Random projections

Optimization-Based Data Analysis

http://www.cims.nyu.edu/~cfgranda/pages/OBDA_spring16

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Introduction

Random projections in data analysis and signal processing

They preserve information embedded in low-dimensional subspaces of high-dimensional spaces

Non-adaptive compression, agnostic to specific data

Dimensionality reduction

- Principal component analysis
- Random projections

Compressed sensing

- Motivation: Magnetic resonance imaging
- Exact recovery
- Robustness

Sampling

- Nyquist-Shannon sampling theorem
- Compressive sampling

Dimensionality reduction

Projection of data onto lower-dimensional space

- Decreases computational cost of processing the data
- Allows to visualize (2D, 3D)

We will focus on linear projections

Linear projection

The linear projection of $x \in \mathbb{R}^n$ onto a subspace $\mathcal{S} \subseteq \mathbb{R}^n$ of dimension $m \leq n$ is the solution to

$$\begin{aligned} \text{minimize} \quad & \|x - u\|_2 \\ \text{subject to} \quad & u \in \mathcal{S} \end{aligned}$$

If the columns of U: $U_1, \ldots, U_m$ are an orthonormal basis of $\mathcal{S}$

$$\mathcal{P}_{\mathcal{S}}(x) = \sum_{i=1}^{m} \langle x, U_i \rangle U_i = UU^T x$$

To reduce the dimension we represent the signal using the coefficients

$$c := U^T x \in \mathbb{R}^m$$

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Adaptive projection

Data: $\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_k$

Preprocessing: Centering the data

$$x_i = \tilde{x}_i - \frac{1}{k} \sum_{i=1}^{k} \tilde{x}_i$$

Aim: Find directions of maximum variation

Principal component analysis (PCA)

1. Group the centered data in a data matrix X

$$X = \begin{bmatrix} x_1 & x_2 & \cdots & x_k \end{bmatrix}$$

2. Compute the SVD of $X = U \Sigma V^T$

3. Extract the first m left singular vectors

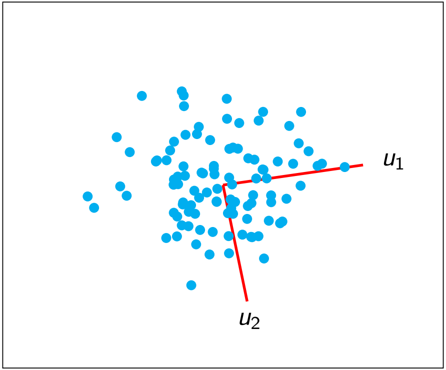
$$\widehat{U} = \begin{bmatrix} U_1 & \cdots & U_m \end{bmatrix}$$

For any n-dimensional subspace $\mathcal{S}'$

$$\sum_{i=1}^{k} ||\mathcal{P}_{\mathcal{S}'} x_i||_2^2 \leq \sum_{i=1}^{k} \left|\left| \widehat{U} \widehat{U}^T x_i \right|\right|_2^2$$

