O}(t)$ we have the estimate*

$$\mathbb{E}^{G_t}_{V_t}\left(A^{h_t,-}_{V_t}-A^{h,-}_{V+uf}\right)=\mathrm{O}(N^{-1/2+3\sigma+\varepsilon}b^{-1})\|f\|_{3,b}+\mathrm{O}(tN^{2\sigma+\varepsilon}b^{-2})\|f\|_{3,b}^2;\tag{4.29}$$

an analogous estimate holds with $\mathbb{E}^{G_t}_{V_t}$ replaced by $\mathbb{E}^{\mathcal{C}}_V$.

Proof. To simplify notation, we set $u=0$ in the following proof as the general case is proved in the same way. By definition,

$$A^{h_t,-}_{V_t}-A^{h,-}_V=\frac{N}{2}\iint_{z\neq w}\Big[\Psi^-_{h_t}(z,w)\,\tilde\mu_{V_t}(\mathrm{d}z)\,\tilde\mu_{V_t}(\mathrm{d}w)-\Psi^-_h(z,w)\,\tilde\mu_V(\mathrm{d}z)\,\tilde\mu_V(\mathrm{d}w)\Big].\tag{4.30}$$

Using that

$$\partial_s\partial h_s(z)=\mathrm{O}\left(\|\nabla f\|_\infty\|\nabla^3f\|_\infty+\|\nabla^2f\|_\infty^2\right)=\mathrm{O}(b^{-4})\|f\|_{3,b}^2,\tag{4.31}$$

we first use that with high probability with respect to the measure $P_{V_s}^{G_s}$,

$$N \iint_{z\neq w} [\Psi_{h_t}^- - \Psi_h^-](z,w)\,\tilde\mu_{V_t}(\mathrm{d}z)\,\tilde\mu_{V_t}(\mathrm{d}w) \tag{4.32}$$

$$\leq N\int_0^t \mathrm{d}s \iint_{z\neq w} \left|\partial_s \partial h_s(z)\right| \mathbb{1}(|z-w|\leq\theta)\,\tilde\mu_{V_t}(\mathrm{d}z)\,\tilde\mu_{V_t}(\mathrm{d}w) \tag{4.33}$$

$$\leq N\int_0^t \mathrm{d}s \iint_{z\neq w} \left|\partial_s \partial h_s(z)\right| \mathbb{1}(f(z)\neq 0)\,\mathbb{1}(|z-w|\leq\theta)\,\tilde\mu_{V_t}(\mathrm{d}z)\,\tilde\mu_{V_t}(\mathrm{d}w) \tag{4.34}$$

$$\leq N^{\varepsilon}\mathrm{O}(tN\theta^2 b^2)(\|\nabla f\|_\infty \|\nabla^3 f\|_\infty + \|\nabla^2 f\|_\infty^2) \leqslant \mathrm{O}(tN^{2\sigma+\varepsilon}b^{-2})\|f\|_{3,b}^2 \tag{4.35}$$

where we used the local density estimate Proposition 4.5, and the factor $\theta^2 b^2$ comes from the integration restriction that z is in the support of F and $|w-z|\lesssim\theta^2$, i.e.,

$$\iint |\ \mathbb{1}(f(z)\neq 0)\ \mathbb{1}(|z-w|\leq\theta)\,m(\mathrm{d}z)\,m(\mathrm{d}w) = \mathrm{O}(b^2\theta^2). \tag{4.36}$$

Similarly, we have the estimates (again with high probability)

$$N\iint \Psi_{h_t}^-(z,w)\left[\tilde\mu_{V_t}(\mathrm{d}z) - \tilde\mu_{V+tf}(\mathrm{d}z)\right]\tilde\mu_{V_t}(\mathrm{d}w) = \mathrm{O}(N)\iint \Psi_{h_t}^-(z,w)\,t\Delta F(z)\,m(\mathrm{d}z)\,\tilde\mu_{V_t}(\mathrm{d}w)$$
$$= \mathrm{O}(N^{1+\varepsilon}\theta^2 b^2)\|\nabla h_t\|_\infty \|t\Delta F\|_\infty = OO(N^{-1/2+3\sigma+\varepsilon})tb^{-3}\|f\|_{b,4}^2$$

and

$$N\iint \Psi_{h_t}^-(z,w)\left[\tilde\mu_{V}(\mathrm{d}z) - \tilde\mu_{V+tf}(\mathrm{d}z)\right]\tilde\mu_{V_t}(\mathrm{d}w) = \mathrm{O}(Nt)\iint \Psi_{h_t}^-(z,w)\,\Delta f(z)\,m(\mathrm{d}z)\,\tilde\mu_{V_t}(\mathrm{d}w)$$
$$= \mathrm{O}(N^{1+\varepsilon}t\theta^2 b^2)\|\nabla h_t\|_\infty \|\Delta f\|_\infty = \mathrm{O}(N^{1+\varepsilon}t\theta^2 b^{-2})\|f\|_{b,4}^2.$$

This completes the proof. ☐

4.4. CLT with angle term: proof of Proposition 4.1. As in [6, Lemma 7.5], using the local law of Proposition 4.5, we can bound the terms $A^{\pm}$ as follows.

Lemma 4.9. *For any $\varepsilon>0$, uniformly in $0\leqslant t\ll 1$ with $t\|\Delta f\|_\infty \ll 1$, we have*

$$\mathbb{E}_{V_t}^{G_t}\left(A_{V_t}^{g,\pm}\right) = \mathrm{O}(N^{\varepsilon})b\|g\|_{2,b}. \tag{4.37}$$

Proof. The proof is exactly the same as that of [6, Lemma 7.5], using the local density estimate Proposition 4.5. Here A^- corresponds to $t\leqslant N^{-1/2+\delta}$ in that proof and A^+ to $t\geqslant N^{-1/2+\delta}$. ☐

Using these bounds in (4.22) and Markov's inequality, the following rigidity estimate follows, again as in [6].

Proposition 4.10. *Assume the same conditions as in Proposition 4.5. For any $\varepsilon>0$, $s\in(0,1/2)$, for any f supported in a ball of radius $b=N^{-s}$ contained in S_V with distance of order 1 to ∂S_V,*

$$X_f \prec \|f\|_{4,b}. \tag{4.38}$$

Proof. The proof is exactly the same as the proof of [6, Theorem 1.2]. ☐

Finally, using this (stronger) rigidity estimate instead of the local law of Proposition 4.5, we obtain the following improved bound on A^+.

Lemma 4.11. *For any $\varepsilon > 0$, uniformly in $0 \leqslant t \ll 1$ with $t\|\Delta f\|_\infty \ll 1$, we have*

$$\mathbb{E}^{G_t}_{V_t}\left(A^{g,+}_{V_t}\right) = \mathrm{O}(N^{-\sigma+\varepsilon}) b \|g\|_{2,b}. \tag{4.39}$$

In particular, when $g = h_t$, the last term is bounded by $\mathrm{O}(N^{-\sigma+\varepsilon})\|f\|_{3,b}$. For a Coulomb gas satisfying (3.1) *a similar estimate holds, i.e.,*

$$\mathbb{E}^{\mathcal{C}}_{V+tf}\left(A^{g,+}_{V+tf}\right) = \mathrm{O}(N^{-\sigma+\varepsilon}) b \|g\|_{2,b}. \tag{4.40}$$

Proof. The proofs will be given in Appendix B.3. □

Proof of Proposition 4.1. Proposition 4.1 follows immediately from (4.21) and Lemma 4.11. □

4.5. Concentration of angle term (macroscopic case): proof of Proposition 4.2. The main input of the proof of Proposition 4.2 is the following estimate of large deviations type, which is a direct consequence of Theorem 1.1.

Corollary 4.12. *Assume that V satisfies the conditions of Remark 3.1. Let $0 < t \ll 1$ and $\kappa < 1/24$. Then for any $f \in \mathscr{C}^5$ whose support is contained in S_V and has distance of order* 1 *to the boundary of S_V, assuming that $t\|\Delta f\|_\infty \ll 1$, we have*

$$\frac{1}{t\beta N}\log \mathbb{E}_V \mathrm{e}^{-\beta t N X^f_V} = \frac{tN}{8\pi}\int |\nabla f|^2 \, \mathrm{d}m + \left(\frac{1}{2}-\frac{1}{\beta}\right) Y^f_V + \mathrm{O}(N^{-\kappa}/t)(1+\|\Delta f\|_3)^2 + \mathrm{O}(t)\|\Delta f\|^2_\infty.$$

Proof. By Theorem 1.1, we have

$$\begin{aligned}\frac{1}{t\beta N}\log \mathbb{E}_V \mathrm{e}^{-\beta t N X^f_V} &= N\int f \, \mathrm{d}\mu_V - \frac{N}{t}\left(I_{V+tf} - I_V\right) \\ &\quad + \left(\frac{1}{2}-\frac{1}{\beta}\right)\frac{1}{t}\left(\int \rho_{V+tf}\log\rho_{V+tf} - \int \rho_V \log \rho_V\right) + \mathrm{O}(t^{-1}N^{-\kappa}),\end{aligned} \tag{4.41}$$

with an f-dependent error term. More precisely, by Remark 3.1, with V fixed, the f-dependence of the error term can be taken to be $\mathrm{O}(t^{-1}N^{-\kappa})(1+\|\Delta f\|_3)^2$.

Using that $\rho_V = \frac{1}{4\pi}\Delta V \mathbb{1}_{S_V}$ and $\rho_{V+tf} = \frac{1}{4\pi}(\Delta V + \Delta f)\mathbb{1}_{S_V}$ for f with compact support contained in S_V such that $t\Delta f < \Delta V$ in its support, an explicit calculation (see, e.g., [6, Proposition 3.1]) shows that

$$I_{V+tf} - I_V = t\int f \, \mathrm{d}\mu_V - \frac{t^2}{8\pi}\int |\nabla f|^2 \, \mathrm{d}m, \tag{4.42}$$

and that

$$\begin{aligned}\frac{1}{t}\left(\int \rho_{V+tf}\log\rho_{V+tf} - \int \rho_V\log\rho_V\right) &= \frac{1}{4\pi}\int \Delta f \log \rho_V + \frac{1}{t}\int \rho_{V+tf}\log\left(\frac{\rho_{V+tf}}{\rho_V}\right) \\ &= \frac{1}{4\pi}\int \Delta f \log\rho_V + \mathrm{O}\left(t\int (\Delta f)^2\right),\end{aligned} \tag{4.43}$$

where for the last equality we expanded $\log(1 + t\Delta f/\Delta V)$ to first order and used $\int \Delta f = 0$. Equations (4.42) and (4.43) in (4.41) conclude the proof. □

Proof of Proposition 4.2. Let κ be as in Corollary 4.12 and write $W = V - tf$. Using the identity as in (4.4), for $t \ll 1$, we have

$$\frac{1}{t\beta N}\log \mathbb{E}_V \mathrm{e}^{t\beta N \hat{A}^f_{V+uf}} = \frac{1}{t\beta N}\left(\log \mathbb{E}_W \mathrm{e}^{-\beta N t (X^f_V - \hat{A}^f_{V+uf})} - \log \mathbb{E}_W \mathrm{e}^{-\beta N t X^f_V}\right).$$

We can replace X^f_V by X^f_W in the two exponents in the above equation since $X^f_V - X^f_W$ is a constant which cancels in the above expression. Also, $\hat{A}^f_{V+uf} = \hat{A}^f_{W+(t+u)f}$. By Proposition 4.1,

$$\frac{1}{t\beta N}\log \mathbb{E}_W \mathrm{e}^{-\beta N t (X^f_W - \hat{A}^f_{W+(t+u)f})} = -\frac{1}{2}Y^f_W + N^{\varepsilon}\mathrm{O}(tN^{2\sigma} + N^{-\sigma} + N^{-1/2})(1+\|f\|_5)^2.$$

By Corollary 4.12 with V replaced by W, we can estimate the last term $\log \mathbb{E}_W \mathrm{e}^{-\beta N t X^f_W}$. Recall from (4.27) that $Y^f_W = Y^f_V + \mathrm{O}\left(t\int |\Delta f|^2 \,\mathrm{d}m\right)$. Putting all these bounds together, we have arrived at

$$\frac{1}{t\beta N}\log \mathbb{E}_V \mathrm{e}^{t\beta N \hat{A}^f_{V+uf}} = -\frac{1}{2}Y^f_V + N^{\varepsilon}\mathrm{O}(tN^{2\sigma} + N^{-\sigma} + t^{-1}N^{-\kappa} + N^{-1/2})(1+\|f\|_5)^2.$$

This proves (4.3) for $t = N^{-4\sigma} = N^{-2/3\kappa}$. Moreover, the bound also holds as claimed for smaller t by the monotonicity of $t \mapsto t^{-1}\log \mathbb{E}(\mathrm{e}^{tX})$ applied with the choice $X = \beta N(\hat{A}^f_{V+uf} + \frac{1}{2}Y^f_V)$. □

4.6. CLT for mesoscopic test functions. To extend the proof of the central limit theorem to test functions on mesoscopic scales, it suffices to prove the estimate for the local angle term.

Proposition 4.13. *Suppose that V satisfies the conditions* (1.4) *and* (1.9). *Let $s \in (0, \frac{1}{2})$ and assume that f is supported in a ball of radius $b = N^{-s}$ contained in S_V with distance of order* 1 *to the boundary ∂S_V. Then there exists $\tau = \tau(s) > 0$ such that with high probability under the measure $P^{\mathcal{C}}_V$,*

$$\left|\hat{A}^f_V + \frac{1}{2}Y^f_V\right| \prec (Nb^2)^{-\tau/3}\|f\|_{5,b}. \tag{4.44}$$

This proposition can be proved by following the strategy used in the proof of Proposition 4.2, after conditioning on the particles outside a mesoscopic ball with radius of order b containing the support of f. Before implementing this, we complete the proof of Theorem 1.2 using (4.44).

Proof of Theorem 1.2 for mesoscopic test functions. We apply (4.22) and we need to estimate the term $\frac{1}{t}\operatorname{Re}\int_0^t \mathbb{E}^{\mathcal{C}}_{V+sf}\left(A^{h_s,-}_{V+sf} + A^{h_s,+}_{V+sf}\right)\mathrm{d}s$ on right hand side of (4.22). The term A^+ is again bounded by Lemma 4.11. To estimate the expectation of A^-, we now use (4.44) which implies that with high probability

$$A^{h_s,-}_{V+sf} = -\frac{1}{2}Y^f_{V+sf} + \mathrm{O}(M^{-\tau/3+\varepsilon})\|f\|_{5,b} = -\frac{1}{2}Y^f_V + \mathrm{O}(M^{-\tau/3+\varepsilon} + sb^{-2})\|f\|_{5,b}$$

where we have used $Y^f_{V+sf} = Y^f_V + \mathrm{O}\left(s\int |\Delta f|^2\,\mathrm{d}m\right)$ as in (4.43). Clearly, the high probability estimate immediately implies the same estimate under expectation. Integrating s from 0 to t, this implies an estimate on the term $\frac{1}{t}\operatorname{Re}\int_0^t \mathbb{E}^{\mathcal{C}}_{V+sf}\left(A^{h_s,-}_{V+sf} + A^{h_s,+}_{V+sf}\right)\mathrm{d}s$. Inserting this estimate into (4.22), we have completed the proof of Theorem 1.2. □

In the remainder of this section, we prove Proposition 4.13. For this, we use the approach of local conditioning of [6], and then proceed as in the proof of Proposition 4.2. (In fact, essentially the same argument also implies a version of Theorem 1.2 which holds under conditioning, which we will state at the end of the section.) Let $B \subset \mathbb{C}$ be a disk of radius b contained in S_V, and consider the Coulomb gas obtained by conditioning on all of the particles outside B. We denote the effective external potential of this system W, and recall from [6] that the potential W may be written down as follows. Let M denote the number of particles in B and let $(\tilde{z}_1, \tilde{z}_2, \dots, \tilde{z}_M)$ denote the collection of particles inside B. Correspondingly, we write $(\hat{z}_1, \hat{z}_2, \dots, \hat{z}_{N-M})$ for the particles outside B. The Hamiltonian $H_{N,V}$ may then be written as

$$H_{N,V}(z) = \sum_{j \neq k} \log \frac{1}{|\tilde{z}_j - \tilde{z}_k|} + N \sum_j \Big(V(\tilde{z}_j) - V_o(\tilde{z}_j|\hat{z}) \Big) + E(\hat{z}),$$

where

$$V_o(w|\hat{z}) = -\frac{2}{N} \sum_k \log \frac{1}{|w - \hat{z}_k|}, \qquad E(\hat{z}) = \sum_{j \neq k} \log \frac{1}{|\hat{z}_j - \hat{z}_k|} + N \sum_j V(\hat{z}_j).$$

The term $E(\hat{z})$ is independent of the particles in B and is thus irrelevant for the conditioned measure. For any configuration of external particles $\hat{z} \in (\mathbb{C} \setminus B)^{N-M}$ and $z \in \mathbb{C}$, we write

$$W(w|\hat{z}) = \begin{cases} \frac{N}{M}(V(w) - V_o(w|\hat{z})) & (w \in B), \\ +\infty & (w \notin B), \end{cases} \tag{4.45}$$

$$P_{N,V,\beta}(\mathrm{d}w|\hat{z}) = P_{M(\hat{z}),W(\cdot|\hat{z}),\beta}(\mathrm{d}w). \tag{4.46}$$

The Coulomb gas given by the potential $W(\cdot|\hat{z})$ is the conditional gas inside B, given the external configuration $\hat{z}$. In [6], it was proven that under our assumptions on V the conditional potential satisfies the following properties. First, since $V_o(\cdot|\hat{z})$ is harmonic in B we have

$$\mu_W(\mathrm{d}z) = \frac{\Delta W(z)}{4\pi} m(\mathrm{d}z) = \frac{N}{M} \mu_V(\mathrm{d}z) \tag{4.47}$$

in the interior of the support $S_W \subset B$. Especially,

$$M \int f \, \mathrm{d}\mu_W = N \int f \, \mathrm{d}\mu_V \tag{4.48}$$

for any function f that has compact support in S_W. Finally, from [6], we also know that the measure $\mathrm{d}\mu_W$ may be expressed as $\frac{N}{M} \mathbb{1}_{S_W} \mathrm{d}\mu_V + v \, \mathrm{d}s$, where $\mathrm{d}s$ is the length measure on ∂B, $v \in L^\infty(\partial B)$, and that the following properties hold. These properties are verified in the proof of [6, Theorem 6.1].

Lemma 4.14. *For any $s \in (0, \frac{1}{2})$, there exists a constant $\tau > 0$ such that the following statements hold with probability at least $1 - \mathrm{e}^{-Nb^2}$ for $b = N^{-s}$,*

(i) $\quad M = N \mu_V(B)(1 + \mathrm{O}(M^{-\tau})),$

(ii) $\quad S_W \supset \{z \in B : d(z, \partial B) > M^{-\tau} b\},$

(iii) $\quad \mu_W(\partial B) = \int v \, \mathrm{d}s \leqslant M^{-\tau},$

(iv) $\quad \|v\|_\infty \leqslant \mathrm{O}(1/b).$

Relative to the conditioned measure, for f compactly supported in $S_W \subset S_V$, the definitions (1.11), (1.12) translate to

$$X^f_{N,V} = X^f_{M,W} = \sum_j f(\tilde z_j) - M \int f \, \mathrm{d}\mu_W, \quad Y^f_V = Y^f_W = \frac{1}{4\pi} \int \Delta f \log \rho_W \, \mathrm{d}m,$$

where ρ_W is the density of the absolutely continuous part of μ_W; inside the support of f, this density equals that of μ_V up to rescaling. The angle term relative to the conditioned measure is

$$\hat A^f_V = \hat A^f_W = \frac{M}{2} \operatorname{Re} \iint_{z \neq w} \Psi^-_{h_W}(z,w)\, \tilde\mu_W(\mathrm{d}z)\, \tilde\mu_W(\mathrm{d}w), \quad h_W = \frac{\bar\partial f(z)}{\partial\bar\partial W(z)}. \tag{4.49}$$

The following proposition is a conditioned version of Proposition 4.2. Note that Lemma 4.14 implies that the assumptions of this proposition holds with high probability. Thus by the Markov inequality, Proposition 4.15 immediately implies Proposition 4.13.

Proposition 4.15. *Let W be the conditional potential defined above and assume that it satisfies the conclusions of Lemma 4.14. Choosing the local angles cutoff $\theta = M^{-1/2+\sigma}$ with $\sigma = \tau/6$, for any $0 \le t \le M^{-2\tau/3}$ we have*

$$\frac{1}{t\beta M} \log \mathbb{E}_W \mathrm{e}^{t\beta M(\hat A^f_W + \frac12 Y^f_W)} = \mathrm{O}(M^{-\tau/3})(1 + \|f\|_{5,b})^2. \tag{4.50}$$

To prove Proposition 4.15, we need a version of Theorem 1.1 for the conditioned measure. Recall that μ_W denotes the unique minimizer of the energy functional

$$\mathcal{I}_W(\mu) = \iint \log \frac{1}{|z-w|}\, \mu(\mathrm{d}z)\, \mu(\mathrm{d}w) + \int W(z)\, \mu(\mathrm{d}z),$$

defined for probability measures supported in B, and that its minimum value is $I_W = \mathcal{I}_W(\mu_W)$.

Theorem 4.16. *Let W be the conditional potential defined above and assume that it satisfies the conclusions of Lemma 4.14. Then there exists $\tau > 0$ (depending on the constant τ in Lemma 4.14 but possibly smaller; here we have abused the notation and use the same symbol τ) such that with $\zeta^{\mathcal{C},\beta} \in \mathbb{R}$ defined in Theorem 1.1,*

$$\begin{aligned} &\frac{1}{\beta M} \log \int_{B^M} \mathrm{e}^{-\beta H^{\mathcal{C}}_W(\mathbf{z})}\, m^{\otimes M}(\mathrm{d}\mathbf{z}) \\ &\qquad = -M I_W + \zeta^{\mathcal{C},\beta} + \frac12 \log M + \left(\frac12 - \frac1\beta\right) \int_B \rho_W(z) \log \rho_W(z)\, m(\mathrm{d}z) + \mathrm{O}(M^{-\tau}), \end{aligned}$$

where ρ_W is the density of the absolutely continuous part of μ_W.

Proof. We apply Theorem 1.1 with the extension in Remark 3.1 to the conditional Coulomb gas satisfying the properties stated in Lemma 4.14. More precisely, to literally apply Theorem 1.2, we first rescale and center the domain B, which is a disk of radius b, to the unit disk $\mathbb{D}$ with center at 0. Since the translation is trivial, we will assume that the center of B is already at the origin. Denote the rescaling by $\mathbf{z} = b\mathbf{u}$ and define the new Hamiltonian $\hat H^{\mathcal{C}}_W(\mathbf{u})$ through the identity

$$\int_{B^M} \mathrm{e}^{-\beta H^{\mathcal{C}}_W(\mathbf{z})}\, m^{\otimes M}(\mathrm{d}\mathbf{z}) = \int_{\mathbb{D}^M} \mathrm{e}^{-\beta \hat H^{\mathcal{C}}_W(\mathbf{u})}\, m^{\otimes M}(\mathrm{d}\mathbf{u}). \tag{4.51}$$

Hence $\hat{H}_W^{\mathcal{C}}(\mathbf{u})$ is a Coulomb gas with external potential $\tilde{W}(u) = W(bu)$ up to a constant. More precisely,

$$\hat{H}_W^{\mathcal{C}}(\mathbf{u}) = H_W^{\mathcal{C}}(\mathbf{u}/b) - 2M\beta^{-1}\log b = H_{\tilde{W}}^{\mathcal{C}}(\mathbf{u}) - M(M-1)\log b - \frac{1}{\beta}M\log b^2. \tag{4.52}$$

By Theorem 1.1 with Remark 3.1, there exists $\tau > 0$ such that

$$\frac{1}{\beta M}\log\int_{B^M} \mathrm{e}^{-\beta H_W^{\mathcal{C}}(\mathbf{z})}\, m^{\otimes M}(\mathrm{d}\mathbf{z}) = \frac{1}{\beta M}\log\int_{\mathbb{D}^M} \mathrm{e}^{-\beta H_{M,\tilde{W}}(\mathbf{u})}\, m^{\otimes M}(\mathrm{d}\mathbf{u}) + M\log b - \Big(\frac{1}{2}-\frac{1}{\beta}\Big)\log b^2$$
$$= -M(I_{\tilde{W}} - \log b) + \frac{1}{2}\log M + \Big(\frac{1}{2}-\frac{1}{\beta}\Big)\Big[\int_{\mathbb{D}} \rho_{\tilde{W}}(u)\log\rho_{\tilde{W}}(u)\, m(\mathrm{d}u) - \log b^2\Big] + \mathrm{O}(M^{-\tau}).$$

Recall the normalization conditions for the densities $\int \rho_{\tilde{W}}(u)m(\mathrm{d}u) = 1 = \int \rho_W(z)m(\mathrm{d}z)$. Hence $\rho_{\tilde{W}}(u) = \rho_W(bu)b^2$ and we have

$$\int_{\mathbb{D}} \rho_{\tilde{W}}(u)\log\rho_{\tilde{W}}(u)\, m(\mathrm{d}u) - \log b^2 = \int_B \rho_W(z)\log\rho_{\tilde{W}}(z)\, m(\mathrm{d}z).$$

A similar argument shows that $(I_{\tilde{W}} - \log b) = I_W$. We have thus proved the theorem. □

Proof of Proposition 4.15. By assumption, the potential W satisfies the conditions of Remark 3.1, and therefore the assumptions of Proposition 4.1. Together with Proposition 4.16 to replace Theorem 1.1, the proposition follows in exactly the same way as Proposition 4.2. □

Similarly as in the proof of Theorem 4.16, one can also derive a conditioned version of the CLT, stated below; we omit the details of the proof.

