Lecture 15

Finishing Continuous Distributions

Review Exercises

1. Suppose you are able to generate draws from a Unif(0, 1) distribution. In other words, you have some procedure in a programming language that will give values that are uniformly distributed on (0, 1). How can you use this procedure to generate draws from an Exp(\( \lambda \)) distribution?

2. We saw that a Cauchy distribution has
   \[ f_X(x) = \frac{C}{1 + x^2} \]
   for some \( C \). What is \( C \)?

3. What is the expectation of a Cauchy random variable?

4. Let \( X \sim \mathcal{N}(\mu, \sigma^2) \).
   (a) What is the PDF of \( Y = e^X \)?
   (b) Compute \( E[Y] \).

Solutions

1. Recall the CDF of an Exp(\( \lambda \)) distribution is
   \[ F(x) = 1 - e^{-\lambda x} \]
   for \( x \geq 0 \). Inverting we have
   \[ F^{-1}(y) = -\frac{\log(1 - y)}{\lambda}. \]
   Thus if \( X \sim \text{Unif}(0, 1) \) then
   \[ -\frac{\log(1 - X)}{\lambda} \sim \text{Exp}(\lambda). \]
   We take each value generated by our procedure, and plug it into \( F^{-1}(y) \) to get our draws.
2. Note that
\[ 1 = \int_{-\infty}^{\infty} \frac{C}{1 + x^2} \, dx = C [\arctan(x)]_{-\infty}^{\infty} = C \pi, \]
so \( C = 1/\pi \).

3. We will show the expectation doesn’t exist. To see this, note that
\[ \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{|x|}{1 + x^2} \, dx = \frac{1}{\pi} \int_{0}^{\infty} \frac{2x}{1 + x^2} \, dx = \frac{1}{\pi} \int_{1}^{\infty} \frac{du}{u} = \frac{[\log(u)]_{1}^{\infty}}{\pi} = \infty. \]

4. (a) Let \( g(x) = e^x \) so that \( Y = g(X) \) and, for \( y > 0 \),
\[ f_Y(y) = f_X(\log(y)) \cdot \frac{1}{y} = \frac{1}{y \sqrt{2\pi \sigma^2}} e^{-\frac{1}{2} \left( \frac{\log(y) - \mu}{\sigma} \right)^2}. \]

(b) By LOTUS we have
\[
E[Y] = \frac{1}{\sqrt{2\pi \sigma^2}} \int_{-\infty}^{\infty} e^x e^{-\frac{1}{2} \left( \frac{x - \mu}{\sigma} \right)^2} \, dx \\
= \frac{1}{\sqrt{2\pi \sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2} \left( x^2 + \mu^2 - 2x\mu - 2x^2 - 2\sigma^2 \right)} \, dx \\
= \frac{1}{\sqrt{2\pi \sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2} \left( (x+\sigma^2)^2 + \frac{x^2}{2} + \mu \right)} \, dx \\
= e^{\frac{\sigma^2}{2} + \mu}.
\]

**Multivariate Distributions**

Up to this point we have spent our time studying a variety of different distributions, both discrete and continuous. In this section we look at joint distributions which show not only how random variables behave, but how they are related. We had a preview of this topic when looking at independent random variables.

**Multivariable Calculus Review**

**Integrating over Regions Exercises**

1. Let \( f : \mathbb{R}^2 \to \mathbb{R} \) be defined as follows:
\[
f(x, y) = \begin{cases} 
1 & \text{if } 0 \leq x \leq y \leq 1, \\
0 & \text{otherwise}.
\end{cases}
\]
What is \( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \, dx \, dy \)?
2. Consider the function \( f(x, y) = 4 - (x^2 + y^2) \). The graph is a surface of revolution of a parabola.

(a) Let \( R = \{(x, y) : 0 \leq x, y \leq 1\} \). What is \( \iint_R f(x, y) \, dx \, dy \)?

(b) Let \( S = \{(x, y) : 0 \leq x^2 + y^2 \leq 4\} \). What is \( \iint_S f(x, y) \, dx \, dy \)?
(c) Let $T$ be the triangular region in the $xy$-plane enclosed by the lines $y = 1 - x$, $x = 0$, and $y = 0$. What is $\iint_T f(x, y) \, dx \, dy$?
3. Let $f(x, y) = e^{-(x+y)}$.

(a) Let $g$ be defined by

$$g(x, y) = \begin{cases} 1 & \text{if } 0 \leq x, y \leq 2, \\ 0 & \text{otherwise}. \end{cases}$$

What is $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) g(x, y) \, dx \, dy$?

(b) Let $R = \{(x, y) : 0 \leq x, y \leq 1\}$ and let $T = 2R = \{(x, y) : 0 \leq x, y \leq 2\}$. Which of the following two integrals is bigger:

$$\int_{R} f(x, y) \, dx \, dy \quad \text{or} \quad \int_{T} f(x/2, y/2) \, dx \, dy?$$
(c) Let $R = \{(x, y) : 1 \leq x, y \leq 4\}$ and $T = \{(x, y) : 1 \leq x, y \leq 2\}$. Which of the following two integrals is bigger:

$$\int_R f(x/10, y/10) \, dx \, dy \quad \text{or} \quad \int_T f(x^2/10, y^2/10) \, dx \, dy?$$

4. Let $g(x, y) = x^2 + 2xy + y^3$ and let $f$ be defined by

$$f(x, y) = \int_{-\infty}^{x} \int_{-\infty}^{y} e^{-\left(2t^2 + u^2\right)} \, dt \, du.$$

(a) What are $\frac{\partial g}{\partial x}(x, y)$ and $\frac{\partial g}{\partial y}(x, y)$?