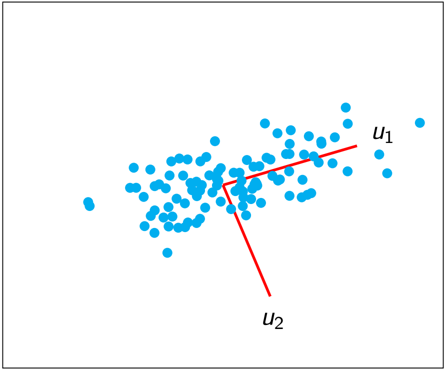
Example: 2D data

$$\frac{\sigma_1}{\sqrt{k}} = 0.705 \qquad \frac{\sigma_2}{\sqrt{k}} = 0.690$$



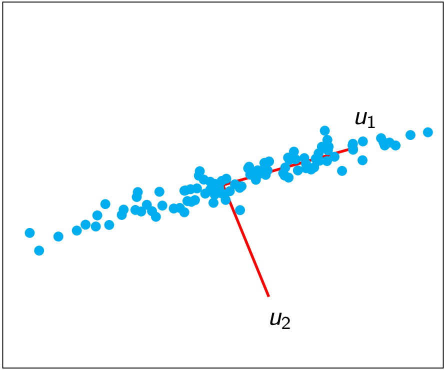
Example: 2D data

$$\frac{\sigma_1}{\sqrt{k}} = 0.9832 \qquad \frac{\sigma_2}{\sqrt{k}} = 0.3559$$



Example: 2D data

$$\frac{\sigma_1}{\sqrt{k}} = 1.3490 \qquad \frac{\sigma_2}{\sqrt{k}} = 0.1438$$



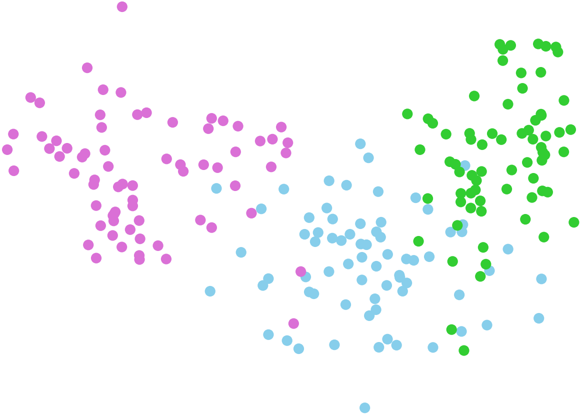
Example

Seeds from three different varieties of wheat: Kama, Rosa and Canadian

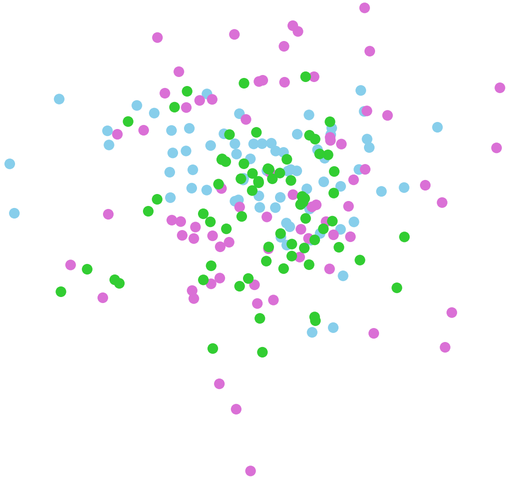
Dimensions:

- Area
- Perimeter
- Compactness
- Length of kernel
- Width of kernel
- Asymmetry coefficient
- Length of kernel groove

Projection onto two first PCs



Projection onto two last PCs



- Dimensionality reduction
  - Principal component analysis
  - Random projections
- Compressed sensing
  - Motivation: Magnetic resonance imaging
  - Exact recovery
  - Robustness
- Sampling
  - Nyquist-Shannon sampling theorem
  - Compressive sampling

Non-adaptive projections

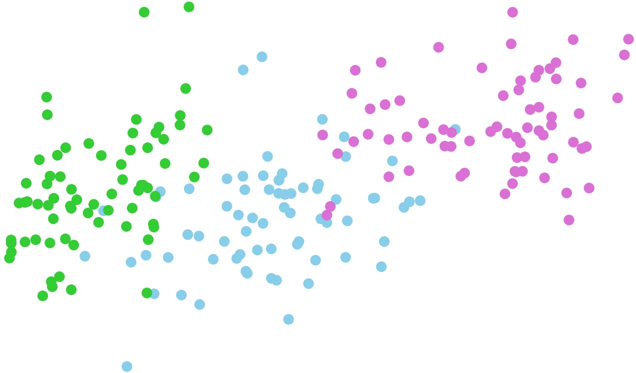
PCA requires processing all of the data before projecting

Idea: Project onto random m-dimensional subspace

Not optimal, but more computationally efficient

Approximate projection: Multiply by a random matrix $A \in \mathbb{R}^{m \times n}$

Approximate projection onto two random directions



Johnson-Lindenstrauss lemma

Random function f preserves distance between points

$$f(x) := \frac{1}{\sqrt{m}} A x$$

A is an $m \times n$ matrix with iid Gaussian entries with mean 0 and variance 1 (can be generalized to Bernouilli ± 1 entries)

Fix $x_1, \ldots, x_k \in \mathbb{R}^n$. For any $x_i \neq x_j$

$$(1-\epsilon) \left|\left|x_i - x_j\right|\right|_2^2 \leq \left|\left|f(x_i) - f(x_j)\right|\right|_2^2 \leq (1+\epsilon) \left|\left|x_i - x_j\right|\right|_2^2$$

with probability at least $\frac{1}{k}$ as long as

$$m \geq \frac{8 \log(k)}{\epsilon^2}$$

Result for fixed vector

For any fixed vector $v \in \mathbb{R}^n$

$$(1-\epsilon)\,||v||_2^2 \leq \frac{1}{m}\,||Av||_2^2 \leq (1+\epsilon)\,||v||_2^2$$

with probability at least

$$1 - 2\exp\left(-\frac{m\epsilon^2}{8}\right)$$

Combining this with the union bound yields the result

Proof of result for fixed vector

Apply concentration bound on chi-square random variable Z with m degrees of freedom

$$Z := \sum_{i=1}^{m} X_i^2$$

$X_1, \ldots, X_m$ are Gaussian with mean 0 and variance 1 and independent

For any $\epsilon > 0$ we have

$$P\left(Z > m\left(1 + \epsilon\right)\right) \leq \exp\left(-\frac{m\epsilon^2}{8}\right)$$

$$P\left(Z < m\left(1 - \epsilon\right)\right) \leq \exp\left(-\frac{m\epsilon^2}{2}\right)$$

