Homework 3 Solutions

Due Thursday, July 9th at the beginning of class

1. Suppose you are randomly dealt 7 cards from a standard 52 card deck without replacement.

(a) Give a sample space and probability measure.

(b) What is the probability of getting 4 of one value and 3 of another?

(c) What is the probability at least 5 of them have the same suit?

(d) (⋆) What is the probability at least 3 cards have the same value?

(e) What is the probability of getting 3 distinct pairs, while not getting 3-of-a-kinds (i.e., can’t get 3 of the same value)?

Solution. Let $D$ denote the set of all cards in the deck.

(a) $S = \{B : B \subseteq D, |B| = 7\}$ and let $P$ be determined by treating every outcome as equally likely. Alternatively, we could use

$$S = \{(d_1, d_2, \ldots, d_7) : d_i \in D, d_i \text{ are distinct}\}.$$ 

(b) $\frac{13 \cdot 12 \cdot \binom{4}{3}}{\binom{52}{7}} = \frac{13 \cdot 12 \cdot 4}{\binom{52}{7}}.$

If you used ordered hands, you would get the same probability but it would be written slightly differently (this goes for the rest of the solutions as well).

(c) The number of hands with at least 5 of the same suit is

$$4 \binom{13}{5} \binom{39}{2} + 4 \binom{13}{6} \binom{39}{1} + 4 \binom{13}{7}$$

thus the answer is

$$4\binom{13}{5} \binom{39}{2} + 4\binom{13}{6} \binom{39}{1} + 4\binom{13}{7}.$$

(d) Let $A_i$ denote the event that at least 3 cards have value $i$, and assume the values are 1, \ldots, 13. Then we have, by inclusion-exclusion,

$$P\left(\bigcup_{i=1}^{13} A_i\right) = \sum_{i=1}^{13} P(A_i) - \sum_{i<j} P(A_iA_j) + \sum_{i<j<k} P(A_iA_jA_k) + \cdots.$$
Note that it is impossible to have more than 2 values with at least 3 cards, so only the first two sums are non-zero. Computing the parts we have

\[ P(A_i) = \binom{4}{3} \binom{48}{4} + \binom{3}{3} \binom{48}{4} \binom{52}{7} \]

and

\[ P(A_iA_j) = \binom{4}{3}^2 \binom{44}{4} + 2 \binom{4}{4} \binom{48}{3} \binom{52}{7} \]

Thus the answer is

\[ 13 \binom{4}{3} \binom{48}{4} + \binom{3}{3} \binom{48}{4} - \left( 13 \right)^2 \binom{4}{3} \binom{44}{4} + 2 \binom{4}{4} \binom{48}{3} \binom{52}{7} \]

(e) \[ \frac{10 \binom{13}{3} \binom{4}{3} \binom{1}{4}}{\binom{52}{7}} \]

2. Suppose you roll a 4-sided die once, and then flip a coin 4 times that is heads with probability \( k/5 \) where \( k \) is the roll of the die. Given that you get 3 heads, what is the probability you rolled a 3?

Solution. Let \( A \) denote the event of rolling a 3, and let \( B \) denote the event of getting 3 heads. Then we have

\[ P(B) = \sum_{k=1}^{4} \frac{1}{4} \binom{4}{3} \left( \frac{k}{5} \right)^3 \left( 1 - \frac{k}{5} \right) \]

\[ P(A \cap B) = P(B|A)P(A) = \binom{4}{3} \left( \frac{3}{5} \right)^3 \left( \frac{2}{5} \right) \frac{1}{4} \]

\[ P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\binom{3}{3} \left( \frac{3}{5} \right)^3 \left( 1 - \frac{k}{5} \right) \frac{1}{4}}{\frac{1}{4} \sum_{k=1}^{4} \binom{4}{3} \left( \frac{k}{5} \right)^3 \left( 1 - \frac{k}{5} \right)} \]

\[ = \frac{\sum_{k=1}^{4} \binom{3}{3} \left( \frac{k}{5} \right)^3 \left( 1 - \frac{k}{5} \right)}{27} \]

\[ = \frac{27}{73} \]