Theory of Probability - Brett Bernstein

Homework 1 Solutions

Due Tuesday, July 7th at the beginning of class

1. Fix $a_0, d \in \mathbb{R}$ and let $a_k = a_{k-1} + d$ for all $k \geq 1$. Show that $\frac{1}{n} \sum_{k=0}^{n-1} a_k$ is given by $(a_{n-1} + a_0)/2$.

   Solution. Since $a_k = a_0 + kd$ and $a_{n-1} = a_0 + (n-1)d$ we have
   
   \[
   \sum_{k=0}^{n-1} a_k = \sum_{k=0}^{n-1} a_0 + kd = na_0 + d \sum_{k=0}^{n-1} k = na_0 + \frac{dn(n-1)}{2}.
   \]
   
   Note that
   
   \[
   \frac{a_0 + a_{n-1}}{2} = \frac{a_0 + a_0 + (n-1)d}{2} = a_0 + \frac{(n-1)d}{2},
   \]
   
   so dividing our above result by $n$ gives the answer.

2. Let $S = \{1, 2, \ldots, n\}$.

   (a) By counting the number of subsets of $S$ in two ways, prove that

   \[
   2^n = \sum_{k=0}^{n} \binom{n}{k}.
   \]

   (b) How many ways are there to choose 2 disjoint subsets $A, B$ from $S$? For example, if $n = 10$ then some distinct choices are:

   \[
   A = \{1, 3\}, \quad B = \{4\}
   \]

   \[
   A = \{4\}, \quad B = \{1, 3\}
   \]

   \[
   A = \{1, 2, 3\}, \quad B = \emptyset
   \]

   \[
   A = \emptyset, \quad B = \emptyset
   \]

   \[
   A = \emptyset, \quad B = \{1, 2, \ldots, 10\}.
   \]
Solution.

(a) Each element of $S$ is either in our subset or not. Thus we have 2 options, and we must make this choice $n$ times giving $2^n$ possible subsets. Alternatively, we can sum over the possible sizes of the subset. For a given size $k$, there are $\binom{n}{k}$ subsets of that size.

Another way to solve this problem is to apply the binomial theorem to $(1 + 1)^n$.

(b) Here each element is in $A$, $B$, or neither. Since we have 3 options and must make this choice $n$ times, the solution is $3^n$.

3. You have a team of 15 (distinguishable) players and must choose which 11 will play. There are two kinds of roles ($A,B$) for each player chosen. Seven will play role $A$, and four will play role $B$.

(a) If 8 are able to play role $A$ and 7 are able to play role $B$, how many ways are there to choose who will play each role?

(b) If 8 are able to play role $A$, 5 are able to play role $B$ and 2 can play either, how many ways are there to choose who will play each role?

Solution.

(a) $\binom{8}{5} \binom{7}{4}$

(b) We can sum over the ways of assigning the 2 special people to roles. Each can play either role $A$, role $B$, or not play at all. Here we count the number of ways to choose roles in each case.

i. Both play role $A$: $\binom{8}{5} \binom{5}{5}$.

ii. Both play role $B$: $\binom{8}{5} \binom{5}{5}$.

iii. First plays role $A$, second plays role $B$ or vice-versa: $2 \binom{8}{5} \binom{5}{3}$.

iv. First plays role $A$ and second doesn’t play, or vice-versa: $2 \binom{8}{5} \binom{5}{4}$.

v. First plays role $B$ and second doesn’t play, or vice-versa: $2 \binom{8}{7} \binom{5}{3}$.

vi. Neither play: $\binom{8}{7} \binom{5}{5}$.

Thus the final answer is

$\binom{8}{5} \binom{5}{4} + \binom{8}{5} \binom{5}{2} + 2 \binom{8}{6} \binom{5}{3} + 2 \binom{8}{6} \binom{5}{4} + 2 \binom{8}{7} \binom{5}{3} + \binom{8}{7} \binom{5}{4}$. 

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