Theorem 4.17. *Suppose W is the conditional potential defined above and assume that it satisfies the conclusions of Lemma 4.14. Then for any $\beta > 0$, $c \in (0,1)$ and large $C > 0$, there a positive constant $\tau > 0$ such that the following holds. For any $f : \mathbb{C} \to \mathbb{R}$ supported in the ball with same center as B and radius $b(1-c)$ and satisfying $\|f\|_{4,b} < C$, and for any $0 \leqslant \lambda \leq M^{1-2\tau}$, we have*

$$\frac{1}{\beta\lambda}\log\left(\mathbb{E}_{M,W,\beta}^{\mathcal{C}}\mathrm{e}^{-\lambda\beta\left(X_W^f - (\frac{1}{\beta}-\frac{1}{2})Y_W^f\right)}\right) = \frac{\lambda}{8\pi}\int |\nabla f(z)|^2\, m(\mathrm{d}z) + \mathrm{O}(M^{-\tau}). \tag{4.53}$$

Note that the measure associated to the external potential $W + \frac{\lambda}{M}f$ is a perturbation of the original measure provided that

$$|\lambda\Delta f| \ll |M\Delta W| = |N\Delta V|.$$

Our assumptions $\|f\|_{4,b} < C$ and $\lambda \leq M^{1-2\tau}$ guarantee this condition. Also note that, in the context of the above Theorem 4.17, our test function has shrinking support so that

$$Y_W^f = \frac{1}{4\pi}\int \Delta f(z)\log\rho_W(z)\,\mathrm{d}m(z) = \frac{1}{4\pi}\int \Delta f(z)\log\frac{\Delta V(z)}{\Delta V(z_0)}\mathrm{d}m(z) = \mathrm{O}(b)\|f\|_{b,2}\|V\|_3 = \mathrm{o}(1),$$

where we used (4.47) and denoted the center of $\mathscr{J}$ by z_0. Thus Theorem 4.17, with λ of order 1, implies there is no shift of the mean in the convergence to the Gaussian free field for mesoscopic observables:

$$X_W^f \xrightarrow[N\to\infty]{(\mathrm{d})} \mathscr{N}\left(0, \frac{1}{4\pi\beta}\int |\nabla f|^2\right).$$

A Local law for Yukawa gas and perturbed Coulomb gas

This appendix is an adaption of our results in [6] to the one-component Yukawa gas, in which the two-dimensional Coulomb potential $\log 1/|z|$ is replaced by the Yukawa potential $Y^\ell(z)$ defined in (2.1), and also to the perturbed Coulomb gas used in Section 4. Our presentation here follows closely that of [6] and we will mainly present the differences.

A.1. Interactions. We first recall the definitions of the different interactions to which we extend the local density estimate.

A.1.1. Yukawa gas on the plane. Throughout this appendix, we denote the range of the Yukawa potential by $\ell > 0$ and write $m = 1/\ell$ for the inverse range (or mass). Given an external potential $Q : \mathbb{C} \to \mathbb{R} \cup \{+\infty\}$ with sufficient growth, for $\mathbf{z} = (z_1, \dots, z_N) \in \mathbb{C}^N$, we define the energy

$$H^\ell_{N,Q}(\mathbf{z}) = \sum_{j \neq k} Y^\ell(z_j - z_k) + N \sum_j Q(z_j), \tag{A.1}$$

as in (1.1), abbreviating H^{Y^ℓ} by H^ℓ. The corresponding probability measure on particle configurations is defined as in (1.3) and the Yukawa variational functional and its infimum are defined for probability measures μ on $\mathbb{C}$ by

$$\mathcal{I}^\ell_Q(\mu) = \int Q(z)\, \mu(\mathrm{d}z) + \int Y^\ell(z-w)\, \mu(\mathrm{d}z)\, \mu(\mathrm{d}w), \quad I^\ell_Q = \inf_\mu \mathcal{I}^\ell_Q(\mu). \tag{A.2}$$

Under mild assumptions on Q, standard arguments imply that the convex variational functional has a unique minimizer over the set of probability measures on $\mathbb{C}$, the equilibrium measure μ^ℓ_Q; see Theorem A.5 below. We always make the following assumptions throughout this section:

(i) The set $\Sigma_Q = \{z : Q(z) < \infty\}$ has positive logarithmic capacity.

(ii) The potential Q is locally in $C^{1,1}$ and satisfies the growth condition: for some $\varepsilon > 0$,

$$\liminf_{|z| \to \infty} (Q(z) - \varepsilon \log |z|) > -\infty. \tag{A.3}$$

(iii) The density of μ^ℓ_Q is bounded below in a neighborhood of the support of all considered test functions f.

Given a test function $f : \mathbb{C} \to \mathbb{R}$, we denote the linear statistic centered by the equilibrium measure by

$$X_f = \sum f(z_k) - N \int f(z)\, \mu^\ell_Q(\mathrm{d}z). \tag{A.4}$$

We remark that, in Section 3, we use the Yukawa gas in two contexts: with range $R \gg 1$ as a regularization of the Coulomb gas, and with range $\ell \ll 1$ to describe a screened Coulomb gas. In the second application, the potential Q is defined in terms of the long-range interaction of a Coulomb gas with external potential V, and then the equilibrium measure μ^ℓ_Q then *equals* the minimizer μ_V of the Coulomb variational functional, for which the density is explicitly given by $\frac{1}{4\pi}\Delta V$ in its support. We remark that in this application, the growth condition (A.3) corresponds to the stronger condition (1.4) assumed of the original Coulomb gas. In the application of the Yukawa gas to approximate the Coulomb gas with range $R \to \infty$ (in Section 3.9), we naturally assume that Q satisfies the stronger growth condition (1.4) required for the Coulomb gas, i.e.,

$$\liminf_{|z| \to \infty} \big(Q(z) - (2+\varepsilon) \log |z|\big) > -\infty.$$

A.1.2. Yukawa gas on the torus. Let $\mathbb{T}$ be the unit torus, i.e., the unit square $[-1/2, 1/2)^2$ with periodic boundary condition. Given $Q : \mathbb{T} \to \mathbb{R}$, for $\mathbf{z} = (z_1, \dots, z_N) \in \mathbb{T}^N$, we define the energy

$$H^\ell_{N,Q}(z) = \sum_{j \neq k} U^\ell(z_j - z_k) + N \sum_j Q(z_j), \tag{A.5}$$

where $U^\ell(z) = \sum_{n \in \mathbb{Z}^2} Y^\ell(z + n)$ is the Yukawa potential of range ℓ on $\mathbb{T}$, i.e., the Green's function of $-\Delta + m^2$. For the Yukawa gas on the torus, we will make the simplifying assumption

$$\ell \leqslant N^{-c} \tag{A.6}$$

for some constant $c > 0$. We are mainly interested in $Q = 0$, but allow non-zero periodic Q to study linear statistics, which can be studied by changing Q to $Q + tf$. For $\beta > 0$, we define the corresponding probability measure on $\mathbb{T}^N$ analogously by

$$P^\ell_{N,Q}(dz) = \frac{1}{Z^\ell_N} \mathrm{e}^{-\beta H^\ell_{N,Q}(z)}\, m(\mathrm{d}\mathbf{z}),$$

where m is the uniform probability measure on $\mathbb{T}$. We define X_f analogously as in (A.4).

A.1.3. Coulomb gas with angle perturbation. We consider the probability measure given by the Hamiltonian defined in (4.13)–(4.14). There the function h is defined in terms of a test function f; however, now, we consider h as a given fixed function, and use f for generic test functions which need not have anything to do with h. We make the same assumption on V as in Section 1, or more generally that it satisfies the conditions stated in Remark 3.1. We denote by μ_V the equilibrium measure of the one-component Coulomb gas with potential V (defined without contribution from the angle perturbation), and again define X_f analogously as in (A.4).

A.2. Local law. The result of this appendix are the following local density estimates, which are an adaptation of Theorems 1.1 in [6].

Theorem A.1 (Local law for the plane)**.** *Consider the interaction defined in Section A.1.1. Let $s \in (0, \frac{1}{2})$, $\varepsilon > 0$, and let B be a disk with radius $b = N^{-s}$ contained in S_Q with distance of order 1 to ∂S_Q. For any $f \in C^2(\mathbb{C})$ such that* $\mathrm{supp}((\Delta - m^2)f) \subset B$, *we have*

$$|X_f| \prec \sqrt{N b^2 (f, (-\Delta + m^2) f)} + b^2 \|\Delta f\|_\infty, \tag{A.7}$$

In fact, the following stronger conditioned statement holds. Let $B' \supset B$ be a disk with the same center as B and radius $2b$. Then, with high probability under the original measure $P_{N,Q}$, the estimate (A.7) *holds with high probability under the conditional measure given the particles outside B'.*

Theorem A.2 (Local law for the torus)**.** *Consider the interaction defined in Section A.1.2. Let $s \in (0, \frac{1}{2})$ and $\varepsilon > 0$. For any $f : \mathbb{T} \to \mathbb{R}$ supported in a ball of radius $b = N^{-s}$ contained in S_Q with distance of order* 1 *to ∂S_Q,*

$$|X_f| \prec \sqrt{N b^2 (f, (-\Delta + m^2) f)} + b^2 \|\Delta f\|_\infty, \tag{A.8}$$

Again, the stronger conditioned statements holds as well.

Theorem A.3 (Local law for Coulomb gas with angle term)**.** *Consider the interaction defined in Section A.1.3. Let* $s \in (0, \frac{1}{2})$ *and* $|t|\theta^2 N \leqslant 1$ *and* $\|\nabla h\|_\infty \leq 1$*. Then for any* $f : \mathbb{C} \to \mathbb{R}$ *supported in a ball of radius* $b = N^{-s}$ *contained in* S_V *with distance of order* 1 *to* ∂S_V*,*

$$|X_f| \prec \sqrt{Nb^2(f, -\Delta f)} + b^2\|\nabla^2 f\|_\infty = \mathrm{O}(\sqrt{Nb^2})\|f\|_{2,b}. \tag{A.9}$$

Remark A.4 (Boundedness of local density)**.** *The estimates* (A.7)*,* (A.8)*,* (A.9) *imply, in particular, that for any disk* $B_r(z)$ *of radius* $r \geqslant N^{-1/2+\varepsilon}$ *(which satisfies the respective support assumptions), with high probability, the number of particles in* $B_r(z)$ *is* $\mathrm{O}(r^2)$*.*

We first focus on the proof of Theorem A.1; Theorem A.2 is then a minor modification. For Theorem A.3, the required modifications compared to the proof for the Coulomb gas in [6] are smaller since the perturbation is relatively small; we consider these in Section A.8.

A.3. Potential theory for the Yukawa potential. To prove Theorem A.1, we require some properties of the equilibrium measure of the Yukawa gas, which are analogous to those used for the Coulomb gas in [6]. For a probability measure μ on $\mathbb{C}$, define the Yukawa potential by

$$U^\ell_\mu(z) = \int Y^\ell(z - w)\, \mu(\mathrm{d}w).$$

The following standard result gives the existence and uniqueness of the equilibrium measure for the Yukawa gas. Its statement and proof are identical to that for the Coulomb case. Denote $\Sigma_Q = \{z : Q(z) < \infty\}$ and let $P(\Sigma_Q)$ be the set of probability measures supported in Σ_Q.

Theorem A.5. *Suppose* Q *satisfies* (A.3) *and that* Σ_Q *has positive capacity. Then there exists a unique* $\mu^\ell_Q \in P(\Sigma_Q)$ *such that*

$$\mathcal{I}^\ell_Q(\mu^\ell_Q) = \inf\{\mathcal{I}^\ell_Q(\mu) : \mu \in P(\Sigma_Q)\}.$$

The support $S^\ell_Q = \operatorname{supp} \mu^\ell_Q$ *is compact and of positive capacity, and* $I^\ell_Q(\mu^\ell_Q) < \infty$*. Furthermore, the energy-minimizing measure* μ_Q *may be characterized as the unique element of* $P(\Sigma_V)$ *for which there exists a constant* $c_Q \in \mathbb{R}$ *such that Euler-Lagrange equation*

$$\begin{aligned} U^\ell_{\mu^\ell_Q} + \tfrac{1}{2}Q &= c_Q \quad \text{q.e. in } S^\ell_Q \quad \text{and} \\ U^\ell_{\mu^\ell_Q} + \tfrac{1}{2}Q &\geqslant c_Q \quad \text{q.e. in } \mathbb{C} \end{aligned} \tag{A.10}$$

holds. The equilibrium measure μ^ℓ_Q *in the set* S^ℓ_Q *is given by*

$$\mu_Q = \frac{1}{4\pi}(\Delta Q + m^2(2c_Q - Q)) = \frac{1}{4\pi}\left((\Delta - m^2)Q + 2m^2 c_Q\right), \tag{A.11}$$

where the Laplacian is understood in the distributional sense. We will drop the subscript ℓ *when it is understood.*

Proof. The proof is identical to that of the Coulomb case; see e.g. [40]. □

Remark A.6. *By the same argument, under the additional assumption that* Q *satisfies* (1.4)*, the support of* μ^ℓ_Q *is compact uniformly in* ℓ*.*

The next theorem characterizes the Yukawa potential of the equilibrium measure in terms of an obstacle problem. Again, the theorem is similar to the Coulomb case, but requires a slightly different characterization of the admissible potentials than the one stated for the Coulomb case in [24], for example. We give a proof for completeness, as we were unable to locate a suitable reference.

Proposition A.7. *Under the assumptions of Theorem A.5, the following holds. Define*

$$u_{Q,\ell}(z) = \sup_{\nu,c}\{-U^\ell_\nu(z) + c : -U^\ell_\nu + c \leqslant \tfrac{1}{2}Q, \nu \geqslant 0, \nu(\mathbb{C}) \leqslant 1\}, \tag{A.12}$$

where the supremum is over measures ν and constants c. Then $u_{Q,\ell} = -U^\ell_{\mu_Q} + c_Q$ where c_Q is the constant in (A.10)*.*

Proof. By definition, $u_{Q,\ell} \geqslant -U^\ell_{\mu_Q} + c_Q$ since the right-hand side is a subsolution of the same form as inside the supremum in (A.12). To prove that in fact equality holds, suppose otherwise that $u_{Q,\ell}(z_0) > -U^\ell_{\mu_Q}(z_0) + c_Q$ for some $z_0 \in \mathbb{C}$. Then there exists some positive measure $\tilde\eta$ with $\tilde\eta(\mathbb{C}) \leqslant 1$ and constant $c \in \mathbb{R}$ for which $-U^\ell_{\tilde\eta}(z_0) + c > -U^\ell_{\mu_Q}(z_0) + c_Q$. By considering $\tilde\eta|_{B_R}$ for $R > 0$ large enough we may suppose that $\tilde\eta$ is compactly supported, and by convolving with a smooth mollifier we may suppose $\tilde\eta$ has a smooth density. Consider the function

$$g(z) = \max(-U^\ell_{\tilde\eta}(z) + \tilde c, -U^\ell_{\mu_Q}(z) + c_Q).$$

By writing $\max(a, b) = \frac{a+b}{2} + \frac{|a-b|}{2}$ and convolving the absolute value by a smooth, compactly supported, symmetric mollifier, we may check that $g(z) = -U^\ell_\eta(z) + c$ for some positive measure η, and necessarily $c = \max(\tilde c, c_Q)$. To show that g is a subsolution of the form in (A.12) we need to show that $\eta(\mathbb{C}) \leqslant 1$. For this, suppose without loss of generality that $c = \tilde c$. Denote $D = \{z : -U^\ell_{\tilde\eta}(z) + \tilde c < -U^\ell_{\mu_Q}(z) + c_Q\}$. Then

$$\begin{aligned}
\eta(\partial D) &= \int_{\partial D} \partial_n(-U^\ell_{\tilde\eta} - (-U^\ell_{\mu_Q})) = \int_D \Delta(-U^\ell_{\tilde\eta} - (-U^\ell_{\mu_Q})) \\
&= \int_D (\tilde\eta - \mu_Q) + m^2 \int_D (-U^\ell_{\tilde\eta} - (-U^\ell_{\mu_Q})) \\
&= \int_D (\tilde\eta - \mu_Q) + m^2 \int_D (-U^\ell_{\tilde\eta} + \tilde c - (-U^\ell_{\mu_Q} + c_Q)) + m^2 \int_D (c_Q - \tilde c) \leqslant \tilde\eta(D) - \mu_Q(D).
\end{aligned}$$

Thus $\eta(D \cup \partial D) = \eta(\partial D) + \mu_Q(D) \leqslant \tilde\eta(D)$. Since clearly $\eta(\mathbb{C} \setminus (\partial D \cup D)) = \tilde\eta(\mathbb{C} \setminus (\partial D \cup D))$, we have $\eta(\mathbb{C}) \leqslant \tilde\eta(C) \leqslant 1$. Now,

$$\begin{gathered}
g - (-U^\ell_{\mu_Q} + c_Q) \geqslant 0, \\
(\Delta - m^2)(-U^\ell_\eta - (-U^\ell_{\mu_Q}) = \eta - \mu \geqslant m^2(c - c_Q) \geqslant 0.
\end{gathered}$$

Since strict inequality holds in the first inequality for z_0 and the functions involved are continuous, equality (as distributions) cannot hold on the second line. But this implies $\eta(\mathbb{C}) > \mu_Q(\mathbb{C}) = 1$, a contradiction. ☐

We also require the following properties of the Yukawa potential (2.1). Recall that

$$Y^\ell(z) = g(a), \quad a = \frac{|z|}{2\ell}, \qquad \text{where } g(a) = \int_1^\infty \mathrm{e}^{-a(s+1/s)} \frac{\mathrm{d}s}{s}. \tag{A.13}$$

In fact, $g(a) = K_0(2a)$ where K_0 is a modified Bessel function of the second kind. In particular, the gradient of the Yukawa potential has the expression:

$$\nabla Y^\ell(z) = g'(\frac{|z|}{2\ell})\frac{\nabla|z|}{2\ell} = g'(\frac{|z|}{2\ell})\frac{|z|}{2\ell}\frac{1}{\bar{z}} = -\frac{1}{\bar{z}}f(a), \quad a = \frac{|z|}{2\ell}, \tag{A.14}$$

where

$$f(a) = \int_1^\infty a(s+1/s)\mathrm{e}^{-a(s+1/s)}\frac{\mathrm{d}s}{s} = \int_a^\infty (1+a^2/s^2)\mathrm{e}^{-(s+a^2/s)}\mathrm{d}s.$$

The function f is smooth in $a > 0$, satisfies $f(0) = 1$, and is positive and decreasing. As a consequence we have $|\nabla Y^\ell(2r)| \leqslant |\nabla Y^\ell(r)|/2$.

Since $\nabla Y^\ell(z) \sim \nabla \log \frac{1}{|z|}$ for $z \to 0$, the following formula can be proven as in [6, (3.21)]. Let $\gamma \subset \mathbb{C}$ be a C^1 curve and η a measure supported on γ for which the potential U_η^ℓ is continuous on $\mathbb{C}$. Then for $z \in \gamma$ we have

$$\partial_n^- U_\eta^\ell(z) = \pi \lim_{r\to 0^+} \frac{\eta(B_r(z))}{s(B_r(z))} + \int_\gamma \nabla Y^\ell(z-w)\cdot \bar{n}\,\eta(dw), \tag{A.15}$$

where ∂_n^- denotes a one-sided derivative in the normal direction $\bar{n} = \bar{n}(z)$ and s denotes the arclength measure of γ, if the limit on the right-hand side exists.

The formula (A.15) implies the following estimate for the density of a measure supported on $\partial\mathbb{D}$. For the statement, define

$$I^\ell := \frac{1}{2\pi}\int_{\partial\mathbb{D}} f(\frac{|1-w|}{2\ell})\, s(\mathrm{d}w) \in (0,1), \tag{A.16}$$

and note that I^ℓ is increasing in ℓ with $I^\ell = 1 + O(1/\ell)$ as $\ell \to \infty$ and $I^\ell = O(\ell)$ as $\ell \to 0$. The proofs of the following Lemma A.8 and A.9 are based on elementary potential theory and can be skipped in the first reading.

Lemma A.8. *For any (signed) measure ω supported on $\partial\mathbb{D}$, denote by $\bar{\omega} = \frac{1}{2\pi}\int \mathrm{d}\omega$ the constant part of ω. Then*

$$\left\|\frac{d\omega}{ds} - \bar{\omega}\right\|_\infty \leqslant \frac{2}{\pi I^\ell}\|\partial_n^- U_\omega^\ell\|_{\infty,\partial\mathbb{D}} \tag{A.17}$$

$$\left\|\frac{d\omega}{ds}\right\|_\infty \leqslant \frac{1}{\pi(1-I^\ell)}\|\partial_n^- U_\omega^\ell\|_{\infty,\partial\mathbb{D}} \tag{A.18}$$

and

$$\partial_n^- U_{\bar{\omega}}^\ell(1) = \frac{1}{2\pi}\int_{\partial\mathbb{D}} \partial_n^- U_\omega^\ell(z)\, s(\mathrm{d}z) \leqslant \|\partial_n^- U_\omega^\ell\|_{\infty,\partial\mathbb{D}}. \tag{A.19}$$

Proof. By (A.15), we have

$$\frac{d\omega}{ds}(z) = \frac{1}{2\pi}\left(2\partial_n^- U_\omega^\ell(z) - 2\int \nabla Y^\ell(z-w)\cdot\bar{n}(z)\,\omega(\mathrm{d}w)\right). \tag{A.20}$$

For z, w with $|z| = |w| = 1$ and $z \neq w$,

$$\frac{z-w}{|z-w|^2}\cdot\frac{z}{|z|} = \mathrm{Re}\left(\frac{z-w}{|z-w|^2}\bar{z}\right) = \mathrm{Re}\left(\frac{1-w/z}{|1-w/z|^2}\right) = \frac{1}{2}, \tag{A.21}$$

and, by (A.14), therefore

$$-2\nabla Y^{\ell}(z-w)\cdot n(z) = f(\frac{|z-w|}{2\ell}). \tag{A.22}$$

It follows that

$$\frac{d\omega}{ds}(z) = \frac{1}{2\pi}\left(2\partial_n^- U_\omega^\ell(z) + \int f(\frac{|z-w|}{2\ell})\,\omega(\mathrm{d}w)\right). \tag{A.23}$$

Integrating (A.23), we obtain the identity

$$(1-I^\ell)\int \omega = \frac{2}{2\pi}\int_{\partial\mathbb{D}} \partial_n^- U_\omega^\ell(z)\, s(\mathrm{d}z). \tag{A.24}$$

Applying this identity to $\bar{\omega}$, since $\int \mathrm{d}\omega = \int \mathrm{d}\bar{\omega}$, we obtain

$$\partial_n^- U_{\bar{\omega}}^\ell(1) = \frac{1}{2\pi}\int_{\partial\mathbb{D}} \partial_n^- U_{\bar{\omega}}^\ell(z)\, s(\mathrm{d}z) = \frac{1}{2\pi}\int_{\partial\mathbb{D}} \partial_n^- U_\omega^\ell(z)\, s(\mathrm{d}z). \tag{A.25}$$

This shows (A.19). Similarly, from (A.23), we obtain

$$(1-I^\ell)\left\|\frac{d\omega}{ds}\right\|_\infty \leqslant \frac{2}{2\pi}\|\partial_n^- U_\omega^\ell\|_\infty, \tag{A.26}$$

which shows (A.18), and also similarly,

$$\left\|\frac{d\omega}{ds} - \frac{1}{2\pi}\int f(\frac{|\cdot - w|}{2\ell})\,\omega(\mathrm{d}w)\right\|_\infty \leqslant \frac{2}{2\pi}\|\partial_n^- U_\omega^\ell\|_\infty. \tag{A.27}$$

To show (A.17), i.e.,

$$I^\ell\left\|\frac{d\omega}{ds} - \frac{1}{2\pi}\int d\omega\right\|_\infty \leqslant \frac{4}{2\pi}\|\partial_n^- U_\omega^\ell\|_\infty, \tag{A.28}$$

write

$$\begin{aligned}
&\frac{d\omega}{ds} - \frac{1}{2\pi}\int f(\frac{|\cdot - w|}{2\ell})\,\omega(dw) \\
&= \frac{d\omega}{ds} - \frac{1}{2\pi}\int d\omega + \frac{1}{2\pi}\int \left(1 - f(\frac{|\cdot - w|}{2\ell})\right)\,\omega(dw) \\
&= \frac{d\omega}{ds} - \frac{1}{2\pi}\int d\omega + \frac{1}{2\pi}\int \left(1 - f(\frac{|\cdot - w|}{2\ell})\right)\left(\frac{d\omega}{ds}(w) - \frac{1}{2\pi}\int d\omega\right) s(dw) \\
&\quad + \frac{1}{2\pi}\int \left(1 - f(\frac{|\cdot - w|}{2\ell})\right) s(dw)\cdot\frac{1}{2\pi}\int d\omega,
\end{aligned}$$

Taking absolute values on the supremum over $\partial\mathbb{D}$, and using (A.26), therefore

$$\begin{aligned}
&\|\frac{d\omega}{ds} - \frac{1}{2\pi}\int f(\frac{|\cdot - w|}{2\ell})\,\omega(dw)\|_\infty \\
&\geqslant \|\frac{d\omega}{ds} - \frac{1}{2\pi}\int d\omega\|_\infty - (1-I^\ell)\|\frac{d\omega}{ds} - \frac{1}{2\pi}\int d\omega\|_\infty - (1-I^\ell)\left|\frac{1}{2\pi}\int d\omega\right|, \\
&\geqslant I^\ell\|\frac{d\omega}{ds} - \frac{1}{2\pi}\int d\omega\|_\infty - \frac{2}{2\pi}\|\partial_n^- U_\omega^\ell\|_\infty.
\end{aligned}$$

Together with (A.27), we obtain (A.28). □

We will also need the following properties of the function

$$l_r(z) = \left(Y^\ell * \frac{1}{\pi r^2} 1_{B(0,r)}\right)(z). \tag{A.29}$$

Clearly, $l_r(z)$ is radial, so we can write $\nabla l_r(z) = -(z/|z|)h_r(|z|)$ for $z \neq 0$.