(b) What are $\frac{\partial f}{\partial x}(x, y)$ and $\frac{\partial f}{\partial y}(x, y)$?

(c) What is $\frac{\partial^2 f}{\partial x^2}(x, y)$?

Solutions

1. By inspection, we can see the volume under the surface is $1/2$. More precisely,

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \, dx \, dy = \int_{0}^{1} \int_{x}^{1} 1 \, dy \, dx$$

$$= \int_{0}^{1} [y]_{x}^{1} \, dx$$

$$= \int_{0}^{1} 1 - x \, dx$$

$$= [x - x^2/2]_{0}^{1}$$

$$= 1/2.$$
2. (a) Integrating we have

\[ \int_0^1 \int_0^1 4 - (x^2 + y^2) \, dx \, dy = \int_0^1 \left[ 4x - \frac{x^3}{3} - xy^2 \right]_0^1 \, dy \]
\[ = \int_0^1 4 - \frac{1}{3} - y^2 \, dy \]
\[ = \left[ \frac{11y}{3} - \frac{y^3}{3} \right]_0^1 \]
\[ = \frac{10}{3}. \]

(b) If we change to polar coordinates, we have

\[ \int \int_{S} 4 - (x^2 + y^2) \, dx \, dy = \int_{0}^{2\pi} \int_{0}^{2} (4 - r^2) r \, dr \, d\theta \]
\[ = \int_{0}^{2\pi} \left[ 4r^2 - \frac{r^4}{4} \right]_0^2 \, d\theta \]
\[ = \int_{0}^{2\pi} 8 - 4 \, d\theta \]
\[ = 8\pi. \]

Note above that we converted \((x, y)\) into \((r, \theta)\). This is the multivariate form of a \(u\)-substitution. To understand how this works, let \(T : (0, \infty) \times (0, 2\pi) \rightarrow \mathbb{R}^2\) be defined by

\[ T(r, \theta) = (r \cos \theta, r \sin \theta) = (x, y). \]

Then we write the Jacobian:

\[ \frac{d(x, y)}{d(r, \theta)} = \det \begin{pmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{pmatrix} = r \cos^2 \theta + r \sin^2 \theta = r. \]

This leads to

\[ \int \int_{S} f(x, y) \, dx \, dy = \int_{0}^{2\pi} \int_{0}^{2} f(T(r, \theta)) \left| \frac{d(x, y)}{d(r, \theta)} \right| \, dr \, d\theta \]

as we did above.

Alternatively, we can try to integrate over cartesian coordinates:

\[ \int \int_{S} 4 - (x^2 + y^2) \, dx \, dy = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 4 - x^2 - y^2 \, dy \, dx, \]

but the resulting integral is much harder to evaluate.
(c) Integrating we have
\[
\int_{0}^{1} \int_{0}^{1-x} 4 - x^2 - y^2 \, dy \, dx = \int_{0}^{1} \left[ 4y - x^2y - \frac{y^3}{3} \right]_{0}^{1-x} \, dx
\]
\[
= \int_{0}^{1} 4(1-x) - x^2(1-x) - \frac{(1-x)^3}{3} \, dx
\]
\[
= \int_{0}^{1} 4 - 4x - x^2 - x^3 - \frac{1}{3} + x - x^2 + \frac{x^3}{3} \, dx
\]
\[
= \left[ \frac{11x}{3} - \frac{3x^2}{2} - \frac{2x^3}{3} + \frac{x^4}{12} \right]_{0}^{1}
\]
\[
= 22 - 9 - 4 + 2
\]
\[
= \frac{11}{6}.
\]

3. (a)
\[
\int_{0}^{2} \int_{0}^{2} e^{-(x+y)} \, dx \, dy = \int_{0}^{2} e^{-x} \, dx \int_{0}^{2} e^{-y} \, dy = (1 - e^{-2})^2.
\]
(b) The second is larger. If we let \((u, v) = G(x, y) = (x/2, y/2)\) then we have
\[
\int_{T} f(u, v) \, du \, dv = \int_{T} f(G(x, y)) \left| \frac{d(u, v)}{d(x, y)} \right| \, dx \, dy = \int_{T} f(x/2, y/2) \cdot \frac{1}{4} \, dx \, dy.
\]
(c) The first is larger. If we let \((u, v) = G(x, y) = (x^2, y^2)\) then we have
\[
\int_{T} f(u/10, v/10) \, du \, dv = \int_{T} f(G(x, y)/10) \left| \frac{d(u, v)}{d(x, y)} \right| \, dx \, dy = \int_{T} f(x^2/10, y^2/10)4xy \, dx \, dy.
\]

4. (a) We have
\[
\frac{\partial g}{\partial x}(x, y) = 2x + 2y \quad \text{and} \quad \frac{\partial g}{\partial y}(x, y) = 2x + 3y^2.
\]
(b) We have
\[
\frac{\partial f}{\partial x}(x, y) = \int_{-\infty}^{y} e^{-(2t^2 + x^2)} \, dt \quad \text{and} \quad \frac{\partial f}{\partial y}(x, y) = \int_{-\infty}^{x} e^{-(2y^2 + u^2)} \, du.
\]
(c) We have
\[
\frac{\partial^2 f}{\partial x \partial y}(x, y) = e^{-(2y^2 + x^2)}.
\]