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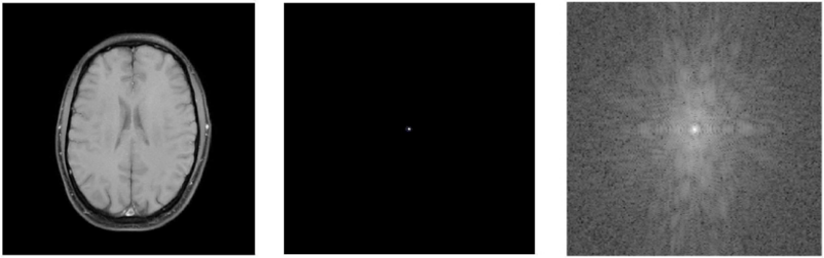
Magnetic resonance imaging



Magnetic resonance imaging

Data: Samples from spectrum

Problem: Sampling is time consuming (annoying, kids move . . .)

Images are compressible (sparse in wavelet basis)

Can we recover compressible signals from less data?

Idea

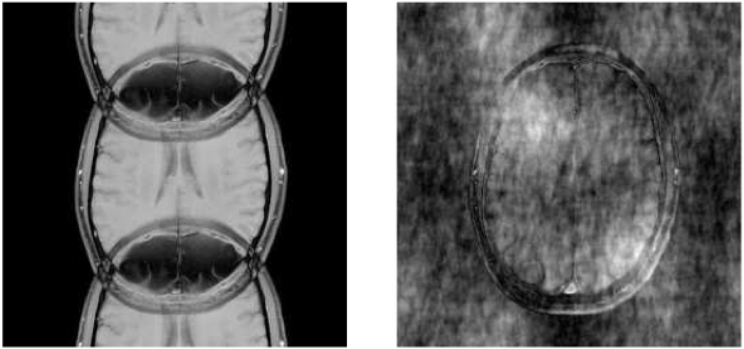
By now (hopefully) we know that ℓ_1-norm induces sparsity

1. Undersample data
2. Solve the optimization problem

$$\begin{array}{ll} \textit{minimize} & ||\text{wavelet transform of estimate}||_1 \\ \textit{subject to} & \text{frequency samples of estimate} = \text{data} \end{array}$$

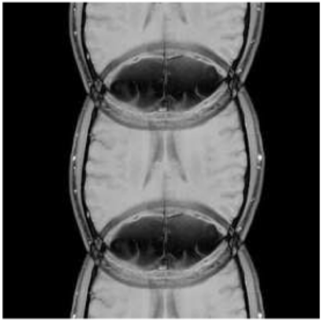
Regular vs random undersampling

Minimum ℓ_2-norm estimate

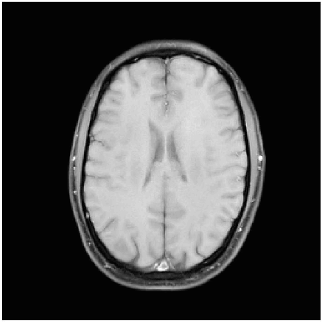


Minimum ℓ_1-norm estimate

Regular



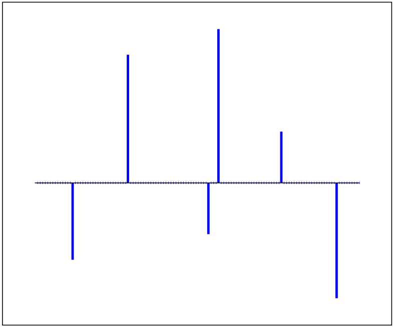
Random



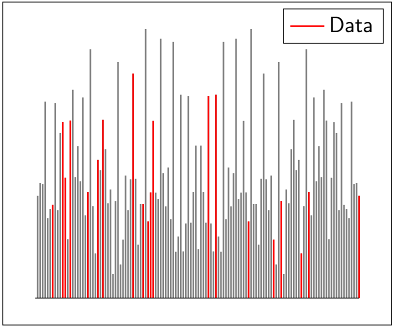
Random samples

1. Undersample the spectrum randomly

Signal



Spectrum

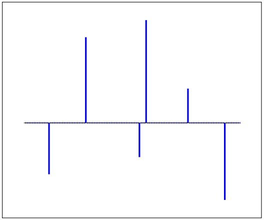


ℓ_1-norm minimization

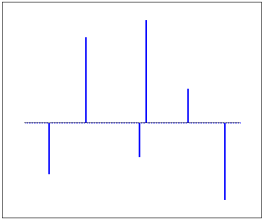
2. Solve the optimization problem

$$\begin{array}{ll} \textit{minimize} & ||\text{estimate}||_1 \\ \textit{subject to} & \text{frequency samples of estimate} = \text{data} \end{array}$$