Lemma A.9. *For any $\ell > 0$, the function $h_r(t)$ is positive, increasing for $t \leqslant r$, decreasing for $t \geqslant r$, and*

$$h_r(t) \geqslant |\nabla Y^\ell(t)| \quad \textit{for } t \geqslant r. \tag{A.30}$$

Proof. That $h_r(t)$ is increasing for $t < r$ can be seen as follows. For $t > 0$, since Y^ℓ is symmetric,

$$\nabla l_r(t) = \int_{|z|\leqslant r} \nabla Y^\ell(z-t)\, m(\mathrm{d}z) = \int_{U_r(t)} \mathrm{Re}\, \nabla Y^\ell(z-t)\, m(\mathrm{d}z) = \int_{U_r(t)-t} \mathrm{Re}\, \nabla Y^\ell(z)\, m(\mathrm{d}z)$$

where $U_r(t)$ is $\{|z| \leqslant r\}$ minus the region $\{|z| \leqslant r : \mathrm{Re}\, z > t\}$ and the reflection of the latter region about the axis $\mathrm{Re}\, z = t$. In particular, the region $U_r(t) - t$ is increasing in t.

To prove (A.30) and that $h_r(t)$ is decreasing for $t > r$, we use the Yukawa version of Newton's shell theorem: there is $M^\ell(r) \geqslant 1$ such that for $t \geqslant r$,

$$\frac{1}{2\pi r} \int_{|z|=r} Y^\ell(t-z)\, s(\mathrm{d}z) = M^\ell(r) Y^\ell(t). \tag{A.31}$$

Denote the left-hand side by $f(t)$. Then f is a bounded and radially symmetric solution to $(-\Delta + 1/\ell^2) f(z) = 0$ for $|z| > r$. Therefore, for $t > r$,

$$f''(t) + \frac{1}{t} f'(t) - \frac{1}{\ell^2} f(t) = 0, \tag{A.32}$$

and the solutions to this ODE are of the form

$$f(t) = A I_0(t/\ell) + B K_0(t/\ell), \tag{A.33}$$

where the I_n are the modified Bessel functions of the first kind and the K_n are the modified Bessel functions of the second kind, and A, B are constants depending on r. The Yukawa potential equals $Y^\ell(z) = K_0(|z|/\ell)$. Since $I_0(t) \to \infty$ as $t \to \infty$, therefore $A = 0$ and thus $f(t) = B K_0(t/\ell) = B Y^\ell(t)$ for some constant $B = M^\ell(r)$.

To see that $B \geqslant 1$, we assume that $r = 1$ and $\ell = 1/2$ to simplify the notation (the general case is analogous). Denote by θ the angle of z with respect to the real axis so that $|t - z|^2 = t^2 - 2t\cos\theta + 1$. Recall (2.1) and note that the function $\tilde{g}(x) = \int_1^\infty \mathrm{e}^{-\sqrt{x}(s+1/s)} \frac{\mathrm{d}s}{s}$ is convex for $x \geqslant 1$. With $x = t^2 - 2t\cos\theta + 1$ and using the Jensen inequality, we have

$$f(t) = \mathbb{E}\tilde{g}(t^2 - 2t\cos\theta + 1) \geq \tilde{g}(t^2 - 2t\mathbb{E}\cos\theta + 1) = \tilde{g}(t^2+1), \quad \mathbb{E} = (2\pi)^{-1}\int \mathrm{d}\theta. \tag{A.34}$$

It is elementary to check that

$$\lim_{t\to\infty} \frac{\tilde{g}(t^2+1)}{\tilde{g}(t^2)} = 1. \tag{A.35}$$

Hence we have proved that $B \geq 1$. (In fact, $B > 1$ for any r, ℓ fixed, but we will not need this.)

In particular, for $t \geqslant r$,

$$l_r(t) = \frac{1}{\pi r^2} \int_{|z|\leqslant r} Y^\ell(z-t) = \frac{1}{\pi r^2} \int_0^r \int_{|z|=s} Y^\ell(z-t)\, s(\mathrm{d}z)\, \mathrm{d}r = \tilde{M}^\ell(t) Y^\ell(t) \tag{A.36}$$

with $\tilde{M}^\ell(t) = \frac{1}{\pi r^2} \int_0^r (2\pi r) M^\ell(r)\, \mathrm{d}r \geqslant 1$. Thus, for $t \geqslant r$,

$$|\nabla l_r(t)| = \tilde{M}^\ell(r) |\nabla Y^\ell(t)| \geqslant |\nabla Y^\ell(t)|. \tag{A.37}$$

The first equality implies that $|\nabla l_r(t)|$ is decreasing for $t > r$ since $|\nabla Y^\ell(t)|$ is decreasing. The inequality implies that (A.30) holds. □

In Section A.4 below, we require the following two technical lemmas to locate the bulk of the support of a perturbed equilibrium measure. Lemma A.10 is a small adaption of [6, Lemma 3.6] to the Yukawa case; Lemma A.11 is a similar statement that applies to a radially symmetric potential on the boundary of a disk instead of a point charge outside a disk.

Lemma A.10. *For any $z_0 \in \mathbb{C}, w \in \mathbb{C}$, $\sigma > \frac{1}{2}$, and $r \in (0,1)$ such that that $|z_0 - w| \geqslant 2r$, there exist $\tilde{z} \in \mathbb{C}$ and $k \in \mathbb{R}$ such that*

$$\sigma\big(l_r(z_0 - \tilde{z}) + k\big) = \frac{1}{2} Y^\ell(z_0 - w) \quad \textit{and} \quad \sigma\big(l_r(z - \tilde{z}) + k\big) \leqslant \frac{1}{2} Y^\ell(z - w) \textit{ for all } z \in \mathbb{C}. \tag{A.38}$$

Moreover, the point $\tilde{z}$ lies on the line passing through z_0 and w at distance at most r from z_0 between z_0 and w.

Proof. By (A.30) and since $\sigma \geqslant \frac{1}{2}$, the map $z \mapsto \sigma \nabla l_r(z_0 - z)$ takes $B_r(z_0)$ onto $B_{\sigma|\nabla l_r(r)|}(0) \supset B_{\sigma|\nabla Y^\ell(r)|} \supset B_{|\nabla Y^\ell(2r)|}(0)$, where we also used $|\nabla Y^\ell(2r)| \leqslant \frac{1}{2}|\nabla Y^\ell(r)|$. Therefore, as in [6, Lemma 3.6], it follows there exists a unique choice of $\tilde{z} \in B_r(z_0)$ so that the gradients of $\sigma l_r(\cdot - \tilde{z})$ and $\frac{1}{2} Y^\ell(\cdot - w)$ match at z_0. By choice of k, we can in addition arrange

$$\sigma\big(l_r(z_0 - \tilde{z}) + k\big) = \frac{1}{2} Y^\ell(z_0 - w). \tag{A.39}$$

It remains to be shown that with the above choice it is in fact true that

$$\sigma\big(l_r(z - \tilde{z}) + k\big) \leqslant \frac{1}{2} Y^\ell(z - w) \quad \text{for all } z \in \mathbb{C}. \tag{A.40}$$

As in the Coulomb case, the point must $\tilde{z}$ lie on the line between the points z_0 and w, and it suffices to show the inequality on this line (by the same argument as in the Coulomb case, [6, Lemma 3.6]). Moreover, without loss of generality, we can assume that $w = 0$, $z_0 > 0$, $\tilde{z} > 0$, so that this line is $\mathbb{R}$. Thus it needs to be shown that

$$f(x) := \frac{1}{2} Y^\ell(x) \geqslant \sigma\big(l_r(x - \tilde{z}) + k\big) =: g(x), \quad x \in \mathbb{R}.$$

As in the Coulomb case, denote by h the common tangent of the graphs of f and g drawn at $x = z_0$. Since f is convex and g is concave on $[\tilde{z} - r, \tilde{z} + r]$, the graph of f lies above h and the graph of g lies below h on this interval. Especially $g(x) \leqslant f(x)$ on $[\tilde{z} - r, \tilde{z} + r]$. Moreover, since $f'(x) < 0$ and $g'(x) > 0$ for $x \in (0, \tilde{z})$, the inequality $g(x) \leqslant f(x)$ holds by these observations for $x \in (0, \tilde{z} + r]$.

To prove the inequality for $x \in [\tilde{z}+r,\infty)$, we have $g'(t) \leqslant f'(t+\tilde{z}) \leqslant f'(t)$ by (A.30), for $t \in [\tilde{z}+r,\infty)$. It follows that

$$g(x) - g(\tilde{z}+r) = \int_{\tilde{z}+r}^{x} g'(t)\,dt \leqslant \int_{\tilde{z}+r}^{x} f'(t)\,dt = f(x) - f(\tilde{z}+r),$$

which by $g(\tilde{z}+r) \leqslant f(\tilde{z}+r)$ implies the desired inequality $g(x) \leqslant f(x)$, now proven for $x \in (0,\infty)$. The case $x < 0$ is actually not required for the application, but true. Indeed, for $x \in (-\infty,0)$ it also holds that $g'(x) \leqslant f'(x)$ and it is clear that $f(x) \geqslant g(x)$ as $x \to 0^-$, so it remains to check the inequality as $x \to -\infty$. As in the Coulomb case, this follows from $k < 0$, which follows from

$$\sigma k = \frac{1}{2}Y^{\ell}(z_0) - \sigma l_r(z_0 - \tilde{z}) < \frac{1}{2}Y^{\ell}(2r) - \sigma l_r(r) < 0.$$

This completes the proof. □

Lemma A.11. *Let $r \in (0,\frac{1}{2})$ and $\sigma \geqslant \sigma_0$ and $\ell \geqslant \ell_0$, where σ_0 and ℓ_0 are sufficiently large absolute constants. Then for any $z_0 \in \mathbb{C}$ with $|z_0| < 1-2r$, there exists a constant $k \in \mathbb{R}$ and $\tilde{z} \in \mathbb{C}$ with $|\tilde{z}| < 1-r$ on the line through 0 and z_0 such that*

$$\sigma\big(l_r(z_0-\tilde{z})+k\big) = \pm\ell^2 I_0(|z_0|/\ell) \quad \textit{and} \quad \sigma\big(l_r(z-\tilde{z})+k\big) \leqslant \pm\ell^2 I_0(|z|/\ell) \textit{ for all } z \in \mathbb{D}, \tag{A.41}$$

where $\pm$ is either always $+$ or always $-$, and I_0 is a modified Bessel function of the first kind.

Proof. Throughout the proof, $x \gg 1$ means that x is larger than a large absolute constant. Let

$$I(z) = \ell^2(I_0(|z|/\ell) - 1). \tag{A.42}$$

Replacing k by $k - \ell^2/\sigma$, the claim (A.41) is equivalent to the claim

$$\sigma\big(l_r(z_0-\tilde{z})+k\big) = I(z_0) \quad \text{and} \quad \sigma\big(l_r(z-\tilde{z})+k\big) \leqslant I(z) \text{ for all } z \in \mathbb{D}. \tag{A.43}$$

For the right-hand side, for $\ell \gg 1$, we have

$$I(z) = \frac{1}{4}|z|^2(1+\mathrm{O}(|z|/\ell)), \quad \nabla I(z) = \left(\frac{1}{2} + O(1/\ell)\right) z, \quad \nabla^2 I(z) = \frac{1}{2}\mathbf{1}_{2\times 2} + \mathrm{O}(1/\ell). \tag{A.44}$$

For $\ell \gg 1$, the map $z \mapsto \sigma\nabla l_r(z)$ takes $B_r(0)$ onto $B_{\sigma|\nabla Y^{\ell}(r)|}(0) \supset B_{\sigma(1-\varepsilon)/r}(0) \supset B_1(0)$. Thus, by appropriate choice of $\tilde{z}$ and k, the derivatives of $\sigma l_r(z-\tilde{z})$ and $\pm I$ can be matched at any $|z_0| < 1$. It remains to show the inequality in (A.41). By definition of l_r and since, by (A.14), the derivatives of $Y^{\ell}(z)$ are well approximated by those of $-\log|z|$ for $\ell \gg 1$, we have

$$\nabla^2 l_r(z) = -\frac{1}{r^2}(\mathbf{1}_{2\times 2} + \mathrm{O}(1/\ell)) \quad \text{for } |z| < r. \tag{A.45}$$

Together with (A.44), using that $1/r^2 > 1 > 1/2$, it follows that the function $l_r(z-\tilde{z})+k$ stays below $\pm I(z)$ for $|z-\tilde{z}| < r$, provided that $\ell \gg 1$. Using further that $l_r(0) - l_r(r) = \frac{1}{2} + \mathrm{O}(1/\ell)$, we can choose $\sigma \geqslant \sigma_0$ and $\ell \geqslant \ell_0$ large enough that

$$\sigma(l_r(0) - l_r(r)) > \frac{1}{4}(1+\mathrm{O}(1/\ell)) = \sup_{\mathbb{D}}(\pm I) - \inf_{\mathbb{D}}(\pm I).$$

Since $\sigma(l_r(0)+k) \leqslant \sup_{\mathbb{D}}(\pm I)$, it follows that $\sigma(l_r(z-\tilde{z})+k) \leqslant \inf_{\mathbb{D}}(\pm I)$ for $|z-\tilde{z}| = r$. Since $l_r(z-\tilde{z})$ is decreasing in $|z-\tilde{z}|$ the inequality then holds on all of $\mathbb{D}$. □

A.4. Perturbed equilibrium measure. As in [6], to prove the local law, we will condition on the particles outside small disks. To handle this conditioning, we next state adaptations of the results of [6, Section 3.3] to the Yukawa case. As in [6, Section 3.3], we can assume here that $S_Q = \rho\overline{\mathbb{D}}$ for some $\rho > 0$, where $\mathbb{D} \subset \mathbb{C}$ is the open unit disk. Furthermore, we assume the density of μ_Q is bounded below by $\frac{1}{4\pi}\alpha$ in $\rho\mathbb{D}$ for some parameter $\alpha > 0$. The class of perturbed potentials W that we consider is as follows. Let ν be a positive measure with $\operatorname{supp}\nu \cap \rho\mathbb{D} = \varnothing$, $t > 0$ and let $R \in \mathscr{C}(\rho\overline{\mathbb{D}})$ satisfy $(\Delta - m^2)R = 0$ in $\rho\mathbb{D}$. Then W is given by

$$W(z) = \begin{cases} tQ(z) + 2U_\nu^\ell(z), +2R(z), & z \in \rho\overline{\mathbb{D}}, \\ \infty, & z \in \rho\mathbb{D}^*, \end{cases} \tag{A.46}$$

where we write $\mathbb{D}^* = \mathbb{C} \setminus \overline{\mathbb{D}}$ for the open complement of the unit disk. Both perturbations U_ν^ℓ and R are m-harmonic inside $\rho\mathbb{D}$, i.e., $(\Delta - m^2)R = 0$ and analogously for U_ν^ℓ. In particular, by (A.10), this implies that the density of μ_W is equal to $t\mu_Q +$ constant in S_W. For $z \in \partial(\rho\mathbb{D})$ we write $\bar{n} = \bar{n}(z) = z/|z|$ for the outer unit normal, and we write $\partial_n^- f(z) = \lim_{\varepsilon\downarrow 0} \frac{f(z)-f(z-\varepsilon\bar{n})}{\varepsilon}$ for the derivative in the direction $\bar{n}$ taken from inside $\rho\mathbb{D}$.

The next two propositions show that the bulk of the equilibrium measure μ_Q is stable under suitable perturbations W of the form (A.46), and that the density of μ_W on the boundary remains bounded. To prove the stability of the bulk we use the obstacle problem characterization (A.12) of the support.

Proposition A.12. *Suppose that Q and W are as above* (A.46)*. Then, for any $\ell > 0$, the support S_W of the equilibrium measure with Yukawa interaction of range ℓ and potential W satisfies*

$$S_W \supset \{z \in \rho\mathbb{D} : \operatorname{dist}(z, \rho\mathbb{D}^*) \geqslant \kappa\}, \quad \text{where } \kappa = C\sqrt{\frac{\max(\|\nu\|, \rho\|\partial_n^- R\|_{\infty,\partial\rho\mathbb{D}} + (t-1))}{\alpha t}}. \tag{A.47}$$

Proof. As in the proof of [6, Proposition 3.3], except that we must now replace ℓ by ℓ/ρ, we may assume that $\rho = 1$, and we define $D = \{z \in \mathbb{D} : \operatorname{dist}(z, \mathbb{D}^*) \geqslant \kappa\}$. The replacement of ℓ does not matter since the estimate is uniform in ℓ. By Proposition A.7, to prove the proposition, it suffices to exhibit, for any $z_0 \in D$, a test function $v_{z_0} = v = -U_\nu^\ell(z) + c$ with $v(z_0) = \frac{1}{2}W(z_0)$ and satisfying the requirements for the potential in (A.12) with W instead of Q.

This test function is chosen almost exactly as in the Coulomb case, with the small difference in the handling of the perturbation R. Indeed, recall that by assumption $R = U_\mu^\ell$ for a (signed) charge distribution μ supported in $\mathbb{D}^*$. Up to an additive constant, we may replace μ by its *balayage* ω onto $\partial\mathbb{D}$, i.e., we choose the measure ω supported on $\partial\mathbb{D}$ such that $R = U_\omega^\ell + c$ in $\mathbb{D}$. The existence of ω follows as in the Coulomb case; see e.g. [40]. We choose ℓ_0 to be the sufficiently large absolute constant from Lemma A.11. For $\ell \geqslant \ell_0$, we decompose $\omega = \omega_0 + \omega_+ - \omega_-$ with ω_0 a measure of constant density with respect to the arclength measure on $\partial\mathbb{D}$ such that $\int \mathrm{d}\omega = \int \mathrm{d}\omega_0$ and with $\omega_\pm$ positive measures. For $\ell < \ell_0$, we simply decompose $\omega = \omega_+ - \omega_-$ with $\omega_\pm$ positive measures and set $\omega_0 = 0$. In both cases, Lemma A.8 implies that the total charge of $\omega_\pm$ is estimated by

$$\|\omega_\pm\| = \mathrm{O}(1)\|\partial_n^- R\|_{\infty,\partial\mathbb{D}}. \tag{A.48}$$

Then, similarly as in [6, Proposition 3.3], we will choose the function v of the form

$$v(z) = tu_{Q,\ell}(z) + \sigma L(z) + \gamma L_0(z) - U^{\omega_-}(z), \quad L(z) = \int \big(l_r\big(z - \tilde{z}(w)\big) + k(w)\big)\,(\nu + \omega_+)(\mathrm{d}w), \tag{A.49}$$

where $\sigma > 0$, $r > 0$, $k : \operatorname{supp}\nu \to \mathbb{R}$ and $\tilde{z} : \operatorname{supp}\nu \to \mathbb{D}$ are parameters, and the function l_r is now defined by (A.29), and $L_0(z)$ is chosen of the form

$$L_0(z) = l_r(z - \tilde{z}_0) - k_0.$$

for some $\tilde{z}_0 \in \mathbb{C}$ and $k_0 \in \mathbb{R}$ to be chosen later.

Step 1. With the choice

$$\gamma = \mathrm{O}(1)\|\partial_n^- R\|_{\infty,\partial\mathbb{D}}, \quad \sigma = \max\left(\frac{1}{2}, \frac{(t-1) - \gamma + \|\omega_-\|}{\|\nu + \omega_+\|}\right), \quad r = 2\sqrt{\frac{\|\nu + \omega_+\|\sigma + \gamma}{\alpha t}} = \frac{1}{2}\kappa,$$

the function v is of the form $-U_\mu^\ell + c$ for a positive measure μ of total mass at most $t + \|\omega_-\| - \gamma - \sigma\|\nu + \omega_+\| \leqslant 1$. Indeed, by definition, $-tu_{Q,\ell} + U_{\omega_-}$ is the potential of a positive measure of mass $t + \|\omega_-\|$ and $-\sigma L - \gamma L_0$ is the potential of a negative measure of total mass $-\sigma\|\nu + \omega_+\| - \gamma$. Their sum is the potential of a positive measure since

$$(\Delta - m^2)(tu_{Q,\ell} - U^{\omega_-} + \sigma L + \gamma L_0) \geqslant 2\pi t\rho_{Q,\ell} + 2\pi\omega_- - \frac{2\sigma}{r^2}\|\nu + \omega_+\| - \frac{2\gamma}{r^2} \geqslant 0, \tag{A.50}$$

where we used the assumption $\rho_{Q,\ell} \geqslant \alpha/(4\pi)$.

