Homework 7
Due Thursday, July 16th at the beginning of class

1. Depicted below is the PMF \( p_X \) for a random variable \( X \):

\[
\begin{array}{c|c|c|c|c|c|c|c|c}
\text{y} & 1 & 0.5 & \text{y} & \text{−1} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\hline
\text{x} & & & 0.5 & \text{y} & \text{−1} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7
\end{array}
\]

(a) Draw the CDF \( F_X \).
(b) What is \( E[X] \)?
(c) If \( g(x) = e^{(x-2)^2} \), what is \( E[g(X)] \)?

Solution.
(a) CDF:

\[
\begin{array}{c|c|c|c|c|c|c|c|c}
\text{y} & 1 & 0.5 & \text{y} & \text{−1} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\hline
\text{x} & & & & & 0.5 & \text{y} & \text{−1} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7
\end{array}
\]

(b) We have

\[
E[X] = 1 \cdot p_X(1) + 3 \cdot p_X(3) + 6 \cdot p_X(6) = .5 + 3(.25) + 6(.25) = \frac{11}{4} = 2.75.
\]

(c) Letting \( Y = g(X) \) we have

\[
E[X] = e \cdot p_Y(e) + e^{16} \cdot p_Y(e^{16}) = e(.75) + e^{16}(.25).
\]

2. We have an urn containing 20 black balls and 10 white balls.
(a) Suppose we draw 7 balls from the urn with replacement. Compute \( p_X(x) \) for all \( x \in \mathbb{R} \) where \( X \) is the number of black balls drawn.

(b) Suppose instead we draw 7 balls from the urn without replacement. Compute \( p_Y(y) \) for all \( y \in \mathbb{R} \) where \( Y \) is the number of black balls drawn.

**Solution.**

(a) If \( k \notin \{0, 1, \ldots, 7\} \) we have \( p_X(k) = 0 \). Otherwise,

\[
p_X(k) = \binom{7}{k} \left( \frac{2}{3} \right)^k \left( \frac{1}{3} \right)^{n-k}.
\]

(b) Again, for \( k \notin \{0, 1, \ldots, 7\} \) we have \( p_Y(k) = 0 \). Otherwise,

\[
p_Y(k) = \frac{\binom{20}{k} \binom{10}{7-k}}{\binom{30}{7}}.
\]

3. \( \star \) Prove the following theorem.

**Theorem 1.** Let \( X \) be a discrete random variable that only takes positive integer values (i.e., \( \text{im}(X) \subset \mathbb{Z}_{>0} \)). Then we have

\[
E[X] = \sum_{k=1}^{\infty} P(X \geq k).
\]

[Hint: Use the technique of introducing an inner summation, and then swapping summations like our first computation of \( \sum \frac{k}{2^k} \).]

**Solution.**

**Proof.**

\[
\sum_{k=1}^{\infty} P(X \geq k) = \sum_{k=1}^{\infty} \sum_{j=k}^{\infty} p_X(j) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} p_X(j) = \sum_{j=1}^{\infty} j p_X(j) = E[X].
\]