Homework 14

Due Monday, August 3rd at the beginning of class

1. Suppose $X$ has PDF $f_X$ given by

$$f_X(x) = \begin{cases} \alpha x^\alpha / x_0^{\alpha+1} & \text{if } x \geq x_0, \\ 0 & \text{if } x < x_0, \end{cases}$$

where $x_0 > 0$ and $\alpha > 0$ are given fixed parameters. What is the distribution of
$log(X/x_0)$? Give PDF and the name of the distribution.

Solution. Let $Y = g(X)$ where $g(x) = \log(x/x_0)$. Since $x \geq x_0$ we see that $g(x)$ takes
values in $[0, \infty)$ on which $g^{-1}(y) = x_0 e^y$. Then we have, for $y \geq 0$,

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{1}{g'(g^{-1}(y))} \right| = \frac{\alpha x_0^\alpha}{(x_0 e^y)^{\alpha+1}} e^y = \frac{\alpha e^{-(\alpha+1)y}}{e^y} = \alpha e^{-\alpha y}.$$

Thus $Y \sim \text{Exp}(\alpha)$.

2. Suppose the temperature in a location (measured in Fahrenheit) is normally distributed
with mean 68 degrees and a standard deviation of 4 degrees. What is the distribution
of the temperature measured in Celsius?

Solution. Let $X$ denote the temperature in Fahrenheit so that $X \sim \mathcal{N}(68, 16)$. Then
$Y$, the temperature in Celsius, is given by

$$Y = \frac{5(X - 32)}{9}.$$ 

Thus $Y \sim \mathcal{N}(20, 400/81)$ (i.e., mean is 20, variance is 400/81 and standard deviation is
20/9).

3. Let $Y$ have a lognormal distribution with parameters 3 and 1.44 (i.e., $Y = e^X$ where
$X \sim \mathcal{N}(3, 1.44)$). Compute $P(Y \leq 6.05)$ approximately. [Hint: Don’t use the lognormal PDF/CDF.]
Solution. Since $\sigma_X = \sqrt{1.44} = 1.2$ we have

$$P(Y \leq 6.05) \approx P(X \leq 1.8) = P(X \leq \mu_X - \sigma_x) \approx 0.16.$$  

4. Given that $X \sim \text{Exp}(\lambda)$, compute $E[e^{-(X-\lambda/2)^2}]$. Your answer should not be left as an integral.

Solution. Using LOTUS we have

$$E[e^{-(X-\lambda/2)^2}] = \int_0^\infty \lambda e^{-(x-\lambda/2)^2} e^{-\lambda x} \, dx$$

$$= \int_0^\infty \lambda e^{-x^2 - \lambda^2/4} \, dx$$

$$= \lambda e^{-\lambda^2/4} \int_0^\infty e^{-x^2} \, dx$$

$$= \lambda e^{-\lambda^2/4} \cdot \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{1/2} \cdot \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{1/2} \cdot \frac{1}{\sqrt{2\pi}}$$

$$= \frac{\lambda e^{-\lambda^2/4}}{2} \sqrt{\pi},$$

since the rest is the integral of a $\mathcal{N}(0,1/2)$ PDF.

5. Let $f(x, y) = x^2 y$ for $0 \leq x^2 \leq y \leq 1$, and 0 otherwise.

(a) Compute $\int_{\infty}^{\infty} \int_{\infty}^{\infty} f(x, y) \, dx \, dy$. [Hint: Integrate over region below.]

(b) Compute $\int \int_R f(x, y) \, dx \, dy$ where $R = \{(x, y) : x \geq y\}$. [Hint: Integrate over region below.]
Solution.

(a) We have
\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \, dx \, dy = \int_{-1}^{1} \int_{x^2}^{1} x^2 y \, dy \, dx \\
= \int_{-1}^{1} x^2 \left( \frac{1}{2} - \frac{x^4}{2} \right) \, dx \\
= \left[ \frac{x^3}{6} - \frac{x^7}{14} \right]_{-1}^{1} \\
= \frac{8}{42}.
\]

(b) We have
\[
\int_{0}^{1} \int_{x^2}^{x} x^2 y \, dy \, dx = \int_{0}^{1} x^2 \left( \frac{x^2}{2} - \frac{x^4}{2} \right) \, dx \\
= \left[ \frac{x^5}{10} - \frac{x^7}{14} \right]_{0}^{1} \\
= \frac{1}{35}.
\]

6. Let \( f(x, y) = y^2 \) for \((x, y) \in [0, 2] \times [0, 1]\) and 0 otherwise. Compute \( \iint_{R} f(x, y) \, dx \, dy \) where \( R \) is given by:

(a) \( R = \{(x, y) : x + y > 2\} \).
(b) \( R = \{(x, y) : y < 1/2\} \).
(c) \( R = \{(x, y) : x \leq 1\} \).
(d) \( R = \{(x, y) : x = 3y\} \). [Hint: What is the volume under the graph of \( f \) when you are integrating over a line?]

Solution.
(a) 
\[
\int_1^2 \int_{2-x}^1 y^2 \, dy \, dx = \int_1^2 \frac{1}{3} - \frac{(2-x)^3}{3} \, dx \\
= \frac{1}{3} - \int_1^2 \frac{u^3}{3} \, du \\
= \frac{1}{3} - \frac{1}{12} \\
= \frac{1}{4}.
\]

You could have also used the other order of integration to get the same answer:
\[
\int_0^1 \int_{2-y}^2 y^2 \, dx \, dy.
\]

(b) 
\[
\int_0^2 \int_0^{1/2} y^2 \, dy \, dx = \int_0^2 \frac{1}{24} \, dx = \frac{1}{12}.
\]

(c) 
\[
\int_0^1 \int_0^1 y^2 \, dx \, dy = \int_0^1 \frac{1}{3} \, dx = \frac{1}{3}.
\]

(d) The integral is zero.