Step 2. For appropriate choice of the parameters $\tilde{z}$ and k (depending on z_0), we have $v(z_0) = \frac{1}{2}W(z_0)$ and $v \leqslant \frac{1}{2}W$ in $\overline{\mathbb{D}}$. Indeed, replacing [6, Lemma 3.6] by Lemma A.10 stated below the proof, we choose the parameters $\tilde{z}$ and k exactly as in the proof of [6, Proposition 3.3] to achieve

$$\sigma L(z) \leqslant \frac{1}{2}\int Y^\ell(z - w)\,(\nu + \omega_+)(dw) \quad \text{for all } z \in \overline{\mathbb{D}}, \tag{A.51}$$

$$\sigma L(z_0) = \frac{1}{2}\int Y^\ell(z_0 - w)\,(\nu + \omega_+)(dw). \tag{A.52}$$

This concludes the proof for $\ell < \ell_0$. For $\ell \geqslant \ell_0$, it remains to handle the remaining part of the perturbation, which is the potential $U_{\bar{\omega}}^\ell$ generated by the constant part $\bar{\omega}$ of ω. Since the Yukawa potential of $\bar{\omega}$ is m-harmonic in $|z| < 1$, radially symmetric and bounded as $|z| \to 0$, as in (A.33), it is explicitly given inside $\mathbb{D}$ by

$$U_{\bar{\omega}}^\ell(z) = \pm A\ell^2 I_0(|z|/\ell), \quad (|z| < 1),$$

for some constant $A > 0$ depending on ℓ and $\bar{\omega}$, where I_n are the modified Bessel functions of the first kind. Using that $I_0' = I_1$ by general relations between Bessel functions,

$$\nabla U_{\bar{\omega}}^\ell(z) = \pm A2\ell I_1(|z|/\ell)\frac{z}{|z|} = \partial_n^- U_{\bar{\omega}}^\ell(1)\frac{I_1(|z|/\ell)}{I_1(1/\ell)}\frac{z}{|z|}.$$

The modified Bessel functions satisfy the asymptotics

$$I_0(t) \sim 1 + \frac{1}{4}t^2, \quad I_1(t) \sim \frac{1}{2}t, \quad \text{as } t \to 0. \tag{A.53}$$

Therefore, with (A.19), the constant A is given by

$$A = \pm\frac{\partial_n^- U_{\bar{\omega}}^\ell(1)}{2\ell I_1(1/\ell)} = \pm(1 + \mathrm{O}(1/\ell))\partial_n^- U_{\bar{\omega}}^\ell(1) \leqslant (1 + \mathrm{O}(1/\ell))\|\partial_n^- U_\omega^\ell\|_{\infty,\partial\mathbb{D}} = \mathrm{O}(1)\|\partial_n^- R\|_{\infty,\partial\mathbb{D}}. \tag{A.54}$$

By Lemma A.11, there exists a large constant σ such that we can choose k_0 and $\tilde{z}_0$ and $\gamma = \mathrm{O}(1)\|\partial_n^- R\|_{\infty,\partial\mathbb{D}}$ such that with $\gamma = \sigma A$,

$$\gamma L_0(z_0) = U^\ell_{\tilde{\omega}}(z_0), \quad \gamma L_0(z) \leqslant U^\ell_{\tilde{\omega}}(z) \quad \text{for all } z \in \mathbb{D}. \tag{A.55}$$

This concludes the proof. □

Proposition A.13. *Suppose that Q and W are as above* (A.46) *and assume in addition that μ_Q is absolutely continuous with respect to the 2-dimensional Lebesgue measure. Then $\mu_W = \mu + \eta$, where μ is absolutely continuous with respect to μ_Q, and η absolutely continuous with respect to the arclength measure s on $\partial\rho\mathbb{D}$ with the Radon–Nikodym derivative bounded by*

$$\rho\Big\|\frac{d\eta}{ds}\Big\|_\infty \leqslant C\Big(\|\eta\| + \|\nu\| + 2\rho\|\partial_n^- R\|_{\infty,\partial\rho\mathbb{D}} + |1-t|\rho\|\partial_n^- Q\|_{\infty,\partial\rho\mathbb{D}}\Big). \tag{A.56}$$

Proof. The only change in the proof of Proposition A.13 compared to [6] is the change of the logarithmic potentials to Yukawa potentials. In particular, the formula (A.15) holds and $\nabla Y^\ell(z)$ is proportional to $\nabla \log \frac{1}{|z|}$. □

A.5. One-step estimate. As in [6, Proposition 4.1], we use a simple mean-field partition function estimate to obtain a bound on the fluctuations of smooth linear statistics.

Proposition A.14. *Let $\Sigma = \Sigma_W$ be a smooth domain. Given a potential $W \in C^{1,1}_{loc}(\Sigma_W)$ possibly depending on the number of particles M, assume that there exist $u : \Sigma_W \to \mathbb{R}_+$ and $v : \partial\Sigma_W \to \mathbb{R}_+$ such that $d\mu_W = u\,\mathrm{d}m + v\,\mathrm{d}s$, where $\mathrm{d}m$ is the 2-dimensional Lebesgue measure and $\mathrm{d}s$ is the arclength measure on $\partial\Sigma_W$. Assume the conditions (i)-(iv) as in [6] but replace the bounds on $\frac{1}{4\pi}\Delta W$ (which is the density in the Coulomb case) by the same bound on the density u (which is now slightly less explicit), and also modify the assumption (iv) by replacing ζ by $\zeta^\ell = U^\ell_{\mu_W} + \frac{1}{2}Q - c_Q$, where c_Q is as in* (A.10)*. Then, for any constant A, for any bounded $f \in C^2(\mathbb{C})$ with compactly supported $(\Delta - m^2)f$,*

$$\log\int \mathrm{e}^{-\beta H_{M,W}(\mathbf{z}) + \sum f(z_j)}\, m(\mathrm{d}\mathbf{z}) \leqslant -\beta M^2 I^\ell_W(\mu_Q) + M(f,\mu_W) + \tfrac{1}{8\pi\beta}(f, -(\Delta - m^2)f) + \mathrm{O}(M^{-A})\|\Delta f\|_\infty + \mathrm{O}(M\log M), \tag{A.57}$$

$$\log\int \mathrm{e}^{-\beta H_{M,W}(\mathbf{z})}\, m(\mathrm{d}\mathbf{z}) \geqslant -\beta M^2 I^\ell_W(\mu_Q) + \mathrm{O}(M\log M), \tag{A.58}$$

and consequently for any $\xi \geqslant 1 + 1/\beta$,

$$\Big|\sum_j f(z_j) - M\int f\,\mathrm{d}\mu^\ell_W\Big| = \mathrm{O}(\xi)\left(\sqrt{M\log M}(f, (-\Delta + m^2)f)^{1/2} + M^{-A}\|\Delta f\|_\infty\right), \tag{A.59}$$

with probability at least $1 - \mathrm{e}^{-\xi\beta M\log M}$, with the implicit constant depending only on the numbers A in the assumptions (i)-(iv).

Proof. The probability estimate is obtained as in [6] from the partition function bounds (A.57) and (A.58), which are analogous to [6, Lemmas 4.3 and 4.4] except that $\|\nabla f\|_2 = (f, (-\Delta)f)^{1/2}$ is replaced by $(f, (-\Delta + m^2)f)^{1/2}$. The lower bound can be proved exactly the same way; for the upper bound we may bound the energy slightly differently from below, as follows, avoiding the need that the support of $(\Delta - m^2)f$ is contained in S_V.

All the properties of the Coulomb potential used in the proof of [6, Lemmas 4.3] also hold for the Yukawa potential. Replacing the point charges by charged disks of radius ε, and denoting by $D^\ell(\cdot,\cdot)$ the Yukawa analog of $D(\cdot,\cdot)$, we get the bound

$$\begin{aligned} H_M^\ell(\mathbf{z}) - \frac{1}{\beta M}\sum_j f(z_j) &\geqslant M^2 D^\ell(\hat\mu^{(\varepsilon)},\hat\mu^{(\varepsilon)}) + M^2(W - \tfrac{1}{\beta M}f,\hat\mu) + \mathrm{O}(M\log\frac{1}{\varepsilon}) \\ &= M^2\Big(D^\ell(\hat\mu^{(\varepsilon)},\hat\mu^{(\varepsilon)}) + (W,\hat\mu) - (\tfrac{1}{\beta M}f,\hat\mu^{(\varepsilon)})\Big) + M^2(\tfrac{1}{\beta M}f,\hat\mu^{(\varepsilon)}-\hat\mu) + \mathrm{O}(M\log\frac{1}{\varepsilon}). \end{aligned}$$

Writing

$$D^\ell(\hat\mu^{(\varepsilon)},\hat\mu^{(\varepsilon)}) = D^\ell(\mu_W,\mu_W) + 2D^\ell(\mu_W,\hat\mu^{(\varepsilon)}-\mu_W) + D^\ell(\hat\mu^{(\varepsilon)}-\mu_W,\hat\mu^{(\varepsilon)}-\mu_W)$$

and further using the Euler–Lagrange equation (A.10) to write

$$2D^\ell(\mu_W,\hat\mu^{(\varepsilon)}-\mu_W) + (W,\hat\mu) = (W,\mu_W) + 2(\zeta^\ell,\hat\mu-\mu_W) + 2(U^\ell_{\mu_W},\hat\mu^{(\varepsilon)}-\hat\mu),$$

where $\zeta^\ell = U^\ell_{\mu_W} + \frac12 W - c_W = 0$ on S_W, we therefore can the bound $H_M^\ell(\mathbf{z}) - \frac{1}{\beta M}\sum_j f(z_j)$ by

$$\begin{aligned} M^2\Big(I_W^\ell(\mu_W) + D^\ell(\hat\mu^{(\varepsilon)}-\mu_W,\hat\mu^{(\varepsilon)}-\mu_W) - (\tfrac{1}{\beta M}f,\hat\mu^{(\varepsilon)})\Big) + 2M^2(\zeta^\ell,\hat\mu-\mu_W) \\ + M^2(\tfrac{1}{\beta M}f,\hat\mu^{(\varepsilon)}-\hat\mu) + 2M^2(U^\ell_{\mu_W},\hat\mu^{(\varepsilon)}-\hat\mu) + \mathrm{O}(M\log\varepsilon^{-1}). \end{aligned}$$

We write

$$D^\ell(\hat\mu^{(\varepsilon)}-\mu_W,\hat\mu^{(\varepsilon)}-\mu_W) - (\tfrac{1}{\beta M}f,\hat\mu^{(\varepsilon)}) = \tfrac{1}{2\pi}(\tfrac{1}{\beta M}f + U^\ell_{\hat\mu^{(\varepsilon)}-\mu_W}, -(\Delta-m^2)U^\ell_{\hat\mu^{(\varepsilon)}-\mu_W}) - (\tfrac{1}{\beta M}f,\mu_W).$$

The Yukawa potentials decay exponentially at infinity, so we may integrate by parts and use the elementary inequality $-|ab| + |b|^2 \geqslant -|a|^2/4$ to get

$$\begin{aligned} \tfrac{1}{2\pi}(\tfrac{1}{\beta M}f + U^\ell_{\hat\mu^{(\varepsilon)}-\mu_W},(-\Delta)U^\ell_{\hat\mu^{(\varepsilon)}-\mu_W}) &= \tfrac{1}{2\pi}(\tfrac{1}{\beta M}\nabla f + \nabla U^\ell_{\hat\mu^{(\varepsilon)}-\mu_W},\nabla U^\ell_{\hat\mu^{(\varepsilon)}-\mu_W}) \\ &\geqslant -\tfrac{1}{8\pi\beta^2M^2}(\nabla f,\nabla f) = -\tfrac{1}{8\pi\beta^2M^2}(f,(-\Delta)f). \end{aligned}$$

By the same inequality we have

$$\tfrac{1}{2\pi}(\tfrac{1}{\beta M}f + U^\ell_{\hat\mu^{(\varepsilon)}-\mu_W}, m^2U^\ell_{\hat\mu^{(\varepsilon)}-\mu_W}) \geqslant -\tfrac{m^2}{8\pi\beta^2M^2}(f,f).$$

In conclusion,

$$\begin{aligned} &M^2D^\ell(\hat\mu^{(\varepsilon)},\hat\mu^{(\varepsilon)}) + M^2(W-\tfrac{1}{\beta M}f,\hat\mu) \\ &\geqslant M^2\Big(I_W^\ell(\mu_W) - \tfrac{1}{\beta M}(f,\mu_W) - \tfrac{1}{8\pi\beta^2M^2}(f,-(\Delta-m^2)f)\Big) \\ &\quad + 2M^2(\zeta^\ell,\hat\mu-\mu_W) + M^2(\tfrac{1}{\beta M}f,\hat\mu^{(\varepsilon)}-\hat\mu) + 2M^2(U^\ell_{\mu_W},\hat\mu^{(\varepsilon)}-\hat\mu) + \mathrm{O}(M\log\frac{1}{\varepsilon}). \end{aligned}$$

In the same way as in [6], for the error terms on the last line,

$$\frac{M}{\beta}|(f^{(\varepsilon)}-f,\hat\mu)| \leqslant \frac{M}{\beta}C\varepsilon^2\|\Delta f\|_\infty \leqslant M^{-A}\|\Delta f\|_\infty$$

and

$$2M^2|(U_{\mu_W^\ell},\hat\mu^{(\varepsilon)}-\hat\mu)| \leqslant C\varepsilon^2M^{A_u} + C\sqrt{\varepsilon}M^{A_v} \leqslant 1,$$

by choosing ε sufficiently small depending on A and such that $\log\frac{1}{\varepsilon} = \mathrm{O}(\log M)$. Finally, we use that $2M^2(\zeta^\ell,\hat\mu-\mu_W) \geqslant 0$ by the Euler–Lagrange equation to conclude the proof. □

Remark A.15. *For test functions f supported in S_Q^ℓ and satisfying the condition $\int f\,\mathrm{d}m = 0$,*

$$\int f\,\mathrm{d}\mu_Q^\ell = \int f \frac{1}{4\pi}(\Delta Q - m^2 Q)\,\mathrm{d}m.$$

Consequently, if Q is replaced by $Q+R$ with $(\Delta - m^2)R = 0$, and assuming that f is supported in the intersection of the supports of the equilibrium measures of Q and $Q+R$, and that $\int f\,dm = 0$, we have

$$\int f\,\mathrm{d}\mu_Q^\ell = \int f\,\mathrm{d}\mu_{Q+R}^\ell.$$

Since we are ultimately interested in test functions without the condition $\int f\,\mathrm{d}m = 0$, some additional care is required. (The condition was not necessary in the Coulomb case in [6].) This problem will be addressed at the beginning of the proof of Proposition A.16.

A.6. Yukawa gas on the plane: proof of Proposition A.1. To improve the estimate of Proposition A.14 to that asserted by Theorem A.1, we now follow the strategy the proof of Theorem [6, Theorem 1.1], using local conditioning.

First, we note that [6, Section 5] applies without changes except that the Coulomb potential $\log 1/|z|$ is replaced by the Yukawa potential $Y^\ell(z)$ in all expressions, and with the additional condition that $\int f\,\mathrm{d}m = 0$ in the assumption of [6, Proposition 5.3]. This condition is necessary because, with the m-harmonic perturbation V_o, inside the support of μ_W we now have

$$\mu_W = \frac{N}{M}\mu_Q + \text{const}.$$

by (A.11). As explained in Remark A.15, the additional constant has no effect if both sides are integrated against a test function f with support in the support of μ_W that satisfies $\int f\,\mathrm{d}m = 0$.

Next, we adapt [6, Section 6] to the Yukawa case. Here two modifications are required. First, the scaling of the Yukawa gas is different, which leads to a different recursion of scales. Second, in the case of the Yukawa gas, as noted above, the density of the equilibrium is only stable under m-harmonic perturbations up to a constant, and thus an small extra argument is required to remove the mean zero condition.

As previously, we write $\ell = N^{-1/2+\delta}$ for the range of the Yukawa potential. Given $\varepsilon > 0$ (and assuming $\varepsilon < \delta$), we set $s_0 = 0$ and

$$s_{j+1} = \left(\left(\frac{1}{4} + \frac{s_j}{2}\right) \wedge (s_j + \delta)\right) - \varepsilon,$$

for $\varepsilon > 0$ fixed sufficiently small. As long as the second term in the minimum dominates, the sequence s_j grows linearly as $j(\delta - \varepsilon)$ until the scale $s = \frac{1}{2} - 2\delta$ is reached. After that, the first term dominates. Then s_j evolves according to $\frac{1}{2} - \delta - \varepsilon$; then $\frac{1}{2} - \frac{1}{2}\delta$; then $\frac{1}{2} - \frac{1}{4}\delta - \frac{3}{2}\varepsilon$ and converges geometrically to $\frac{1}{2} - 2\varepsilon$. In particular, given $s \in (0, \frac{1}{2})$ as in Theorem A.1, we can fix $\varepsilon > 0$ and $n < \infty$ such that $s_n = s$, and we will assume such a choice from now on. We also recall from [6] that by replacing $Q(z)$ by $Q(z - z_0)$ we consider functions supported in the balls $B_s^o = B(0, \frac{1}{2}N^{-s}) \subset B_s = B(0, N^{-s})$. The induction assumption (A_r) is modified as follows (as a formal remark, note that compared to [6], we changed the index of the condition A_t into A_r as, in the current paper, t refers to the argument of the Laplace transform).

Assumption (A_r). For any bounded $f \in C^2(\mathbb{C})$ with $\operatorname{supp}(\Delta - m^2)f \subset B_r^\circ \cap S_Q$, we have

$$\left|\frac{1}{N}\sum_j f(z_j) - \int f\,\mathrm{d}\mu_Q\right| \prec N^{-\frac{1}{2}-r}(f, (-\Delta + m^2)f)^{\frac{1}{2}} + N^{-1-2r}\|\Delta f\|_\infty \tag{A.60}$$

Proposition A.16. *For arbitrary $\varepsilon > 0$, (A_r) implies (A_s) for any $0 \leqslant r \leqslant s \leqslant (\frac{1}{4}+\frac{1}{2}r)\wedge(r+\delta)-\varepsilon$ (with the implicit constants depending on ε).*

Proof. First, we show that, for any s as asserted in the proposition, it suffices to prove that (A_r) implies (A'_s), where (A'_s) is defined exactly as (A_s) except that the test functions f are required to obey the additional mean zero condition $\int f \,\mathrm{d}m = 0$. Indeed, assume (A_r) and that we have proved (A'_s) for all s as in the statement of the proposition. As in [6, Section 6], let $B = B_s$ be a disk of radius N^{-s} and B_s° the disk with the same center and half the radius. For any test function f supported on B_s° we define $f_i(z) = 2^{-2i} f(2^{-i}z)$, and write

$$f = f_k + \sum_{i=0}^{k-1}(f_i - f_{i+1}),$$

where k is the largest integer such that $2^k N^{-s} \leqslant N^{-t}$. Then

$$\|\Delta f_i\|_\infty = 2^{-4i}\|\Delta f\|_\infty, \quad (f_i, (-\Delta + m^2)f_i) \leqslant 2^{-2i}(f, (-\Delta + m^2)f). \tag{A.61}$$

Therefore, with $s_i = s - (i+1)/\log_2 N$ for $i = 0, 1, \dots, k-1$, applying (A'_{s_i}) to the mean zero function $f_i - f_{i+1}$, we obtain

$$\begin{aligned}
&\frac{1}{N}\sum_j (f_i(z_j) - f_{i+1}(z_j)) - \int (f_i - f_{i+1})\,\mathrm{d}\mu_Q \\
&\prec 2^{2i} N^{-1-2s}\|\Delta(f_i - f_{i+1})\|_\infty + 2^i N^{-\frac{1}{2}-s}(f_i - f_{i+1}, (-\Delta + m^2)(f_i - f_{i+1}))^{1/2} \\
&\prec 2^{-2i} N^{-1-2s}\|\Delta f\|_\infty + N^{-\frac{1}{2}-s}(f, (-\Delta + m^2)f)^{1/2}.
\end{aligned}$$

Similarly, applying (A_r) to f_k, we have

$$\begin{aligned}
\frac{1}{N}\sum_j f_k(z_j) - \int f_k \,\mathrm{d}\mu_Q &\prec \big(N^{-1-2r}\|\Delta f_k\|_\infty + N^{-\frac{1}{2}-r}(f_k, (-\Delta + m^2)f_k)^{1/2}\big) \\
&\prec \big(2^{-4k} N^{-1-2r}\|\Delta f\|_\infty + 2^{-k} N^{-\frac{1}{2}-r}(f, (-\Delta + m^2)f)^{1/2}\big) \\
&\prec \big(2^{-4k} N^{-1-2s}\|\Delta f\|_\infty + 2^{-k} N^{-\frac{1}{2}-s}(f, (-\Delta + m^2)f)^{1/2}\big).
\end{aligned}$$

Then

$$\begin{aligned}
\frac{1}{N}\sum_j f(z_j) - \int f &= \frac{1}{N}\sum_j \Big(f_k(z_j) + \sum_{i=0}^{k-1}(f_i(z_j) - f_{i+1}(z_j))\Big) - \int \Big(f_k + \sum_{i=0}^{k-1}(f_i - f_{i+1})\Big) \\
&\prec \sum_{i=0}^{k}\big(2^{-2i} N^{-1-2s}\|\Delta f\|_\infty + N^{-\frac{1}{2}-s}(f, (-\Delta + m^2)f)^{1/2}\big) \\
&\prec N^{-1-2s}\|\Delta f\|_\infty + N^{-\frac{1}{2}-s}(f, (-\Delta + m^2)f)^{1/2}.
\end{aligned}$$

It remains to prove that (A_r) implies (A'_s) for s as in the statement of the proposition. This proof proceeds exactly as in [6, Section 6.1], with the only essential changes in [6, Lemmas 6.2–6.3], due to $m^2 > 0$ in (A.60). Indeed, the required properties of the conditional equilibrium measure follow from Propositions A.12–A.13, as soon as [6, Lemmas 6.2–6.3] are adapted.

In [6, Lemma 6.2], which states that $\tau = 1 + \mathrm{O}(N^{-c\varepsilon})$ (where we recall that $\tau = \frac{N}{M}\mu_Q(B)$) and $\nu(\mathbb{C}) = \mathrm{O}(N^{-c\varepsilon})$, with high probability, the following changes are necessary. Recall that $\chi_\pm$ are smooth cutoff functions with

$$\chi_+|_B = 1, \quad \chi_+|_{B_+^c} = 0, \qquad \chi_-|_{B^c} = 0, \quad \chi_-|_{B_-} = 1,$$

obeying $\|\nabla^k \chi_\pm\|_\infty = \mathrm{O}(N^{ks}/\eta^k)$ for $k = 0, 1, 2$ (see [6] for the definitions of the expressions). We replace the estimates on $(\chi_\pm, -\Delta\chi_\pm)$ by

$$(\chi_\pm, (-\Delta + m^2)\chi_\pm) = \mathrm{O}(\eta N^{-2s} N^{2s}/\eta^2) + \mathrm{O}(N^{1-2\delta} N^{-2s}) = \mathrm{O}(1/\eta) + \mathrm{O}(N^{1-2\delta-2s}),$$

and thus

$$N^{-1-2r}(\chi_\pm, (-\Delta + m^2)\chi_\pm) = \mathrm{O}(N^{-4s-4\varepsilon}/\eta) + \mathrm{O}(N^{-4s-2\varepsilon}) = \mathrm{O}(N^{-4s-c\varepsilon}).$$

Using this, the rest of the proof of [6, Lemma 6.2] proceeds as in [6].

In [6, Lemma 6.3], which states the estimate $N^{-s}\|\nabla \hat{R}\|_{L^\infty(B)} = \mathrm{O}(N^{-c\varepsilon})$, with high probability, we make the following changes. We change the definition of f from $f(w) = N^{-s}\nabla(\psi(w)\log\frac{1}{|z-w|})$ to $f(w) = N^{-s}\nabla(\psi(w) Y^\ell(z-w))$. In particular, the property that $\Delta f = 0$ on A^c is replaced by $(\Delta - m^2) f = 0$ on A^c, and using this, the estimate on $(f, -\Delta f)$ is replaced by (here again we use a notation from [6], namely $a = N^{-c\varepsilon}$),

$$\begin{aligned} &N^{-1-2r}(f, (-\Delta + m^2) f) \\ &= N^{-1-2r}\mathrm{O}(N^{-2s}N^{2s}|\log a|/a^2) + N^{-1-2r}\mathrm{O}(N^{1-2\delta}N^{2s}|\log a|^2) = \mathrm{O}(N^{-4s-c\varepsilon}), \end{aligned}$$

so that, again, the rest of the proof of [6, Lemma 6.3] proceeds as in [6]. □

Proof of Theorem A.1. Proposition A.14 with $W = Q$ and $M = N$ verifies Assumption (A_0). By inductive application of Proposition A.16, the assumption (A_s) is verified for all $s \in (0, \frac{1}{2})$. This completes the proof. □

A.7. Yukawa gas on the torus: proof of Theorem A.2. The proof of Theorem A.2 is a straightforward adaption of that of Theorem A.1; we only mention the required changes here. As in the proof of Theorem A.2, we condition on the particles outside a ball. As in the full plane case, the equilibrium measure given by replacing the charges outside by the equilibrium measure is the restricted measure. Then after conditioning, using that $\ell \leqslant N^{-c}$, we may replace the torus Yukawa potential by the full plane Yukawa potential. Indeed, we have

$$H_{N,0}(z) = \sum_{j \neq k} U^\ell(z_j - z_k) + \mathrm{O}(N^2 e^{-cN^\varepsilon}) = \sum_{j \neq k} Y^\ell(z_j - z_k) + \mathrm{O}(N^{-2}),$$

with error bound uniform in $z \in \mathbb{T}^N$. From this point on, the remaining argument is identical to the proof of Theorem A.1.

A.8. Coulomb gas with angle term: proof of Theorem A.3. As in the proof of the local density estimate for the Yukawa gas, Theorem A.1, we follow the strategy originally used in [6]. However, we will see that due to the assumption $|t|N\theta^2 \leqslant 1$ the modifications needed are much smaller than those needed for the Yukawa gas. In particular, the required equilibrium functionals and potential theory are those of the unperturbed Coulomb gas used in [6].

As a preliminary step towards Theorem A.3, we prove the following estimate, which provides a weaker fluctuation bound than Theorem A.3 does. However, using this bound for all scales, Theorem A.3 then follows from the same estimates.

Proposition A.17. *Assume the same conditions as in Theorem A.3. Let $t = N^{-2a}$ and suppose that* $\operatorname{supp} f$ *has diameter at most N^{-s}. Then*

$$\frac{X_f}{N} \prec (N^{-\frac{1}{2}-s} + N^{-a-2s})\|\nabla f\|_2 + (N^{-1-2s} + N^{-2a-4s})\|\Delta f\|_\infty.$$

To prove this bound, we proceed as in the proof of [6, Theorem 1.1]. The first ingredient is the following generalization of the one-step estimate [6, Proposition 4.1].

Proposition A.18. *Assume that the potential W and the number of particles M satisfy the assumptions of [6, Proposition 4.1]. Consider the probability measure on Σ_W^M with density proportional to $\mathrm{e}^{-\beta H(\mathbf{z})}$ where the Hamiltonian $H : \Sigma_W^M \to \mathbb{R}$ satisfies the uniform estimate*

$$|H(\mathbf{z}) - H^{\mathcal{C}}_{M,V}(\mathbf{z})| \leqslant tMK. \tag{A.62}$$

Then for any bounded $f \in C^2(\mathbb{C})$ with $\operatorname{supp} \Delta f$ *compact,*

$$\sum_j f(z_j) - M \int f \, \mathrm{d}\mu_W = \mathrm{O}(\xi) \left((tMK + M \log M)^{1/2} \|\nabla f\|_2 + M^{-A} \|\Delta f\|_\infty \right) \tag{A.63}$$

with probability at least $1 - \mathrm{e}^{-\xi\beta(tMK + M\log M)}$ for any $\xi \geqslant 1 + 1/\beta$.

Proof of Proposition A.18. By the assumption (A.62), we may trivially estimate

$$\log \int \mathrm{e}^{-\beta H^{\mathcal{C}}_{M,W}} \, m^{\otimes M}(\mathrm{d}\mathbf{z}) - tMK \leqslant \log \int \mathrm{e}^{-\beta H} \, m^{\otimes M}(\mathrm{d}\mathbf{z}) \leqslant \log \int \mathrm{e}^{-\beta H^{\mathcal{C}}_{M,W}} \, m^{\otimes M}(\mathrm{d}\mathbf{z}) + tMK.$$

By [6, Lemmas 4.3 and 4.4], the partition function of the Coulomb Hamiltonian (without angle perturbation) can be estimated as

$$\frac{1}{\beta} \log \int \mathrm{e}^{-\beta H^{\mathcal{C}}_W} \, m^{\otimes M}(\mathrm{d}\mathbf{z}) \geqslant M^2 I_W + \mathrm{O}(M \log M),$$
$$\frac{1}{\beta} \log \int \mathrm{e}^{-\beta H^{\mathcal{C}}_{W+f}} \, m^{\otimes M}(\mathrm{d}\mathbf{z}) \leqslant M^2 I_W + \frac{1}{8\pi\beta^2}(f, -\Delta f) + \mathrm{O}(M^{-A})\|\Delta f\|_\infty + \mathrm{O}(M \log M).$$

Here we used the improvement commented on in the proof of Proposition A.14, which gives the improved factor for the error term proportional to $\|\Delta f\|_\infty$ and avoids the restriction on $\|\Delta f\|_\infty$. From this, we obtain the estimate

$$\frac{1}{\beta} \log \mathbb{E}^{G_t}_{V_t} \mathrm{e}^{X_f/s} = \frac{1}{8\pi s^2 \beta^2}(f, -\Delta f) + \mathrm{O}(M^{-A}) \frac{1}{s} \|\Delta f\|_\infty + \mathrm{O}(E),$$

with $E = tMK + M \log M$. As in the proof of [6, Lemmas 4.1], choosing

$$s = E^{-\frac{1}{2}} \|\nabla f\|_2 + M^{-A} E^{-1} \|\Delta f\|_\infty,$$

this implies

$$\frac{1}{\beta} \log \mathbb{E}^{G_t}_{V_t} \mathrm{e}^{X_f/s} = \mathrm{O}(E).$$

By Markov's inequality, $\mathbb{P}(X_f \geqslant \mathrm{O}(sE)) \leqslant \mathrm{e}^{-E}$, and since the same estimate also holds with f replaced by $-f$, we have

$$\mathbb{P}\Big(X_f = \mathrm{O}\left(E^{1/2} \|\nabla f\|_2 + M^{-A} \|\Delta f\|_\infty \right) \Big) \geqslant 1 - 2\mathrm{e}^{-E},$$

which implies the claim (A.63). □

As in the proof of [6, Theorem 1.1], the proof of Proposition A.17 follows from iterated applications of Proposition A.18 to the conditioned measures associated to increasingly small balls. This induction proceeds almost exactly as in [6, Section 6], with the additional element that, in each step, we improve also the bound K for the conditioned measure. We first give an outline of this induction now. We write $t = N^{-2a}$.

First step. In the first step, using that $\|\nabla h\|_\infty \leqslant 1$, the difference $H - H^{\mathcal{C}}_{V,N}$ is bounded uniformly by

$$t\left|\sum_{j\neq k} \mathrm{e}^{-\frac{|z_j-z_k|^2}{2\theta^2}} \frac{h(z_j)-h(z_k)}{z_j-z_k}\right| \leqslant tMK, \tag{A.64}$$

with $M = N$ and the trivial bound $K = N$. From Proposition A.18, we therefore get the high probability estimate

$$\frac{X_f}{N} \prec N^{-A-1}\|\Delta f\|_\infty + (N^{-a} + N^{-\frac{1}{2}})\|\nabla f\|_2.$$

This estimate proves an effective estimate on the number of particles on scales N^{-s} for $s < a/2$.

Induction. By induction, supposing we can control particle numbers on the distance scale N^{-r}, in Proposition A.18 applied to the conditional measure in a ball of the former scale, we have $M \approx N^{1-2r}$ and $\alpha = N/M \approx N^{2r}$. Furthermore, (A.62) holds (see Lemma A.20 below) with

$$K = \mathrm{O}(M \vee N^{2\sigma}).$$

Using this estimate, by conditioning exactly as in the proof of [6, Theorem 1.1], for any f whose support has diameter at most N^{-r}, we obtain from Proposition A.18 the estimate

$$\begin{aligned}\frac{X_f}{N} &\prec \big(t\frac{M\vee N^{2\sigma}}{\alpha N} + \alpha^{-1}N^{-1}\big)\|\Delta f\|_\infty + \big(t^{\frac{1}{2}}\frac{M\vee (MN^{2\sigma})^{\frac{1}{2}}}{N} + M^{\frac{1}{2}}N^{-1}\big)\|\nabla f\|_2\\ &\prec \big(N^{-2a-4r} + N^{-1-2r}\big)\|\Delta f\|_\infty + \big(N^{-a-2r} + N^{-\frac{1}{2}-r}\big)\|\nabla f\|_2.\end{aligned}$$

This is an effective estimate on particle numbers on scales N^{-s} for $s < (r + \frac{a}{2}) \vee (\frac{1}{4} + \frac{r}{2}) + \varepsilon$, improving the assumed estimate. We remark that, as far as the scales are concerned, this is the same recursion as in the case of the Yukawa gas, with δ replaced by $a/2$.

To set up the induction formally, we replace the assumption (A_r) of [6] by the following one. (Note also that as before we changed the index t from condition A_t from [6] into A_r as, in the current paper, t refers to the argument of the Laplace transform).

Assumption ($\mathbf{A}_r$). For any bounded $f \in C^2(\mathbb{C})$ with $\operatorname{supp}\Delta f \subset B_r^\circ \cap S_V$, we have

$$\frac{X_f}{N} \prec (N^{-1-2r} + N^{-2a-4r})\|\Delta f\|_\infty + (N^{-\frac{1}{2}-r} + N^{-a-2r})\|\nabla f\|_2. \tag{A.65}$$

As shown above, for $r = 0$ this is (A.63) applied with $M = N$ and $V = W$ and the trivial estimate $K = N$. To prove Proposition A.17, it is enough to prove the next proposition.

Proposition A.19. *For arbitrary $\varepsilon > 0$, (A_r) implies (A_s) for any*

$$0 \leqslant r < s \leqslant \big(\frac{1}{4} + \frac{r}{2}\big) \wedge \big(\frac{a}{2} + r\big) - \varepsilon, \tag{A.66}$$

with the implicit constant in (A.65) *depending only on ε.*

To prove Proposition A.19, exactly as in [6, Sections 5-6], we condition on the outside of a ball B_s on scale s and replace the Coulomb potential of the outside charges with the Coulomb potential of the equilibrium measure. To ensure that the equilibrium measure of the conditional system inside B_s does not move much under this replacement, we use [6, Propositions 3.3 and 3.4] and the analogues of [6, Lemmas 6.2 and 6.3], where the input assumption is replaced by our new assumption (A_r); the lemmas are checked exactly as in the case of the Yukawa gas. The additional required estimate is the bound K on (A.62), which is given by the following lemma.

Lemma A.20. *Assume (A_r). Then, with high probability, uniformly for all configurations of the M charges inside B_s, the estimate* (A.62) *holds with*

$$K = \mathrm{O}(N^{1-2r} \vee N\theta^2).$$

In particular, if B is at scale N^{-r} then the right-hand side is $\mathrm{O}(M \vee N^{2\sigma})$.

Proof. We split the interaction term

$$\sum_{j \neq k} \mathrm{e}^{-\frac{|z_j - z_k|^2}{2\theta^2}} \frac{h(z_j) - h(z_k)}{z_j - z_k}$$

into the three contributions: (1) both particles are inside B, (2) one particle is in B and one outside B, and (3) both particles are outside B. The contribution (3) with both particles outside B is a constant for the conditioned measure and thus irrelevant for the estimate on the conditioned measure. Contribution (1) is trivially estimated by $M^2 \|\nabla h\|_\infty \leqslant M^2$. Contribution (2) is bounded by $\mathrm{O}(M(N\theta^2 + N^{1-2r}))$ by the local density estimate, with r-HP for the configurations outside B. This gives the claimed estimate. □

Proof of Theorem A.3. As in the proof of Proposition A.19, we condition on the particles outside a ball B of radius N^{-s} and assume that f is supported in the ball with the same center and half of the radius. However, since ($\mathrm{A}_{1/2-\sigma}$) has already been proved, by Lemma A.20, we now have the optimal estimate $K = \mathrm{O}(N^{2\sigma})$. The theorem then follows directly from the one-step bound (A.63) on any scale b as in the assumption of the theorem using this bound on K, implying that $tK = t\mathrm{O}(N^{2\sigma})\|\nabla h\| = \mathrm{O}(1)$. □

B Rigidity for Yukawa gas on the torus and perturbed Coulomb gas

B.1. Rigidity for Yukawa gas on the torus. Next we derive the following rigidity estimate for linear statistics of the Yukawa gas on the torus without external potential (the case with potential is analogous but not needed for our application). We will denote the expectation w.r.t. to this measure by $\mathbb{E}_0^{U^\ell}$ and the equilibrium measure by μ_0. In the proof of the theorem, we actually need to consider an external potential Q, for which we will denote the expectation by $\mathbb{E}_Q^{U^\ell}$ and the equilibrium measure by μ_Q, and we assume that $Q \in C^2$. The proof of the following theorem is parallel to that of [6, Theorem 1.2].

Theorem B.1 (Rigidity Estimate)**.** *Consider the Yukawa gas of range $1 \gg \ell \gg N^{-1/2}$ on the unit torus without external potential. Let $s \in (0, \frac{1}{2})$ and $\varepsilon > 0$. For any $f : \mathbb{T} \to \mathbb{R}$ supported in a ball of radius $b = N^{-s}$ with high probability*

$$X_f = \sum_j f(z_j) - N \int f \,\mathrm{d}\mu_0 \prec \left(\frac{b}{\ell} + 1\right)^2 \|f\|_{3,b}. \tag{B.1}$$

Remark B.2. *The previous estimate also holds for the Coulomb gas (i.e., $\ell = \infty$) with a similar proof provided that the external potential satisfying the assumptions (1.9, 1.4). This improves the error bound from depending on four derivatives of f in* (1.8) *and* (4.38) *to three derivatives. However, we will not need this result in this paper.*

The rigidity estimate also implies the following useful proposition for functions of two points. For any function G on $\mathbb{T} \times \mathbb{T}$ we denote

$$|\nabla^j G(x,y)| = \sup_{\sum_{i=1}^4 k_i = j, a_i \in \{x,\bar{x},y,\bar{y}\}} \left| \prod_{i=1}^4 (\partial_{a_i})^{k_i} G(x,y) \right|, \tag{B.2}$$

and

$$G_s^{(j)}(z,w) = \sup_{(x,y)\in \mathrm{B}(z,s)\times \mathrm{B}(w,s)} |\nabla^j G(x,y)|. \tag{B.3}$$

Proposition B.3. *Consider the Yukawa gas of range $N^{-1/2} \ll \ell \ll 1$ on the unit torus. Let G be smooth on $\mathbb{T}\times\mathbb{T}$ and $G_s^{(j)}$ as defined in Lemma B.6, for some additional scale $N^{-1/2} \ll s \ll 1$. Then for any fixed $p \in \mathbb{N}$ and $\varepsilon > 0$, the following bound holds with high probability:*

$$N^2 \iint G(z,w)\, \tilde{\mu}(\mathrm{d}z)\, \tilde{\mu}(\mathrm{d}w)$$
$$= \mathrm{O}(N^\varepsilon)\Big(\big(\frac{1}{s^4} + \frac{1}{\ell^4}\big) \sum_{j=0}^{p-1} s^j \| |\nabla^j G| \|_{\mathrm{L}^1(\mathbb{T}\times\mathbb{T})} + N^2 s^p \|G_s^{(p)}\|_{\mathrm{L}^1(\mathbb{T}\times\mathbb{T})}\Big). \tag{B.4}$$

The following application of Proposition B.3 is useful. Suppose that $G(z,w)$ is a function smooth at the scale R and supported in $|z-w| \leq \mathrm{O}(R)$. Then we choose $s = RN^{-\varepsilon}$ so that the error term is of order $N^{-\varepsilon p}$. Choosing p large enough and noting that

$$\| |\nabla^j G| \|_{\mathrm{L}^1(\mathbb{T}\times\mathbb{T})} \leq C_j R^{-j} R^2 \|G\|_{R,i}$$

we obtain that for any $A > 0$ there is a $p > 0$ such that

$$N^2 \iint G(z,w)\, \tilde{\mu}(\mathrm{d}z)\, \tilde{\mu}(\mathrm{d}w) = \mathrm{O}(N^\varepsilon)\big(\frac{1}{R^2} + \frac{R^2}{\ell^4}\big) \|G\|_{R,p} + N^{-A} \|G\|_{R,p}. \tag{B.5}$$

B.2. Proof of Theorem B.1 and Proposition B.3. The proof uses the same ideas as the proof of [6, Theorem 1.2]. As a first step, we state the loop equation for the Yukawa gas on the torus. As in (4.6), given a function $v : \mathbb{T} \to \mathbb{R}$, set

$$W_Q^v(\mathbf{z}) = -\sum_{j\neq k} (v(z_j) - v(z_k)) \partial U^\ell (z_j - z_k) + \frac{1}{\beta} \sum_j \partial_j v(z_j) - N \sum_j v(z_j) \partial Q(z_j) \tag{B.6}$$

and recall that by Lemma 4.3 we have $\mathbb{E}_Q^{U^\ell}(W_Q^v) = 0$, because the Yukawa interaction Y^ℓ (and therefore U^ℓ) are functions of $z_j - z_k$. Given $q : \mathbb{C} \to \mathbb{R}$ supported in S_Q, further abbreviate

$$h(z) = \frac{1}{\pi} \frac{\bar{\partial} q(z)}{\rho_Q(z)}, \quad g(z) = \frac{1}{\pi} \frac{q(z)}{\rho_Q(z)}, \tag{B.7}$$

where ρ_Q denotes the density of μ_Q from (A.11), with respect to the Lebesgue measure.

Lemma B.4. *For any $q:\mathbb{T}\to\mathbb{R}$ of class $\mathscr{C}^2$, supported on S_Q, and $\mathbf{z}\in\mathbb{T}^N$, we have*

$$X_Q^q = -\frac{1}{N}W_Q^h(\mathbf{z}) + \frac{1}{N\beta}\sum_k \partial h(z_k) + N\iint_{z\neq w}(h(z)-h(w))\partial U^\ell(z-w)\tilde{\mu}_Q(\mathrm{d}z)\tilde{\mu}_Q(\mathrm{d}w)$$
$$+\frac{Nm^2}{2}\iint_{z\neq w} g(w)U^\ell(z-w)\tilde{\mu}_Q(\mathrm{d}z)\mu_Q(\mathrm{d}w), \tag{B.8}$$

where $m=\ell^{-1}$. Thus, for any smooth enough $f:\mathbb{T}\to\mathbb{R}$ with $\int_{\mathbb{T}} f\mathrm{d}m = 0$ that satisfies the condition that $q = f - m^2\Delta^{-1}f$ has support in S_Q, where Δ is the Laplacian on the torus,

$$X_Q^f = -\frac{1}{N}W_Q^h(\mathbf{z}) + \frac{1}{N\beta}\sum_k \partial h(z_k) + N\iint_{z\neq w}(h(z)-h(w))\partial U^\ell(z-w)\tilde{\mu}_Q(\mathrm{d}z)\tilde{\mu}_Q(\mathrm{d}w). \tag{B.9}$$

The last identity holds without the assumption that $\int_{\mathbb{T}} f\,\mathrm{d}m = 0$ if q in (B.7) *is defined in terms of the mean 0 part of f.*

Proof. As in the proof of Lemma 4.4, we have

$$2\int \partial U^\ell(z-w)\,\mu_Q(\mathrm{d}w) = \partial Q(z), \tag{B.10}$$
$$q(z) = \frac{1}{2\pi}\int(-4\partial\bar{\partial}q(w)+m^2q(w))U^\ell(z-w)m(\mathrm{d}w)$$
$$= \frac{1}{2\pi}\int(4\bar{\partial}q(w)\partial U^\ell(z-w)+m^2q(w)U^\ell(z-w))m(\mathrm{d}w), \tag{B.11}$$

where again the first equation holds for $z\in S_Q$ by the Euler-Lagrange equation, the second equation holds by the definition of the Yukawa potential as the Green's function of $-\Delta+m^2$ and integration by parts – the boundary term vanishes by periodicity. We therefore have

$$\begin{aligned}
X_Q^q &= 2\sum_j\int h(w)\partial U^\ell(z_j-w)\mu_Q(\mathrm{d}w) + \frac{m^2}{2}\sum_j\int g(w)U^\ell(z_j-w)\mu_Q(\mathrm{d}w)\\
&\quad - 2N\iint h(w)\partial U^\ell(z-w)\mu_Q(\mathrm{d}w)\mu_Q(\mathrm{d}z) - \frac{Nm^2}{2}\iint g(w)U^\ell(z-w)\mu_Q(\mathrm{d}w)\mu_Q(\mathrm{d}z)\\
&= 2N\iint(h(w)-h(z))\partial U^\ell(z-w)\hat{\mu}(\mathrm{d}z)\mu_Q(\mathrm{d}w) + \sum_j h(z_j)\partial Q(z_j)\\
&\quad - N\iint(h(w)-h(z))\partial U^\ell(z-w)\mu_Q(\mathrm{d}w)\mu_Q(\mathrm{d}z) + \frac{Nm^2}{2}\iint g(w)U^\ell(z-w)\tilde{\mu}_Q(\mathrm{d}z)\mu_Q(\mathrm{d}w).
\end{aligned}$$

In the first equation we used (B.7) and (B.11), and in the second equation we used (B.10). Since the integrands in the double integrals are symmetric, we arrive at

$$\begin{aligned}
X_Q^q = -\frac{1}{N}\sum_{j\neq k}(h(z_j)-h(z_k))\partial U^\ell(z_j-z_k) + N\iint_{z\neq w}(h(z)-h(w))\partial U^\ell(z-w)\tilde{\mu}_Q(\mathrm{d}z)\tilde{\mu}_Q(\mathrm{d}w)\\
+\sum_j h(z_j)\partial Q(z_j) + \frac{Nm^2}{2}\iint_{z\neq w} g(w)U^\ell(z-w)\tilde{\mu}_Q(\mathrm{d}z)\mu_Q(\mathrm{d}w),
\end{aligned}$$

which is equivalent to (B.8).

For the consequence, note that moving the last term on the right-hand side to the left-hand side, the left-hand side becomes X^f with

$$f(z) = q(z) - \frac{m^2}{2\pi}\int q(w)U^\ell(z-w)\,m(\mathrm{d}w) = ((1-K)q)(z),$$

where

$$1 - K = 1 - (1-\ell^2\Delta)^{-1} = \frac{\ell^2\Delta}{\ell^2\Delta - 1}, \quad (1-K)^{-1} = 1 - m^2\Delta^{-1}.$$

Therefore, given f as in the assumption, we can choose $q = f - m^2\Delta^{-1}f$.

Finally, since $\int \mu_Q = 1$, we have $X_Q^f = X_Q^{f-c}$ for any constant c. Hence the assumption $\int_{\mathbb{T}} f \mathrm{d}m = 0$ is trivial to remove. □

Lemma B.5. *For any $A > 0$, there is a constant C such that for any smooth $f : \mathbb{T} \to \mathbb{R}$ supported in a ball of radius b with $b \geq N^{-1/2}$, there exists f_s support in a ball of radius $b_s := b + Cs\log N$ for $0 \leq s \leq \log N$ such that $h(z) = \frac{1}{\pi\rho_Q(z)}\bar\partial(1-m^2\Delta^{-1})f(z)$ can be written as*

$$h(z) = \frac{1}{\pi\rho_Q(z)}\left(\bar\partial f(z) - m^2\int_0^{\log N}\frac{\mathrm{d}s}{s}\bar\partial f_s(z)\right) + \mathrm{O}(N^{-A}\|f\|_{1,b}), \tag{B.12}$$

$$\|f_s\|_{k,b_s} \leq C(b\wedge s)^2 N^\varepsilon \|f\|_{k,b}. \tag{B.13}$$

It is useful to keep in mind that if f is dimensionless then f_s has linear dimension 2.

Proof. We write

$$\bar\partial\Delta^{-1}f(z) = \int_0^\infty \mathrm{d}t\,\mathrm{e}^{t\Delta}\bar\partial f(z) = \int_0^M \mathrm{d}t\,\mathrm{e}^{t\Delta}\bar\partial f(z) + \int_M^\infty \mathrm{d}t\,\mathrm{e}^{-t\Delta}\bar\partial f(z). \tag{B.14}$$

Since Δ has a spectral gap of order one w.r.t mean zero function and $\bar\partial f$ is mean zero for any f with compact support, there is $c > 0$ such that

$$\int_M^\infty \mathrm{d}t\,\mathrm{e}^{t\Delta}\,\bar\partial f(z) \leq \int_M^\infty \mathrm{d}t\,\mathrm{e}^{-ct}\|\bar\partial f(z)\|_2 \leq \|f\|_{1,b}\mathrm{e}^{-cM}.$$

We choose $M = (\log N)^2$ so that this term is an error term of the form $N^{-A}\|f\|_{1,b}$ for any $A > 0$.

The heat kernel on the unit circle is given by

$$g_t(x) = 2\pi\sum_{k\in\mathbb{Z}} k_t(x + 2\pi k) = \sqrt{\frac{\pi}{t}}\mathrm{e}^{-x^2/4t}\Big[1 + 2\sum_{k\geq 1} e^{-\pi^2k^2/t}\cosh(\pi kx/t)\Big]$$

where $k_t(x) = (4\pi t)^{-1/2}\mathrm{e}^{-x^2/4t}$. The heat kernel on the two dimensional unit torus is given by $G_t(z) := g_t(x)g_t(y)$. Now change variables $s^2 = t$. Clearly, the function G_{s^2} decays exponentially at scale s. Rewrite $G_{s^2} = G_s^1 + G_s^2$ with $G_s^1(z) = G_{s^2}(z)\eta_{sC\log N}(z)$ where C is a large constant and $\eta_a(z)$ is a mollifier in a ball of radius a with $\eta_a(z) = 1$ if $|z| \leq a/2$. Define

$$f_s(z) = s^2\int G_s^1(z-w)f(w)\,m(\mathrm{d}w).$$

Clearly, f_s is supported in a ball of radius b_s and satisfies the bound (B.13). The error term involving G_s^2 can be trivially bounded and the constant A can be arbitrary large by choosing C large depending on A. □

The following simple estimate based on Taylor expansion and the boundedness of local density will be useful later on. Recall the definitions (B.2) and (B.3).

Lemma B.6. *Consider the Yukawa gas on the unit torus with range $N^{-1/2+\sigma} \leqslant \ell \leqslant 1$. Fix a scale s with $N^{-1/2+\sigma} \leqslant s \leqslant N^{-\sigma}$. Then for any fixed $p \in \mathbb{N}$ and $\varepsilon > 0$, the following bound holds with high probability:*

$$\iint G(z,w)\,\tilde{\mu}(\mathrm{d}z)\,\tilde{\mu}(\mathrm{d}w) = \sum_{j=1}^{p} \sum_{\sum k_i = j, a_i \in \{x,\bar{x},y,\bar{y}\}} \iint F^{(j,\mathbf{k})}(x,y) m(\mathrm{d}x) m(\mathrm{d}y) \tag{B.15}$$

$$\times \left(\iint \varphi_{\mathbf{k}}(x,y,z,w) g_s(z-x) g_s(w-y)\,\tilde{\mu}(\mathrm{d}z)\,\tilde{\mu}(\mathrm{d}w) \right) + \mathrm{O}(s^p \|G_s^{(p)}\|_{\mathrm{L}^1(\mathbb{T}\times\mathbb{T})}), \tag{B.16}$$

for some functions $|F^{(j,\mathbf{k})}(x,y)| = \mathrm{O}(\|\nabla^j G(x,y)\|)$, where

$$\varphi_{\mathbf{k}}(x,y,z,w) = c_{\mathbf{k}} (x-z)^{k_1} (\bar{x}-\bar{z})^{k_2} (y-w)^{k_3} (\bar{y}-\bar{w})^{k_4}.$$

Remark B.7. *This lemma uses only that the density is locally bounded w.r.t. the Yukawa gas. In its application, we choose s such that $s^p \|G_s^{(p)}\|_{\mathrm{L}^1(\mathbb{T}\times\mathbb{T})} \leq N^{-\varepsilon p}$. If G is a function smooth at the scale R, say, then $s = RN^{-\varepsilon}$ is such a choice.*

Proof. Let g_s be a nonnegative mollifier such that $\int g_s(z)\,\mathrm{d}z = 1$, supported in a square of side length s and satisfying $\|g_s^{(n)}\|_\infty \leqslant C_n s^{-2-n}$ for all $n \geqslant 0$. By Taylor expansion, for any $(x,y) \in \mathrm{B}(z,s) \times \mathrm{B}(w,s)$ we can write

$$G(z,w) = \sum_{j=0}^{p-1} \sum_{\sum k_i = j, a_i \in \{x,\bar{x},y,\bar{y}\}} \left(\prod_{i=1}^{4} (\partial_{a_i})^{k_i} G(x,y) \right) \varphi_{\mathbf{k}}(x,y,z,w) + \mathrm{O}\left(s^p G_s^{(p)}(z,w) \right). \tag{B.17}$$

We now rewrite

$$\iint G(z,w)\,\tilde{\mu}(\mathrm{d}z)\,\tilde{\mu}(\mathrm{d}w) = \iint G(z,w)\,\tilde{\mu}(\mathrm{d}z)\,\tilde{\mu}(\mathrm{d}w)\, g_s(z-x)\, g_s(w-y)\, m(\mathrm{d}x)\, m(\mathrm{d}y), \tag{B.18}$$

and insert the equation (B.17) into this identity. To bound the error $\iint G_s^{(p)}(z,w)\,\tilde{\mu}(\mathrm{d}z)\,\tilde{\mu}(\mathrm{d}w)$, we use the local law, Theorem A.2, which implies that the empirical density is bounded with high probability, i.e., for any function k supported in a square of size $w \gg N^{-1/2}$, we have

$$\int |k(z)|\,\hat{\mu}(\mathrm{d}z) \leq C w^2 \|k\|_\infty, \tag{B.19}$$

and the same estimate holds with $\hat{\mu}$ replaced by $\tilde{\mu}$. This proves the lemma with some functions $|F^{(j,\mathbf{k})}(x,y)| = \mathrm{O}(\|\nabla^j G(x,y)\|)$. □

In next lemma, which is parallel to [6, Lemma 7.5], we estimate the last term in (B.9). (In the later application, we only need $Q = f/N$.)

Lemma B.8. *For any $f : \mathbb{T} \to \mathbb{R}$, define h as in Lemma B.5, and $G(z,w) = (h(z)-h(w))\partial U^\ell(z-w)$. Then, for the Yukawa gas on the unit torus, we have*

$$\mathbb{E}_Q^{U^\ell}\left(N \iint_{z\neq w} G(z,w)\,\tilde{\mu}_Q(\mathrm{d}z)\,\tilde{\mu}_Q(\mathrm{d}w) \right) = \mathrm{O}\left(N^\varepsilon \left(1 + \frac{b^2}{\ell^2} \right) \right) \|f\|_{3,b}. \tag{B.20}$$

The proof of this lemma requires only on the local law (A.9).

Proof. We first write

$$\partial U^{\ell}(z)=\sum_{i\le m}U_i(z)+U^{(m)}(z) \tag{B.21}$$

where U_i is supported in $\ell_i/2\le |z|\le 2\ell_i$ with $\ell_i=2^{-i}\ell$ and $U^{(m)}$ is supported in $|z|\le 2\ell_m=N^{-1/2+\varepsilon}$.

Case 1: $U^{(m)}$. For any function k supported in a ball of radius b, using the fact that the empirical density is locally bounded up to a factor N^{ε}, with high probability we have

$$N\iint(\bar\partial k(z)-\bar\partial k(w))U^{(m)}(z-w)\,\tilde\mu(dz)\,\tilde\mu(dw)\le N^{1+\varepsilon}\ell_m^2b^2\|\nabla^2k\|_{L_\infty}\le N^{1+\varepsilon}\ell_m^2\|k\|_{2,b}, \tag{B.22}$$

where the factor $\ell_m^2b^2$ comes from the volume in the integration.

Recall that h is defined in Lemma B.5. We can apply the previous inequality to $k=f$. To bound the other contribution due to the integral of $\ell^{-2}\bar\partial f_s$, we use that f_s is supported in a ball of size b_s and apply (B.22) and (B.13) to have

$$\begin{aligned}&\ell^{-2}N\int_0^\infty\frac{\mathrm{d}s}{s}\iint(\bar\partial f_s(z)-\bar\partial f_s(w))U^{(m)}(z-w)\,\tilde\mu(dz)\,\tilde\mu(dw)\\&\le N\ell_m^2\ell^{-2}\int_0^{\log N}\frac{\mathrm{d}s}{s}\|f_s\|_{2,b_s}+N^{-A}\|f\|_{1,b}\\&\le\frac{N\ell_m^2}{\ell^2}\int_0^{\log N}\frac{\mathrm{d}s}{s}(b\wedge s)^2\|f\|_{2,b}+N^{-A}\|f\|_{1,b}\le N^{1+\varepsilon}\ell_m^2\frac{b^2}{\ell^2}\|f\|_{2,b}+N^{-A}\|f\|_{1,b}.\end{aligned}$$

Case 2: U_i for a scale $\ell_i:=q\le\ell$. Suppose that h is supported in a ball of size r (note that r can be either samller or bigger than ℓ). Let $M_i(z,w)=(h(z)-h(w))U_i(z-w)$. Our goal is to bound $N\iint_{z\neq w}M_i(z,w)\,\tilde\mu_Q(\mathrm{d}z)\,\tilde\mu_Q(\mathrm{d}w)$. Treating $(h(z)-h(w))$ as a multiplicative factor, we can apply Lemma B.6 to the function $U_i(z-w)$ with the scale s in the lemma replaced by $qN^{-\varepsilon}$. Hence we only have to estimate

$$\begin{aligned}&\iint F^{(j,\mathbf{k})}(x,y)\Omega_{x,y}m(\mathrm{d}x)m(\mathrm{d}y),\\&\Omega_{x,y}=\iint(h(z)-h(w))\varphi_{\mathbf{k}}(x,y,z,w)g_q(z-x)g_q(w-y)\ \tilde\mu(\mathrm{d}z)\tilde\mu(\mathrm{d}w),\end{aligned} \tag{B.23}$$

where g_q is a smooth mollifier at scale q (in fact, it should be $qN^{-\varepsilon}$ as remarked in Remark B.7. The reader can follow through this minor change in the following proof).

Rewrite $h(z)-h(w)=(h(z)-h(x))+(h(y)-h(w))+(h(x)-h(y))$. Consider first the term $(h(z)-h(x))$ and $j=0$ so that $\varphi_{\mathbf{k}}(x,y,z,w)=1$. Let $R_x(z)=(h(z)-h(x))g_q(z-x)$. Applying the local law (A.9), we have

$$\begin{aligned}&N\iint(h(z)-h(x))g_q(z-x)g_q(w-y)\ \tilde\mu(\mathrm{d}z)\tilde\mu(\mathrm{d}w)\\&\le q^2N^{2\varepsilon}\Big[\|\nabla_zR_x(z)\|_{L_2(z)}+\|R_x(z)\|_{L_2(z)}\ell^{-1}+N^{-1/2}q\|\Delta_zR_x(z)\|_\infty\Big]\\&\times\Big[\|\nabla_wg_q(w-y)\|_{L_2(w)}+\|g_q(w-y)\|_{L_2(w)}\ell^{-1}+N^{-1/2}q\|\Delta_wg_q(w-y)\|_\infty\Big]\end{aligned} \tag{B.24}$$

The last bracket is bounded by $q^{-2}A(q)$ where $A(q):=1+q\ell^{-1}+(\sqrt{N}q)^{-1}$. Using

$$\iint\mathbb{1}(|z-w|\le q)\mathbb{1}(R_x(z)\neq0\text{ for some }z)\ m(\mathrm{d}w)m(\mathrm{d}z)\le\mathrm{O}(r^2q^2), \tag{B.25}$$

we can bound

$$\iint U_i(x,y)m(\mathrm{d}x)m(\mathrm{d}y)\Big[\|\nabla_z R_x(z)\|_{L_2(z)} + \|R_x(z)\|_{L_2(z)}\ell^{-1} + N^{-1/2}q\|\Delta_z R_x(z)\|_\infty\Big]$$
$$\le q^2r^2q^{-1}\Big[q^{-1}+\ell^{-1}+N^{-1/2}q^{-2}\Big]r^{-1}\|h\|_{2,r} = A(q)r\|h\|_{2,r} \tag{B.26}$$

where we have used that $|U_i(x,y)| \le q^{-1}$ and, e.g.,

$$\|\nabla_z R_x(z)\|_{L_2(z)} \le \|(h(z)-h(x))\nabla g_q(z-x)\|_{L_2(z)} + \|g_q(z-x)\nabla h(z)\|_{L_2(z)} \le Cq^{-1}\|\nabla h\|_\infty$$

The other two terms involving $(h(y)-h(w))$ or $(h(x)-h(y))$ can be estimated in the same way. Hence we can bound (B.23) for $j=0$ by $N^{2\varepsilon}A(q)^2r\|h\|_{2,r}$.

One can check in a similar way that the same bound holds for any j since the factor q^{-j} induced by the derivatives on U_i is compensated by the size of the function $\varphi_{\mathbf{k}}$. Notice that for all j, we need at most two derivatives on h; all other derivatives will apply to explicit functions depending on U_i. Summing over all j and i and using $q \le C\ell$, we have thus proved that

$$\sum_{i\le m} N \iint_{z\ne w} M_i(z,w)\,\tilde\mu_Q(\mathrm{d}z)\,\tilde\mu_Q(\mathrm{d}w) \le N^{2\varepsilon}\sum_{i\le m} A(\ell_i)^2 r\|h\|_{2,r} \le N^{2\varepsilon}\big[1+(\sqrt{N}\ell_m)^{-1}\big]^2 r\|h\|_{2,r} \tag{B.27}$$

From the definition of h in Lemma B.5, we need to consider two contributions of h: one is $\bar\partial f$, the other involves the s integration. Since f is supported in a ball of radius b, the contribution from $\bar\partial f$ can be trivially bounded by

$$N^{2\varepsilon}\big[1+(\sqrt{N}\ell_m)^{-1}\big]^2 b\|h\|_{2,b} \le N^{2\varepsilon}\big[1+(\sqrt{N}\ell_m)^{-1}\big]^2\|f\|_{3,b},$$

where we have replaced r in (B.27) by b. Again, applying (B.27) to the function $\bar\partial f_s$, we can bound the other term of h involving s integration by

$$N^{2\varepsilon}\big[1+(\sqrt{N}\ell_m)^{-1}\big]^2\ell^{-2}\int_0^{\log N}\frac{\mathrm{d}s}{s}b_s\|\partial f_s\|_{2,b_s} + N^{-A}\|f\|_{1,b}$$
$$\le N^{2\varepsilon}\big[1+(\sqrt{N}\ell_m)^{-1}\big]^2\ell^{-2}\int_0^{\log N}\frac{\mathrm{d}s}{s}(b\wedge s)^2\|f\|_{3,b} + N^{-A}\|f\|_{1,b}$$
$$\le N^{2\varepsilon}\big[1+(\sqrt{N}\ell_m)^{-1}\big]^2\frac{b^2}{\ell^2}\|f\|_{3,b} + N^{-A}\|f\|_{1,b},$$

where we have again used (B.13).

Combining Case 1 and 2, we have bounded (B.20) by

$$N^{2\varepsilon}\Big[\big(1+(\sqrt{N}\ell_m)^{-1}\big)^2 + N\ell_m^2\Big]\Big[1+\frac{b^2}{\ell^2}\Big]\|f\|_{3,b} + N^{-A}\|f\|_{3,b}, \tag{B.28}$$

where the term $N\ell_m^2[1+\frac{b^2}{\ell^2}]$ comes from the Case 1 and the other terms come from Case 2. Recalling $\ell_m = N^{-1/2+\varepsilon}$, we have proved (B.20). □

Proof of Theorem B.1. We will assume $Q=0$; the general case can be proved in a similar way.

We again employ the loop equation and calculate

$$
\begin{aligned}
&\frac{1}{\beta}\log \mathbb{E}_0^{U^\ell} \mathrm{e}^{-t\beta X_0^f} = \frac{1}{\beta}\int_0^t \mathrm{d}s\, \frac{\partial}{\partial s}\log \mathbb{E}_0^{U^\ell} \mathrm{e}^{-s\beta X_0^f} = \int_0^t \mathrm{d}s\, \big(-\mathbb{E}_{sf/N}^{U^\ell} X_{sf/N}^f + N\int f(\mu_0-\mu_{sf/N})\big)\\
&= \int_0^t \mathrm{d}s\, \mathbb{E}_{sf/N}^{U^\ell}\bigg(\frac{1}{N}W_{sf/N}^{h_s} - \frac{1}{N\beta}\sum_j \partial h_s(z_j)\\
&- N\iint (h_s(z)-h_s(w))\partial U^\ell(z-w)\tilde\mu_{sf/N}(\mathrm{d}z)\tilde\mu_{sf/N}(\mathrm{d}w) + N\int f(\mu_0-\mu_{sf/N})\bigg),
\end{aligned}
$$

where h_s is associated to f and $\mu_{sf/N}$ as in (B.7), i.e., $h_s(z) = \frac{1}{\pi\rho_{sf/N}(z)}\bar\partial(1-m^2\Delta^{-1})f(z)$. By Lemma B.8,

$$
N\,\mathbb{E}_{sf/N}^{U^\ell}\iint (h_s(z)-h_s(w))\partial U^\ell(z-w)\tilde\mu_{sf/N}(\mathrm{d}z)\tilde\mu_{sf/N}(\mathrm{d}w) = \mathrm{O}\bigg(N^\varepsilon\big(1+\frac{b^2}{\ell^2}\big)\bigg)\|f\|_{3,b}.
$$

By Lemma 4.3, $\mathbb{E}_{sf/N}^{U^\ell} W_{sf/N}^{h_s} = 0$. Using $m = 1/\ell$, we have

$$
\begin{aligned}
\int_0^t \mathrm{d}s\, N\int f(\mu_0-\mu_{sf/N}) &= -N\int_0^t \mathrm{d}s\, \int f(z)\frac{s}{4\pi N}(\Delta-m^2)f(z)\,m(\mathrm{d}z)\\
&= \int_0^t \mathrm{d}s\,\frac{s}{4\pi}\int f(-\Delta+m^2)f = \mathrm{O}\big(t^2(1+b^2\ell^{-2})\big)\|f\|_{2,b}^2.
\end{aligned}
$$

Finally, using the fact that the local density is bounded and (B.13), we have

$$
\int_0^t \mathrm{d}s\,\frac{1}{N\beta}\mathbb{E}_{sf/N}^{U^\ell}\sum_j \partial h_s(z_j) = \mathrm{O}(t)\|\partial\rho_{sf/N}^{-1}\bar\partial(f-m^2\Delta^{-1})f\|_{L_1} = \mathrm{O}(t)\big(1+\frac{b^2}{\ell^2}\big)b^2\|f\|_{2,b}.
$$

Collecting these estimates gives

$$
\frac{1}{\beta}\log \mathbb{E}_0^{U^\ell}\mathrm{e}^{-t\beta X_0^f} = \mathrm{O}\bigg(N^\varepsilon\big(1+\frac{b^2}{\ell^2}\big)\bigg)\big[t\|f\|_{3,b} + t^2\|f\|_{3,b}^2\big]. \tag{B.29}
$$

By Markov's inequality and with the choice of $t = 1/\|f\|_{3,b}$ this implies that

$$
|X_0^f| \leqslant \mathrm{O}(N^\varepsilon)(1+b^2/\ell^2)\|f\|_{3,b}
$$

with probability at least $1-\mathrm{e}^{-N^\varepsilon}$, which proves Theorem B.1. □

Proof of Proposition B.3. Recall the equation (B.16) which expands the left side of (B.4) into a Taylor series with error term. To prove (B.4), we only have to estimate each term in the summation in (B.16). From the rigidity estimate Theorem B.1, we have

$$
N\int (x-z)^{k_1}(\bar x-\bar z)^{k_2} g_s(z-x)\tilde\mu(\mathrm{d}z) = \mathrm{O}(N^\varepsilon)s^{k_1+k_2-2}\left(1+\frac{s}{\ell}\right)^2, \tag{B.30}
$$

and a similar estimate holds around y. This two bounds together imply that each term in the summation in (B.16) is bounded by $\mathrm{O}(N^\varepsilon)\big(\frac{1}{s^4}+\frac{1}{\ell^4}\big)s^j|||\nabla^j G|||_{\mathrm{L}^1(\mathbb{T}\times\mathbb{T})}$ and this completes the proof of the proposition. □

B.3. Proof of Lemma 4.11. Using the rigidity estimate, we can improve the regime of Case 2 in the proof of (B.20), which we state as the following lemma.

Lemma B.9. *Recall the decomposition* (B.21) *and let* G_σ *be the long range part*

$$G_\sigma(z,w) = (h(z) - h(w)) \sum_{i \le m} U_i(z), \tag{B.31}$$

with ℓ_m *set to be* $N^{-1/2+\sigma}$ *for some* $\sigma > 0$ *fixed. Using the rigidity estimate* (B.20)*, we have*

$$\mathbb{E}_Q^{U^\ell}\left(N \iint_{z \neq w} G_\sigma(z,w)\, \tilde\mu_Q(\mathrm{d}z)\, \tilde\mu_Q(\mathrm{d}w)\right) \leqslant N^{-\sigma+\varepsilon} r \|h\|_{2,r} \le N^{-\sigma+\varepsilon}\|f\|_{3,r} \tag{B.32}$$

where the last inequality holds for when h is given as in Lemma B.5. Notice that we only need two derivatives of h.

Moreover, for the Coulomb gas with angle correction defined by the Hamiltonian (4.14) *so that the external potential V satisfying (1.4 , 1.9) and t satisfying the assumptions of Proposition 4.1, the estimate* (B.32) *holds. Similarly, if we replace the Yukawa gas by a Coulomb gas restricted to a disk defined by* (4.46)*, then* (B.32) *holds up to a rescaling by the radius of the disk.*

Proof. We follow the proof of the Case 2 of Lemma B.8. Instead of the local law, we can now apply the rigidity estimate (B.1) so that the right hand side of (B.24) can be replaced by

$$(qN^{-1/2+2\varepsilon})\Big(1+q^2/\ell^2\Big)\|g_q(\cdot - y)\|_{3,q}\left(\|\nabla_z R_x(z)\|_{L_2(z)} + \|R_x(z)\|_{L_2(z)}\ell^{-1} + N^{-1/2}q\|\Delta_z R_x(z)\|_\infty\right) \tag{B.33}$$

Compared with (B.24), we have gained a factor $(\sqrt{N}q)^{-1}$. Using $\|g_q(\cdot - y)\|_{3,q} \le q^{-2}$ and the estimates for the R_x terms as in the proof of Lemma B.8, we can improve (B.27) to

$$(\sqrt{N}\ell_m)^{-1} N^{2\varepsilon}\big[1 + (\sqrt{N}\ell_m)^{-1}\big]^2 r\|h\|_{2,r}, \tag{B.34}$$

gaining a factor $(\sqrt{N}\ell_m)^{-1}$ over (B.27). This proves (B.32).

Finally, for Coulomb gases with or without angle terms, we can apply the rigidity estimates (4.38) or (1.8). Notice that $\|g_q(\cdot - y)\|_{3,q}$ in (B.33) should be replaced by $\|g_q(\cdot - y)\|_{4,q}$ in these cases. Since g is only a mollifier, both norms are of the same order and we have thus proved the extension of (B.32) to these cases. □

Proof of Lemma 4.11. We only have to apply the above lemma and note that the long range interaction in the definition of A^+ in (4.11) can be written in the form (B.31) with $\ell_m \ge N^{-1/2+\sigma}$. This clearly can be done and we have thus proved Lemma 4.11. □

C Proof of Proposition 3.12

This section proves Proposition 3.12. We will actually prove the following more general version, with arbitrary parameter t from (3.34) to be chosen. All notations and conventions in this section are the same as those of Section 3.5.

Proposition C.1. *Let $c>0$ be a small constant. For any $N^{-\frac{1}{3}+c} \ll t < \gamma < N^{-c}$ and small $\varepsilon > 0$, we have the following estimates for the Yukawa gas of range γ on the unit torus:*

$$\mathbb{E}^{\Psi}\mathbb{E}^{\gamma} N^2 \iint (Y^{\gamma}(\Psi_z-\Psi_w) - Y^{\gamma}(z-w))\hat{\mu}(\mathrm{d}z)\hat{\mu}(\mathrm{d}w) = \mathrm{O}(N^{1+\varepsilon}t), \tag{C.1}$$

$$\mathbb{E}^{\Psi}\mathbb{E}^{\gamma} N^2 \iint (Y^{R}(\Psi_z-\Psi_w) - Y^{\gamma}(\Psi_z-\Psi_w))\tilde{\mu}(\mathrm{d}z)\tilde{\mu}(\mathrm{d}w) = \mathrm{O}(N^{\varepsilon})\left(\frac{1}{\gamma^4} + \frac{N^{\frac{4}{5}}}{\gamma^{\frac{2}{5}}} + \frac{1}{t^3} + \frac{\sqrt{N}}{t}\right), \tag{C.2}$$

$$\mathbb{E}^{\Psi}\mathbb{E}^{\gamma} N^2 \iint (U^{\gamma}(z-w) - Y^{\gamma}(\Psi_z-\Psi_w))\tilde{\mu}(\mathrm{d}z)\,\tilde{\mu}(\mathrm{d}w) = \mathrm{O}(N^{\varepsilon})\left(Nt + \frac{1}{\gamma^4} + \frac{\sqrt{N}}{t}\right). \tag{C.3}$$

In particular, for $t = N^{-1/4}$, the left-hand sides of (C.1)–(C.3) *are simultaneously of order* $\mathrm{O}(N^{\varepsilon})\left(\frac{1}{\gamma^4} + \frac{N^{4/5}}{\gamma^{2/5}}\right)$.

All terms we need to bound can be written as the left-hand side of (B.4), so that Proposition B.3 will be our main tool. However, these terms involve interactions for the Euclidean distance on the square while Proposition 3.12 applies to the unit torus. We therefore first need the next lemma to turn the Euclidean interaction into a periodic one; subsequently, we decompose the resulting singularities carefully.

Lemma C.2. *Consider a Yukawa gas on the unit torus $\mathbb{T}$, and assume all integrands below are integrable. Let $G:\mathbb{T}^2\to\mathbb{R}$, and define, for any $h\in\mathbb{C}$, $\mathcal{T}_G(h) = \int_{\mathbb{T}} G(z,[z+h])\,m(\mathrm{d}z)$, where m is the Lebesgue measure on $\mathbb{T}$ and we used the notation (3.33). Then the following holds:*

$$\mathbb{E}\iint_{z\neq w} G(z,w)\,\hat{\mu}(\mathrm{d}z)\,\hat{\mu}(\mathrm{d}w) = \mathbb{E}\iint_{z\neq w} \mathcal{T}_G([z-w])\,\hat{\mu}(\mathrm{d}z)\,\hat{\mu}(\mathrm{d}w).$$

Moreover, if $G(z,w) = g(|z-w|)$ is a function of the Euclidean distance, for any $h=(h_1,h_2)\in\mathbb{T}$ we have

$$\begin{aligned}\mathcal{T}_g(h) = \int_{\mathbb{T}} g(|[z+h]-z|)m(\mathrm{d}z) &= (1-|h_1|)(1-|h_2|)g_1(h) + |h_1|(1-|h_2|)g_2(h)\\ &\quad + |h_2|(1-|h_1|)g_3(h) + |h_1||h_2|g_4(h),\end{aligned} \tag{C.4}$$

where

$$\begin{aligned} g_1(h) &:= g(\sqrt{|h_1|^2+|h_2|^2}), & g_2(h) &:= g(\sqrt{(1-|h_1|)^2+|h_2|^2}),\\ g_3(h) &:= g(\sqrt{|h_1|^2+(1-|h_2|)^2}), & g_4(h) &:= g(\sqrt{(1-|h_1|)^2+(1-|h_2|)^2}).\end{aligned} \tag{C.5}$$

Remark C.3. *The above calculation is stated for $h\in\mathbb{T}$, and it shows that $\mathcal{T}_G$ is not smooth for $h_1=0$ or $h_2=0$. One may wonder if $\mathcal{T}_G$ has additional singularities at $h_1=\pm1/2$ or $h_2=\pm1/2$, as a function on the torus. It has not, as shown by the following argument. Assume $-1/2\leqslant h_2<1/2$ is fixed. The right-hand side of (C.4) admits an obvious smooth extension to $h_1\in(0,1)$, called $\tilde{\mathcal{T}}_G$. One readily sees that for such $h_1\in(0,1)$, we have $\tilde{\mathcal{T}}_G(h_1,h_2) = \tilde{\mathcal{T}}_G(1-h_1,h_2)$: $\tilde{\mathcal{T}}_G$ is smooth and symmetric with respect to $h_1=1/2$, so all its odd derivatives vanish there, meaning $\mathcal{T}_G$ is smooth at $h_1=\pm1/2$. The same reasoning applies on $h_2=\pm1/2$.*

Proof. Remember ρ_2 is the two point correlation function for the Yukawa gas on $\mathbb{T}$. By translation invariance of the distribution of the Yukawa gas, we have

$$\mathbb{E}\iint_{z\neq w} G(z,w)\hat\mu(\mathrm{d}z)\hat\mu(\mathrm{d}w)=\iint_{z\neq w} G(z,w)\rho_2([z-w])m(\mathrm{d}z)m(\mathrm{d}w)$$
$$=\iint G(z,[z+h])\rho_2(h)m(\mathrm{d}z)m(\mathrm{d}h)=\int\rho_2(h)\left(\int G(z,[z+h])m(\mathrm{d}z)\right)m(\mathrm{d}h)$$
$$=\int\rho_2(h)\mathcal{T}_G(h)m(\mathrm{d}h)=\iint\rho_2(h)\mathcal{T}_G(h)m(\mathrm{d}h)m(\mathrm{d}\tilde z)=\mathbb{E}\iint_{z\neq w}\mathcal{T}_G([\tilde z-\tilde w])\hat\mu(\mathrm{d}\tilde z)\hat\mu(\mathrm{d}\tilde w).$$

In the case $G(z,w)=g(|z-w|)$, the assertion follows from a direct calculation of $\mathcal{T}_G(h)=\int_{\mathbb{T}} g(|[z+h]-z|)m(\mathrm{d}z)$. □

Denote by $\mathbb{E}_{(a,b)}$ integration with respect to the shift (a,b) of Ψ (see Section 3.5), and write $\Delta_z^w=[\Phi_z-\Phi_w]$. Then the functional $\mathcal{T}$ defined in Lemma C.2 naturally appears in the following calculation:

$$\mathbb{E}_{(a,b)}\left(g(|\Psi_z-\Psi_w|)\right)=\int_{\mathbb{T}} g(|[\Psi_z+\tilde z]-[\Psi_w+\tilde z]|)\,m(\mathrm{d}\tilde z)$$
$$=\int_{\mathbb{T}} g(|[\tilde z+\Delta_z^w]-\tilde z|)\,m(\mathrm{d}\tilde z)=\mathcal{T}_g(\Delta_z^w). \tag{C.6}$$

This will be useful in our following proof of Proposition 3.12.

Proof of (C.1). We apply (C.4) with $g=Y^\ell$. For $|h|\leqslant 1/2$, the last three terms in (C.4) are exponentially small since $\gamma\ll 1$ so that $|Y^\gamma(z)|<\mathrm{e}^{-N^\varepsilon}$ for $|z|\geqslant 1/2$. Denote by $q(h)$ the first term on the right-hand side of (C.4). Equation (C.6) gives

$$\mathbb{E}_{(a,b)}\left(Y^\gamma(\Psi_z-\Psi_w)\right)=q(\Delta_w^z)+\mathrm{O}(\mathrm{e}^{-N^\varepsilon}),\quad \mathbb{E}_{(a,b)}\left(Y^\gamma(z-w)\right)=q([z-w])+\mathrm{O}(\mathrm{e}^{-N^\varepsilon}).$$

The proof of (C.1) is therefore reduced to proving

$$\mathbb{E}^\Phi\mathbb{E}^\gamma N^2\iint\left(q([\Phi_z-\Phi_w])-q([z-w])\right)\hat\mu(\mathrm{d}z)\,\hat\mu(\mathrm{d}w)=\inf_{r\in[N^{-\frac12+\varepsilon},\gamma]}\left(N^2t^2r^2+\frac{1}{r^2}\right)\mathrm{O}(N^\varepsilon). \tag{C.7}$$

Let $N^{-1/2}\ll r\ll\gamma$ be some intermediate scale. Let $\chi:\mathbb{R}_+\to[0,1]$ be a smooth function such that $\chi(z)=1$ on $[0,1]$, $\chi(z)=0$ on $[2,\infty)$ and define $\widetilde q(z)=q(z)\chi(|z|/r)$. The proof of (C.7) will consist in the following two estimates:

$$\mathbb{E}^\Phi\mathbb{E}^\gamma N^2\iint\left(\widetilde q([\Phi_z-\Phi_w])-\widetilde q([z-w])\right)\hat\mu(\mathrm{d}z)\,\hat\mu(\mathrm{d}w)=N^\varepsilon\,\mathrm{O}\left(N^2t^2r^2\right), \tag{C.8}$$
$$\mathbb{E}^\Phi\mathbb{E}^\gamma N^2\iint\left((q-\widetilde q)([\Phi_z-\Phi_w])-(q-\widetilde q)([z-w])\right)\hat\mu(\mathrm{d}z)\,\hat\mu(\mathrm{d}w)=N^\varepsilon\,\mathrm{O}\left(\frac{1}{r^2}\right). \tag{C.9}$$

Optimization over r shows that (C.7) holds (note that the optimum $r^*=(Nt)^{-1/2}$ is smaller than γ because $t<\gamma$ and $N^{-1/3}\ll\gamma$).

For the proof of (C.8), we consider the Taylor expansion of $[\Phi_z-\Phi_w]$ around $[z-w]$. As $\tilde q$ is supported on $|x|<N^\varepsilon r$, for all z,w contributing to (C.8) we have $[z-w]=z-w$ and $[\Phi_z-\Phi_w]=\Phi_z-\Phi_w$. For such z,w, with the definition $\Phi=\Phi_1\circ\Phi_2$ with (3.34), therefore

$$[\Phi_w-\Phi_z]=[w-z]+\begin{pmatrix} m_1\left(s(w_2+m_2s(w_1))-s(z_2+m_2s(z_1))\right)\\ m_2\left(s(w_1)-s(z_1)\right)\end{pmatrix}. \tag{C.10}$$

Expanding (C.10),

$$\frac{([\Phi_w-\Phi_z])_2-([w-z])_2}{m_2}=s'(w_1)(w_1-z_1)+\mathrm{O}(|w-z|^2),$$
$$\frac{([\Phi_w-\Phi_z])_1-([w-z])_1}{m_1}=\ s'(w_2)(w_2-z_2)+\mathrm{O}(|w-z|^2)+m_2\,\mathrm{O}(|w-z|)+m_2^2\,\mathrm{O}(|w-z|^2).$$

where here and in the following the O error terms are non-random: they are not functions of m_1,m_2. Denoting

$$\Delta=\begin{pmatrix} m_1\left(s'(w_2)(w_2-z_2)+\mathrm{O}(|w-z|^2)+m_2\,\mathrm{O}(|w-z|)+m_2^2\,\mathrm{O}(|w-z|^2)\right)\\ m_2\left(s'(w_1)(w_1-z_1)+\mathrm{O}(|z-w|^2)\right)\end{pmatrix},$$

we have

$$\widetilde q([\Phi_z-\Phi_w])-\widetilde q([z-w])=\nabla\widetilde q([z-w])\cdot\Delta+\mathrm{O}\left(\sup_{[|z-w|/2,2|z-w|]}|\nabla^2\widetilde q|\right)|\Delta|^2.$$

As m_1,m_2 are centered (under the random choice of Φ), this gives, for any fixed small $\varepsilon>0$,

$$\mathbb{E}^{\Phi}\left(\widetilde q([\Phi_z-\Phi_w])-\widetilde q([z-w])\right)=\mathbb{E}(m^2)|(\nabla^2\widetilde q)(z-w)|\mathrm{O}(|z-w|^2)=\mathrm{O}(N^2t^2)\mathbb{1}_{|z-w|\leqslant N^{\varepsilon}r}.$$

We therefore proved that

$$\mathbb{E}^{\Phi}\mathbb{E}^{\gamma}N^2\iint\left(\widetilde q([\Phi_z-\Phi_w])-\widetilde q([z-w])\right)\hat\mu(\mathrm{d}z)\,\hat\mu(\mathrm{d}w)\leqslant\mathrm{O}(N^2t^2)\mathbb{E}\left(\sup_{z\in\mathbb{T}}\hat\mu(\{w:|[z-w]|\leqslant N^{\varepsilon}r\})\right).\tag{C.11}$$

From the local density estimate for the Yukawa gas on the torus implied by Theorem A.2, the above parenthesis is bounded by $(N^{\varepsilon}r)^2$ with high probability. We have therefore proved (C.8).

To prove (C.9), we denote $f=q-\widetilde q$. Since $|\mathrm{d}\Phi|=1$ by (3.35), we have

$$\int\left(f([\Phi_z-\Phi_w])-f([z-w])\right)\mu(\mathrm{d}z)=\int\left(f([z-\Phi_w])-f([z-w])\right)\mu(\mathrm{d}z)=0.$$

Together with the definition $\tilde\mu=\hat\mu-\mu$, the above equation implies

$$\iint\left(f([\Phi_z-\Phi_w])-f([z-w])\right)\left(\hat\mu(\mathrm{d}z)\,\hat\mu(\mathrm{d}w)-\widetilde\mu(\mathrm{d}z)\widetilde\mu(\mathrm{d}w)\right)=0.$$

We therefore just need to prove that

$$\mathbb{E}^{\Phi}\mathbb{E}^{\gamma}N^2\iint\left(f([\Phi_z-\Phi_w])-f([z-w])\right)\tilde\mu(\mathrm{d}z)\,\tilde\mu(\mathrm{d}w)=N^{\varepsilon}\mathrm{O}\left(\frac{1}{r^2}\right).\tag{C.12}$$

This is a consequence of Proposition B.3. Indeed, let $(\chi_k)_{k\geqslant1}$ be a partition of unity in the sense that $\sum_k\chi_k(x)=1$ for any $x>r$, χ_k is supported on $[2^{k-1}r,2^{k+1}r]$, and $\|\chi_k^{(n)}\|_\infty\leqslant C_n(2^kr)^{-n}$. We apply Proposition B.3 to $G(z,w)=G_k(z,w)=f([z-w])\chi_k(|[z-w]|)$ and $s=s_k=N^{-\varepsilon}2^kr$, for some fixed small $\varepsilon>0$. For any k such that $2^kr<\gamma N^{\varepsilon}$, using the notation (B.2) we have

$$|\nabla^jG_k(x,y)|=\mathrm{O}\left(|[x-y]|^{-j}\mathbb{1}_{|[x-y]|\in[2^{k-2}r,2^{k+2}r]}\right)$$

and the same estimate holds for $(G_k)^{(j)}_s(x,y)$, defined in (B.3). Proposition B.3 gives

$$N^2 \iint G_k(z,w)\,\tilde\mu(\mathrm{d}z)\,\tilde\mu(\mathrm{d}w) = \mathrm{O}(N^{\varepsilon})\left(\frac{1}{s_k^4}\sum_{j=0}^{p-1} s_k^j (s_k N^{\varepsilon})^{2-j} + N^2 s_k^p (s_k N^{\varepsilon})^{2-p}\right) = \mathrm{O}\left(\frac{N^{3\varepsilon}}{s_k^2}\right),$$

where we chose $p = \lfloor 10/\varepsilon \rfloor$. Summation of the above estimate over $1 \leqslant k \leqslant \log N$ gives

$$N^2 \iint G(z,w)\,\tilde\mu(\mathrm{d}z)\,\tilde\mu(\mathrm{d}w) = \mathrm{O}\left(\frac{N^{5\varepsilon}}{r^2}\right), \tag{C.13}$$

which proves (C.12) for the term involving $f([z-w])$. The same estimate holds for the integral of $f([\Phi_z - \Phi_w])$, because for any fixed $m_1, m_2 = \mathrm{O}(t)$ the function $(z,w) \mapsto f([\Phi_z - \Phi_w])$ has the same regularity properties as $(z,w) \mapsto f([z-w])$ ($\Phi = \mathrm{Id} + t\phi$ for some function ϕ smooth on a scale 1). This concludes the proof. □

Proof of (C.2). By (C.6) we need to estimate

$$N^2 \mathbb{E}^{\Phi}\mathbb{E}^{\gamma} \iint \mathcal{T}_L([\Delta_z^w])\,\tilde\mu(\mathrm{d}z)\,\tilde\mu(\mathrm{d}w), \quad L = L_R^{\gamma} + c, \tag{C.14}$$

where c is an arbitrary constant. We will choose $c = \log R - \log\gamma$, so that from (2.2) we have

$$L(z) = \mathrm{O}(|z|/\gamma) \tag{C.15}$$

for $|z| \ll \gamma$. Using Lemma C.2, the left-hand side of (C.14) is equal to

$$N^2 \mathbb{E}^{\gamma} \iint D^t([z-w])\,\tilde\mu(\mathrm{d}z)\,\tilde\mu(\mathrm{d}w), \quad D^t(h) = \mathbb{E}^{\Phi}\int \mathcal{T}_L(\Delta_v^{[v+h]}) m(\mathrm{d}v),$$

where we remind that Φ depends on t. The estimate of this term is the most delicate in the proof of Proposition 3.12, so before giving the technical details, we below list the main difficulties and ingredients.

(i) The function $D^0 = \mathcal{T}_L(h)$ is smooth on $\mathbb{T}$ except on $h_1 = 0$ or $h_2 = 0$, as explained in Remark C.3. This prevents a direct application of Proposition B.3 and is the motivation for our averaging over Φ.

(ii) The function D^t now gained some smoothness in neighborhoods of $h_1 = 0$ and $h_2 = 0$ thanks to the convolution with the distribution of tX. For example, around $h_1 = 0$, D^t is smooth on a scale $|th_2|$: for $k \geqslant 1$, $\partial^k_{h_1} D^t(k) = N^{\varepsilon}\mathrm{O}(|th_2|^{-k+1})$, thanks to the definition of Φ_1 in (3.34) and the asymptotics $tXs(h_2) \sim 2\pi tXh_2$.

(iii) This suggests a decomposition $D^t = R^t + H^t$ where R^t is singular but has small support $\Theta = \{|h_2| < N^{-1/2+\varepsilon}/t, |h_1| < N^{-1/2+\varepsilon}\} \cup \{|h_1| < N^{-1/2+\varepsilon}/t, |h_2| < N^{-1/2+\varepsilon}\}$, and H^t is supported on $(\Theta/2)^c$, $H^t = D^t$ on Θ^c. The contribution from H^t can be bounded by the rigidity estimate, Proposition B.3, while the error due to R^t is controlled by the local law, Theorem A.2, which requires less regularity on the test functions.

(iv) The idea described in the above steps works up to the following obstacle: H^t is smooth on scale $N^{-1/2+\varepsilon}$, so a direct application of Proposition B.3 gives a bound $N^{2-4\varepsilon}$ which is far from sufficient. Therefore we need a multiscale decomposition of H^t taking into account different smoothness exponents depending on the distance to the $h_1 = 0$ and $h_2 = 0$ axes. Given a smooth cutoff function χ, supported on $|h_1|, |h_2| \leqslant N^{-1/2+\varepsilon}$, $\|\chi\|_\infty = 1$, $H_t\chi$ is smooth on scale $N^{-1/2+\varepsilon}$ only for $\sup_{|h_1|,|h_2|\leqslant N^{-1/2+\varepsilon}} |H_t| \leqslant N^{-1/2+\varepsilon}$. Due to the g_1 term, H^t is typically of order 1 on $h_1 = 0$ or $h_2 = 0$, so that the smallest scale requires a separate treatment.

(v) Hence we look for a function $\mathcal{A}$ smooth on scale 1 (therefore it will give an error term $1/\gamma^4$ from Proposition B.3) so that $H^t - \mathcal{A}$ vanishes on $h_1 = 0$ and $h_2 = 0$. We could not find such an $\mathcal{A}$ on the full torus mainly due to the logarithmic singularity of g_1 at 0 and the constraint of not creating singularities on $h_1 = \pm 1/2$, $h_2 = \pm 1/2$. However, one can find such an $\mathcal{A}$ if we a priori do not work with D^t but its long-range part, E^t, as explained in the first step below.

(vi) Once the long range part of D^t is shown to have negligible order, the short range can be decomposed into the contributions from g_1, g_2, g_3, g_4 and each of them bounded separately: there are no problems of possible lack of smoothness on $h_1 = \pm 1/2$, $h_2 = \pm 1/2$ anymore.

First step. In this paragraph we prove that the contribution of the long range is of order

$$N^2 \mathbb{E}^\gamma \iint E^t([z-w])\, \tilde{\mu}(\mathrm{d}z)\, \tilde{\mu}(\mathrm{d}w) = \mathrm{O}\left(N^\varepsilon\right) \left(\frac{1}{t^3} + \frac{1}{\gamma^4}\right), \tag{C.16}$$

where $E^t(h) = \mathbb{E}^\Phi \int \mathcal{T}_L(\Delta_v^{[v+h]})(1-\chi)(\Delta_v^{[v+h]})\, m(\mathrm{d}v)$.

The function $\mathcal{T}_L(h)$ has discontinuous derivative on $\{h_1 = 0\} \cup \{h_2 = 0\}$, which imposes a detailed analysis around these axes. We first gain some order of magnitude of $\mathcal{T}_L(1-\chi)$ around these singularities by removing the following function,

$$A(h) = \left(\tilde{\chi}(h_1)\left((1-|h_2|)g_1(h) + |h_2| g_3(h)\right) + \tilde{\chi}(h_2)\left((1-|h_1|)g_1(h) + |h_1| g_2(h)\right)\right)(1-\chi(h)),$$

where $\tilde{\chi}$ is a smooth cutoff equal to 1 on $[0, 1/200]$ and vanishing outside $[0, 1/100]$. The function A is smooth on $\mathbb{T}$ for the following two reasons. First, the function is smooth on $h_1 = 0$ because the following three estimates cannot be simultaneously satisfied: $|h_1| < 1/1000$, $\tilde{\chi}(h_2) \neq 0$ and $(1-\chi)(h) \neq 0$. Similarly, the function is smooth on $h_2 = 0$. Second, A is smooth on $h_1 = \pm 1/2$ and $h_2 = \pm 1/2$. Indeed, assume $-1/2 \leqslant h_2 < 1/2$ is fixed. Then $(1-|h_1|)((1-\chi)g_1)(h) + |h_1|((1-\chi)g_2)(h)$ admits an obvious smooth extension to $h_1 \in (0,1)$, and this extension is symmetric in a neighborhood of $h_1 = 1/2$, hence all its odd derivatives vanish there, so that A is smooth at $h_1 = \pm 1/2$. The same reasoning applies to $h_2 = \pm 1/2$.

We define $\mathcal{A}(h) = \mathbb{E}^\Phi \int A(\Delta_v^{[v+h]})\, m(\mathrm{d}v)$. As A is smooth, from Proposition B.3 we obtain

$$N^2 \mathbb{E}^\gamma \iint \mathcal{A}([z-w])\, \tilde{\mu}(\mathrm{d}z)\, \tilde{\mu}(\mathrm{d}w) = \mathrm{O}\left(\frac{N^\varepsilon}{\gamma^4}\right). \tag{C.17}$$

It thus remains to estimate $N^2 \mathbb{E}^\gamma \iint H^t([z-w])\, \tilde{\mu}(\mathrm{d}z)\, \tilde{\mu}(\mathrm{d}w)$ where $H^t = E^t - \mathcal{A}$. To understand the regularity properties of H^t, assume first that the distorsion vanishes, i.e., $t = 0$. We have $H^0 = \mathcal{T}_L(1-\chi) - A$, so that H^0 is smooth on $\{h_1 \neq 0\} \cap \{h_2 \neq 0\}$, vanishes on $\{h_1 = 0\} \cup \{h_2 = 0\}$ (this is the purpose of removing the contribution from A) and

$$\sup_{k_1+k_2=k} |\partial_{h_1}^{k_1} \partial_{h_2}^{k_2} H^0(h)| \leqslant C_k.$$

The distorsion t smoothes the singularities on $\{h_1=0\}\cup\{h_2=0\}$ as follows:

$$|H^t(h)|\leqslant C(t+\min(|h_1|,|h_2|)),\quad |\partial_{h_1}^{k_1}\partial_{h_2}^{k_2}H^t(h)|\leqslant C_{k_1,k_2}\left(1+\frac{\mathbb{1}_{|h_2|<t}}{t^{k_2-1}}+\frac{\mathbb{1}_{|h_1|<t}}{t^{k_1-1}}\right). \tag{C.18}$$

The above bounds are elementary after writing H^t explicitly in terms of g_1,g_2,g_3,g_4,χ and $\widetilde{\chi}$. It amounts to the observation that the functions $r_t^{(1)}(h)=\mathbb{E}_X|h_1+tX|$ and $r_t^{(2)}(h)=\mathbb{E}_X|h_2+tX|$ satisfy the same bounds as (C.18).

Let $\Omega_t=\{|h_1|<t\}\cup\{|h_2|<t\}$. Consider a partition of unity $1=\sum\chi_i$ on the torus with $\mathrm{O}(\log N)$ summands, χ_0 with support on Ω_t, χ_i $(i>0)$ supported on $(2^{i+1}\Omega_t)\backslash(2^{i-1}\Omega_t)$, and $\|\chi_i^{(n)}\|_\infty\leqslant C_n(2^it)^{-n}$. Note that for $H=H^t\chi_i$ we have $|\nabla^jH(x,y)|=\mathrm{O}((2^it)^{-j+1})$, and the same estimate holds for $H_s^{(j)}$ when $s=(2^it)N^{-\varepsilon}$. Moreover, $(2^{i+1}\Omega_t)\backslash(2^{i-1}\Omega_t)$ has area $\mathrm{O}(2^it)$, so that Proposition B.3 gives (take $p=\lfloor 10/\varepsilon\rfloor$):

$$N^2\mathbb{E}^\gamma\iint H^t([z-w])\chi_i([z-w])\tilde\mu(\mathrm{d}z)\tilde\mu(\mathrm{d}w)=\mathrm{O}(N^\varepsilon)\left(\frac{1}{(2^it)^3}+\frac{2^it}{\gamma^4}\right),\quad \text{for } 2^it<10.$$

Summation of the above equations over i gives

$$N^2\mathbb{E}^\gamma\iint H^t([z-w])(1-\chi)([z-w])\,\tilde\mu(\mathrm{d}z)\,\tilde\mu(\mathrm{d}w)=\mathrm{O}(N^\varepsilon)\left(\frac{1}{t^3}+\frac{1}{\gamma^4}\right). \tag{C.19}$$

Equations (C.17) and (C.19) prove (C.16).

Second step. In this paragraph we prove that the contribution of the short range is

$$N^2\mathbb{E}^\gamma\iint F^t([z-w])\,\tilde\mu(\mathrm{d}z)\,\tilde\mu(\mathrm{d}w)=\mathrm{O}(N^\varepsilon)\left(\frac{1}{\gamma^4}+\frac{N^{1/2}}{t}+\frac{N^{4/5}}{\gamma^{2/5}}\right), \tag{C.20}$$

where $F^t(h)=\mathbb{E}^\Phi\int\mathcal{T}_L(\Delta_v^{[v+h]})\chi(\Delta_v^{[v+h]})\,m(\mathrm{d}v)$. From our expression (C.4) for $\mathcal{T}_L$, we only need to bound $N^2\mathbb{E}^\gamma\iint F_j^t([z-w])\tilde\mu(\mathrm{d}z)\tilde\mu(\mathrm{d}w)$ $(1\leqslant j\leqslant 3)$ where

$$F_j^t(h)=\mathbb{E}^\Phi\int f_j(\Delta_v^{[v+h]})m(\mathrm{d}v), \tag{C.21}$$

and

$$f_1(h)=L(h)\chi(|h|),\quad f_2(h)=|h_1|L(h)\chi(h),\quad f_3(h)=|h_1h_2|L(h)\chi(h).$$

Indeed, the above are all terms involving g_1, and the other ones can be bounded in an easier way, because g_2,g_3,g_4 are smooth on scale 1 with no singularity at 0.

We first consider F_1^t. Let $N^{-1/2}\ll u\ll\gamma$ be some intermediate scale. Let χ be as before and define $\tilde L(h)=f_1(h)\chi(|h|/u)$. Then the local law, Theorem A.2 and the bound (C.15) give

$$N^2\mathbb{E}^\gamma\iint\tilde F_1^t([z-w])\,\tilde\mu(\mathrm{d}z)\,\tilde\mu(\mathrm{d}w)=\mathrm{O}\left(\frac{N^{2+\varepsilon}u^3}{\gamma}\right),\quad\text{where } \tilde F_1^t(h)=\mathbb{E}^\Phi\int\tilde L(\Delta_v^{[v+h]})\,m(\mathrm{d}v).$$

On the other hand, the same reasoning as the paragraph from (C.12) to (C.13) gives

$$N^2\mathbb{E}^\gamma\iint(F_1^t-\tilde F_1^t)([z-w])\,\tilde\mu(\mathrm{d}z)\,\tilde\mu(\mathrm{d}w)=\mathrm{O}\left(\frac{N^\varepsilon}{u^2}\right).$$

Optimization in u in both previous estimates show that the contribution from F_1^t is

$$N^2\mathbb{E}^\gamma \iint F_1^t([z-w])\,\tilde\mu(\mathrm{d}z)\,\tilde\mu(\mathrm{d}w) = \mathrm{O}\left(\frac{N^{4/5+\varepsilon}}{\gamma^{2/5}}\right). \tag{C.22}$$

We now consider the most delicate term F_2^t. The distorsion t smoothes the singularities on $\{h_1=0\}$ so that

$$|\partial_{h_1}^{k_1}\partial_{h_2}^{k_2}F_2^t(h)| \leqslant C_{k_1,k_2}\left(\frac{|h_1|}{|h|^{k_1+k_2}} + \frac{|h_1|}{|th_2|^{k_1}|h_2|^{k_2}}\mathbb{1}_{|h_1|<t|h_2|}\right). \tag{C.23}$$

The above estimates follow easily from explicitly writing F_2^t. Note that the bounds are different from (C.18) because our smoothing by convolution is now of type $s_t(h)=\mathbb{E}_X(|h_1+h_2tX|)$ instead of $r_t(h)=\mathbb{E}_X(|h_1+tX|)$. The above bound (C.23) (and the following ones) are understood up to a $\log N$ factor: for example for $k_1=k_2=0$, we have $F_2^t(h)=\mathrm{O}(|h_1|L(h))=\mathrm{O}(|h_1|)\log N$. For some mesoscopic scale $N^{-1/2+c}\leqslant r\ll t$, define a partition of unity $\mathbb{1}_{[-1,1]}(x)=\sum_{i=-n}^n\chi_i$ where n is of order $\log N$, χ_0 is supported on $[-2r,2r]$, χ_i is supported on $[2^{i-1}r,2^{i+1}r]$ $(i>0)$, $[-2^{-i+1}r,-2^{-i-1}r]$ $(i<0)$, and $\|\chi_i^{(m)}\|_\infty\leqslant C_m(2^{|i|}r)^{-m}$. We define $F_{ij}^t(h)=F_2^t(h)\chi_i(h_1)\chi_j(h_2)$.

First, for $|i|+|j|=0$ (in fact for $|i|+|j|$ bounded), the local density estimate and $\|F_{ij}^t\|_\infty=\mathrm{O}(r)$ give

$$N^2\mathbb{E}^\gamma \iint F_{ij}^t([z-w])\,\tilde\mu(\mathrm{d}z)\,\tilde\mu(\mathrm{d}w) = \mathrm{O}(N^\varepsilon)N^2r^3. \tag{C.24}$$

We now assume $|i|+|j|>0$. Let us also suppose $i,j\geqslant 0$ (the other cases are analogous).

For $2^i>t2^j$ (in other words $|h_1|>t|h_2|$), (C.23) yields

$$|\partial_{h_1}^{k_1}\partial_{h_2}^{k_2}F_{ij}^t(h)| \leqslant C_{k_1,k_2}\frac{2^ir}{\max(2^ir,2^jr)^{k_1+k_2}}.$$

The area of the support of F_{ij} is $\mathrm{O}(r^22^{i+j})$, so that Proposition B.3 in the form (B.5) gives

$$N^2\mathbb{E}^\gamma \iint F_{ij}^t([z-w])\,\tilde\mu(\mathrm{d}z)\,\tilde\mu(\mathrm{d}w) = \mathrm{O}(N^\varepsilon)\left(\frac{1}{\gamma^4}+\frac{1}{(\max(2^ir,2^jr))^4}\right)(2^ir)(r^22^{i+j}),$$

and after summation over i,j then

$$\sum_{2^i>t2^j} N^2\mathbb{E}^\gamma \iint F_{ij}^t([z-w])\,\tilde\mu(\mathrm{d}z)\,\tilde\mu(\mathrm{d}w) = \mathrm{O}(N^\varepsilon)\left(\frac{1}{\gamma^4}+\frac{1}{r}\right). \tag{C.25}$$

For $i>0$ and $2^i<t2^j$ ($|h_1|<t|h_2|$), from (C.23) we have

$$|\partial_{h_1}^{k_1}\partial_{h_2}^{k_2}F_{ij}^t(h)| \leqslant C_{k_1,k_2}\frac{2^ir}{(t2^jr)^{k_1+k_2}}.$$

The area of the support of F_{ij} is still of order r^22^{i+j}, so that Proposition B.3 now yields

$$N^2\mathbb{E}^\gamma \iint F_{ij}^t([z-w])\,\tilde\mu(\mathrm{d}z)\,\tilde\mu(\mathrm{d}w) = \mathrm{O}(N^\varepsilon)\left(\frac{1}{\gamma^4}+\frac{1}{(t2^jr)^4}\right)(2^ir)(r^22^{i+j}).$$

The contribution of such terms is therefore

$$\sum_{2^i<t2^j,i>0} N^2\mathbb{E}^\gamma \iint F_{ij}^t([z-w])\tilde\mu(\mathrm{d}z)\tilde\mu(\mathrm{d}w) = \mathrm{O}(N^\varepsilon)\left(\frac{1}{\gamma^4}+\frac{1}{rt}\right). \tag{C.26}$$

For $i=0$ and $t2^j r \geqslant N^{-1/2+\varepsilon}$ ($|th_2| \geqslant N^{-1/2+\varepsilon}$), we have from (C.23)

$$|\partial_{h_1}^{k_1}\partial_{h_2}^{k_2} F_{ij}^t(h)| \leqslant C_{k_1,k_2}\frac{r}{(t2^j r)^{k_1+k_2}}$$

so that Proposition B.3 gives

$$N^2\mathbb{E}^\gamma \iint F_{0j}^t([z-w])\,\tilde\mu(\mathrm{d}z)\,\tilde\mu(\mathrm{d}w) = \mathrm{O}(N^\varepsilon)\left(\frac{1}{\gamma^4}+\frac{1}{(t2^j r)^4}\right) r^3 2^j$$

and therefore

$$\sum_{N^{-1/2+\varepsilon}<t2^j r} N^2\mathbb{E}^\gamma \iint F_{0j}^t([z-w])\,\tilde\mu(\mathrm{d}z)\,\tilde\mu(\mathrm{d}w) = \mathrm{O}(N^\varepsilon)\left(\frac{1}{\gamma^4}+\frac{r^2N^{3/2}}{t}\right). \tag{C.27}$$

For the terms corresponding to $i=0$, $j>0$ and $t2^j r \leqslant N^{-1/2+\varepsilon}$, we need one more decomposition. Let $r' = N^{-1/2+\varepsilon} \ll r$, and decompose $F_{0j}^t = A_j + B_j$ where A_j is supported on $\{|h_1 < 2r'|\} \cap \{2^{j-1}r < h_2 < 2^{j+1}r\}$, $\|A_j\|_\infty \leqslant r'$, B_j is smooth, supported on $\{|h_1| < 2r\} \cap \{2^{j-1}r < h_2 < 2^{j+1}r\}$, satisfies $\|B_j\|_\infty \leqslant r$, and

$$|\partial_{h_1}^{k_1}\partial_{h_2}^{k_2} B_j(h)| \leqslant C_{k_1,k_2}\left(\frac{r}{|h_2|^{k_1+k_2}}+\frac{r}{(r')^{k_1+k_2}}\mathbb{1}_{|h_1|<r'}\right).$$

The function A_j is supported on a domain of area $\mathrm{O}(2^j r r')$ and $\|A\|_\infty \leqslant r'$, and the local density implied by Theorem A.2 gives

$$N^2\mathbb{E}^\gamma \iint A_j([z-w])\,\tilde\mu(\mathrm{d}z)\,\tilde\mu(\mathrm{d}w) = \mathrm{O}(N^{1+2\varepsilon}2^j r).$$

The contribution of all A_j terms is therefore

$$\sum_{j>0:\, t2^j r\leqslant N^{-1/2+\varepsilon}} N^2\mathbb{E}^\gamma \iint A_j([z-w])\,\tilde\mu(\mathrm{d}z)\,\tilde\mu(\mathrm{d}w) = \mathrm{O}\left(\frac{N^{1/2+3\varepsilon}}{t}\right). \tag{C.28}$$

For the contribution from B_j, consider the following partition of $\mathbb{T}$: $1 = \sum_{-1/(2r)\leqslant a,b\leqslant 1/(2r)} \chi_{ab}$ where χ_{ab} is supported on a disk of radius $10r$ around (ar, br), and $\|\chi_{ab}^{(n)}\|_\infty \leqslant C_n r^{-n}$. The contribution of B_j is of order at most

$$r^{-2}N^2\mathbb{E}^\gamma \iint \sum_{|a|\leqslant 5,2^{j-1}\leqslant b\leqslant 2^{j+1}} B_j([z-w])\chi_{00}(z)\chi_{ab}(w)\tilde\mu(\mathrm{d}z)\tilde\mu(\mathrm{d}w).$$

Let E be the event that all particles at distance $4r$ from 0 are known. Then

$$N^2\mathbb{E}^\gamma \iint B_j([z-w])\chi_{00}(z)\chi_{ab}(w)\tilde\mu(\mathrm{d}z)\tilde\mu(\mathrm{d}w) = \mathbb{E}^\gamma\mathbb{E}^\gamma\left(\int f(z)\chi_{00}(z)N\tilde\mu(\mathrm{d}z)\mid E\right)$$

where $f(z) = \int B([z-w]\chi_{ab}(w)N\tilde\mu(\mathrm{d}w)$. By the local law Theorem A.2, the set of E such that

$$\begin{aligned}
f(z) &= \mathrm{O}(Nr^3),\\
\nabla f(z) &= \int \nabla(B([z-w]\chi_{ab}(w))N\tilde\mu(\mathrm{d}w) = \mathrm{O}(Nr^2),\\
\Delta f(z) &= \int \Delta(B([z-w])\chi_{ab}(w))N\tilde\mu(\mathrm{d}w) = \mathrm{O}(Nr),
\end{aligned}$$

has measure at least $1-N^{-100}$. Using the (conditioned version of the) local law, for E in such a good set we therefore have

$$
\begin{aligned}
&|\mathbb{E}^{\gamma}\left(\int f(z)\chi_{00}(z)N\tilde{\mu}(\mathrm{d}z)\mid E\right)| \\
&\qquad\leqslant \left(Nr^2\left(\int|\nabla(f\chi_{00})|^2+\frac{1}{\gamma^2}\int(f\chi_{00})^2\right)\right)^{1/2}+N^{\varepsilon}r^2\|\Delta(f\chi_{00})\|_{\infty}=\mathrm{O}(N^{3/2}r^4).
\end{aligned}
$$

Hence the contribution of B_j is at most

$$
N^2\mathbb{E}^{\gamma}\iint B_j([z-w])\,\tilde{\mu}(\mathrm{d}z)\,\tilde{\mu}(\mathrm{d}w)=\mathrm{O}(N^{\varepsilon})2^jN^{3/2}r^2.
$$

All B_j terms are therefore bounded by

$$
\sum_{j>0:\,t2^jr\leqslant N^{-1/2+\varepsilon}} N^2\mathbb{E}^{\gamma}\iint B_j([z-w])\,\tilde{\mu}(\mathrm{d}z)\,\tilde{\mu}(\mathrm{d}w)=\mathrm{O}(N^{\varepsilon})\frac{Nr}{t}. \tag{C.29}
$$

Equations (C.24), (C.25), (C.26), (C.27), (C.28) and (C.29) show that $r=N^{-1/2+c}$ for arbitrarily small c is the best choice. We therefore proved

$$
N^2\mathbb{E}^{\gamma}\iint F_2^t([z-w])\,\tilde{\mu}(\mathrm{d}z)\,\tilde{\mu}(\mathrm{d}w)=\mathrm{O}(N^{\varepsilon})\left(\frac{1}{\gamma^4}+\frac{N^{1/2}}{t}\right). \tag{C.30}
$$

The contribution from F_3^t can be bounded following the same method, and the resulting estimate is smaller due to the extra small $|h_2|$ factor in f_3. Equations (C.22) and (C.30) complete the proof of (C.20) which, together with (C.19), concludes the proof of (C.2). □

Proof of (C.3). First note that, by (2.4) we have $U^{\gamma}(z-w)=Y^{\gamma}([z-w])+\mathrm{O}(\mathrm{e}^{-N^c})$, so that it will be sufficient to prove both of the following estimates:

$$
\mathbb{E}^{\Psi}\mathbb{E}^{\gamma}N^2\iint(Y^{\gamma}([z-w])-Y^{\gamma}([\Psi_z-\Psi_w]))\,\tilde{\mu}(\mathrm{d}z)\,\tilde{\mu}(\mathrm{d}w)=\mathrm{O}(N^{1+\varepsilon}t), \tag{C.31}
$$

$$
\mathbb{E}^{\Psi}\mathbb{E}^{\gamma}N^2\iint(Y^{\gamma}([\Psi_z-\Psi_w])-Y^{\gamma}(\Psi_z-\Psi_w))\,\tilde{\mu}(\mathrm{d}z)\,\tilde{\mu}(\mathrm{d}w)=\mathrm{O}(N^{\varepsilon})\left(\frac{1}{\gamma^4}+\frac{\sqrt{N}}{t}\right). \tag{C.32}
$$

The proof of (C.31) is strictly the same as (C.1), except that we compare Taylor expansions not only on $|z-w|<N^{\varepsilon}r$ but also on $|z-w+q|<N^{\varepsilon}r$, where $q=\pm1$ or $q=\pm\mathrm{i}$.

Equation (C.32) is proved analogously to (C.2). We first consider the averaging in the shift (a,b) from Ψ: denoting $|h|=(|h_1|,|h_2|)$, for any $h\in\mathbb{T}$ we have

$$
\begin{aligned}
&\mathbb{E}_{(a,b)}\left(Y^{\gamma}([[h+(a,b)]-(a,b)])\right)=Y^{\gamma}(|h|),\\
&\mathbb{E}_{(a,b)}\left(Y^{\gamma}([h+(a,b)]-(a,b))\right)=(1-|h_1|)(1-|h_2|)Y^{\gamma}(|h|)+\mathrm{O}(\mathrm{e}^{-N^c}),
\end{aligned}
$$

where the second equation comes from (C.6) and the fact that Y^{γ} is essentially supported on $|x|<N^{\varepsilon}\gamma$. Equation (C.32) is therefore equivalent to

$$
\mathbb{E}^{\gamma}N^2\iint K^t([z-w])\,\tilde{\mu}(\mathrm{d}z)\,\tilde{\mu}(\mathrm{d}w)=\mathrm{O}(N^{\varepsilon})\left(\frac{1}{\gamma^4}+\frac{\sqrt{N}}{t}\right)
$$

where $K^t(h) = \mathbb{E}^\Phi \int f(\Delta_v^{[v+h]})m(\mathrm{d}v)$ and $f(h) = (|h_1| + |h_2| - |h_1 h_2|)Y^\gamma(h)$. This K^t is a linear combination of terms of the same type as type as F_2^t and F_3^t defined in (C.21). Therefore, the same estimate as (C.30) holds: the estimate (C.15) was only used to bound F_1^t, not F_2^t and F_3^t. This concludes the proof. □

References

[1] M. Aizenman and P.A. Martin. Structure of Gibbs states of one-dimensional Coulomb systems. *Comm. Math. Phys.*, 78(1):99–116, 1980/81.

[2] A. Alastuey and B. Jancovici. On the classical two-dimensional one-component Coulomb plasma. *J. Physique*, 42(1):1–12, 1981.

[3] Y. Ameur. A density theorem for weighted Fekete sets, 2015. Preprint, arXiv:1510.08605.

[4] Y. Ameur, H. Hedenmalm, and N. Makarov. Fluctuations of eigenvalues of random normal matrices. *Duke Math. J.*, 159(1):31–81, 2011.

[5] Y. Ameur, H. Hedenmalm, and N. Makarov. Random normal matrices and Ward identities. *Ann. Probab.*, 43(3):1157–1201, 2015.

[6] R. Bauerschmidt, P. Bourgade, M. Nikula, and H.-T. Yau. Local density for two-dimensional one-component plasma, 2015. Preprint, arXiv:1510.02074.

[7] F. Bekerman and A. Lodhia. Mesoscopic central limit theorem for general β-ensembles, 2016. Preprint, arXiv:1605.05206.

[8] G. Borot and A. Guionnet. Asymptotic expansion of beta matrix models in the multi-cut regime, 2013. Preprint, arXiv:1303.1045.

[9] G. Borot and A. Guionnet. Asymptotic expansion of β matrix models in the one-cut regime. *Comm. Math. Phys.*, 317(2):447–483, 2013.

[10] P. Bourgade, L. Erdős, and H.-T. Yau. Universality of general β-ensembles. *Duke Math. J.*, 163(6):1127–1190, 2014.

[11] P. Bourgade, L. Erdős, H.-T. Yau, and J. Yin. Fixed energy universality for generalized Wigner matrices. *Commun. Pure Appl. Math.*, 2015.

[12] H.J. Brascamp and E.H. Lieb. *On Extensions of the Brunn-Minkowski and Prékopa-Leindler Theorems, Including Inequalities for Log Concave Functions, and with an Application to the Diffusion Equation*, pages 441–464. Springer Berlin Heidelberg, Berlin, Heidelberg, 2002.

[13] D.C. Brydges and P. Federbush. Debye screening. *Comm. Math. Phys.*, 73(3):197–246, 1980.

[14] D.C. Brydges and P.A. Martin. Coulomb systems at low density: a review. *J. Statist. Phys.*, 96(5-6):1163–1330, 1999.

[15] J.M. Caillol, D. Levesque, J.J. Weis, and J.P. Hansen. A Monte Carlo study of the classical two-dimensional one-component plasma. *Journal of Statistical Physics*, 28(2):325–349, 1982.

[16] L.-L. Chau and Y. Yu. Unitary polynomials in normal matrix models and wave functions for the fractional quantum Hall effects. *Phys. Lett. A*, 167(5-6):452–458, 1992.

[17] J.G. Conlon, E.H. Lieb, and H.-T. Yau. The $N^{7/5}$ law for charged bosons. *Comm. Math. Phys.*, 116(3):417–448, 1988.

[18] J.G. Conlon, E.H. Lieb, and H.-T. Yau. The Coulomb gas at low temperature and low density. *Comm. Math. Phys.*, 125(1):153–180, 1989.

[19] B. Eynard. Topological expansion for the 1-Hermitian matrix model correlation functions. *J. High Energy Phys.*, (11):031, 35 pp. (electronic) (2005), 2004.

[20] C. Fefferman and R. de la Llave. Relativistic stability of matter. I. *Rev. Mat. Iberoamericana*, 2(1-2):119–213, 1986.

[21] P.J. Forrester. *Log-gases and random matrices*, volume 34 of *London Mathematical Society Monographs Series*. Princeton University Press, Princeton, NJ, 2010.

[22] J. Ginibre. Statistical ensembles of complex, quaternion, and real matrices. *J. Mathematical Phys.*, 6:440–449, 1965.

[23] D.P. Hardin, E.B. Saff, B.Z. Simanek, and Y. Su. Next order energy asymptotics for Riesz potentials on flat tori, 2015. Preprint, arXiv:1511.01552.

[24] H. Hedenmalm and N. Makarov. Coulomb gas ensembles and Laplacian growth. *Proc. Lond. Math. Soc. (3)*, 106(4):859–907, 2013.

[25] J.Z. Imbrie. Debye screening for jellium and other Coulomb systems. *Comm. Math. Phys.*, 87(4):515–565, 1982/83.

[26] B. Jancovici, J.L. Lebowitz, and G. Manificat. Large charge fluctuations in classical Coulomb systems. *J. Statist. Phys.*, 72(3-4):773–787, 1993.

[27] K. Johansson. On Szegő's asymptotic formula for Toeplitz determinants and generalizations. *Bull. Sci. Math. (2)*, 112(3):257–304, 1988.

[28] K. Johansson. On fluctuations of eigenvalues of random Hermitian matrices. *Duke Math. J.*, 91(1):151–204, 1998.

[29] H. Kunz. The one-dimensional classical electron gas. *Annals of Physics*, 85(2):303–335, 1974.

[30] R.B. Laughlin. Anomalous quantum hall effect: An incompressible quantum fluid with fractionally charged excitations. *Phys. Rev. Lett.*, 50:1395–1398, May 1983.

[31] T. Leblé. Local microscopic behavior for 2D Coulomb gases, 2015. Preprint, arXiv:1510.01506.

[32] T. Leblé and S. Serfaty. Fluctuations of two-dimensional Coulomb gases. Preprint.

[33] T. Leblé and S. Serfaty. Large Deviation Principle for Empirical Fields of Log and Riesz Gases, 2015. Preprint, arXiv:1502.02970.

[34] E.H. Lieb, N. Rougerie, and J. Yngvason. Rigidity of the laughlin liquid, 2016. Preprint, arXiv:1609.03818.

[35] M. Petrache and S.R. Nodari. Equidistribution of jellium energy for Coulomb and Riesz interactions, 2016. Preprint, arXiv:1609.03849.

[36] B. Rider. Deviations from the circular law. *Probab. Theory Related Fields*, 130(3):337–367, 2004.

[37] B. Rider and B. Virág. The noise in the circular law and the Gaussian free field. *Int. Math. Res. Not. IMRN*, (2):Art. ID rnm006, 33, 2007.

[38] N. Rougerie and S. Serfaty. Higher-dimensional Coulomb gases and renormalized energy functionals. *Comm. Pure Appl. Math.*, 69(3):519–605, 2016.

[39] N. Rougerie and J. Yngvason. Incompressibility estimates for the Laughlin phase. *Comm. Math. Phys.*, 336(3):1109–1140, 2015.

[40] E.B. Saff and V. Totik. *Logarithmic potentials with external fields*, volume 316 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1997. Appendix B by Thomas Bloom.

[41] E. Sandier and S. Serfaty. 2D Coulomb gases and the renormalized energy. *Ann. Probab.*, 43(4):2026–2083, 2015.

[42] S. Serfaty. *Coulomb gases and Ginzburg-Landau vortices*. Zurich Lectures in Advanced Mathematics. European Mathematical Society (EMS), Zürich, 2015.

[43] M. Shcherbina. Fluctuations of linear eigenvalue statistics of β matrix models in the multi-cut regime. *J. Stat. Phys.*, 151(6):1004–1034, 2013.

[44] A. Soshnikov. The central limit theorem for local linear statistics in classical compact groups and related combinatorial identities. *Ann. Probab.*, 28(3):1353–1370, 2000.

[45] P. Wiegmann and A. Zabrodin. Large scale correlations in normal non-Hermitian matrix ensembles. *J. Phys. A*, 36(12):3411–3424, 2003. Random matrix theory.

[46] A. Zabrodin. Random matrices and Laplacian growth. In *The Oxford handbook of random matrix theory*, pages 802–823. Oxford Univ. Press, Oxford, 2011.

[47] A. Zabrodin and P. Wiegmann. Large-N expansion for the 2D Dyson gas. *J. Phys. A*, 39(28):8933–8963, 2006.