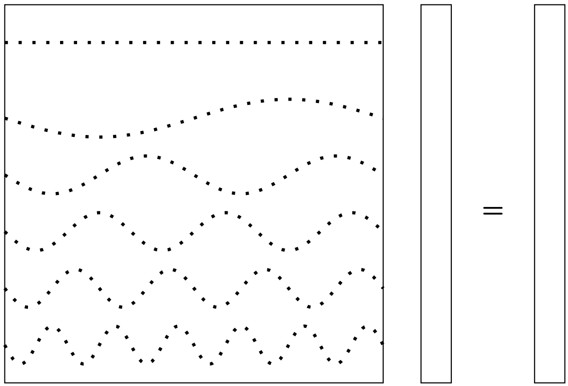
Signal



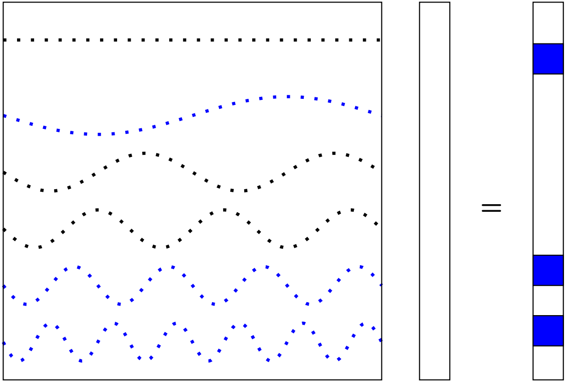
Estimate



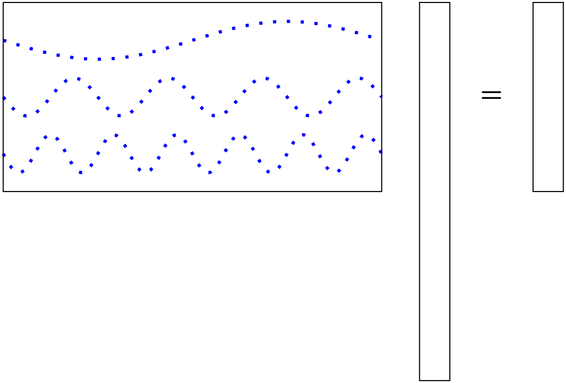
Underdetermined system of equations



Underdetermined system of equations



Underdetermined system of equations



$Ax = y$ where $A \in \mathbb{R}^{m \times n}$ and $m < n$, infinite solutions!

Exact recovery

Assumption: There exists a signal $x \in \mathbb{R}^n$ with s nonzeros such that

$$Ax = y$$

for a random $A \in \mathbb{R}^{m \times n}$ (random Fourier, Gaussian iid, Bernouilli ± 1, ...)

Exact recovery: If the number of measurements satisfies

$$m \geq C' s \log n$$

the solution of the problem

$$\text{minimize} \quad ||\tilde{x}||_1 \qquad \text{subject to} \quad A\,\tilde{x} = y$$

is the original signal with probability at least $1 - \frac{1}{n}$

Incoherent measurements

Generalization: Random rows U_j from orthonormal basis U

Coherence:

$$\mu(U) := \sqrt{n} \max_{1 \le i \le n, 1 \le j \le m} |U_j e_i|$$

Exact recovery is achieved with high probability if

$$m \geq C' \mu(U)\, s \log n$$

Random Fourier: $\mu(F) = 1$

Dual problem

The dual problem is equal to

$$\begin{aligned} &\text{maximize} \quad && y^T \tilde{v} \\ &\text{subject to} \quad && \left|\left|A^T \tilde{v}\right|\right|_\infty \leq 1 \end{aligned}$$

Dual certificate

A dual certificate $v \in \mathbb{R}^m$ associated to x is equal to

$$\left(A^T v\right)_i = \text{sign}\left(x_i\right) \qquad \text{if } x_i \neq 0$$
$$\left|\left|\left(A^T v\right)_i\right|\right|_\infty < 1 \qquad \text{if } x_i = 0$$

Feasible for dual problem, corresponding cost-function value equals

$$y^T v = ||x||_1$$

By weak duality x must be a solution

Dual certificate

By the definition of v

$$q := A^T v$$

is a subgradient of the ℓ_1 norm at x and for any h such that $Ah = 0$

$$h^T q = 0$$

This also implies that x is a solution

If A_T (where T is the support of x) is injective, x is the unique solution

Proof of exact recovery

Prove that dual certificate exists for any s-sparse x

Idea: Choose vector that interpolates the sign and has minimum ℓ_2 norm

$$\begin{aligned} \text{minimize} \quad & \|\tilde{v}\|_2 \\ \text{subject to} \quad & A_T^T \tilde{v} = \operatorname{sign}(x_T) \end{aligned}$$

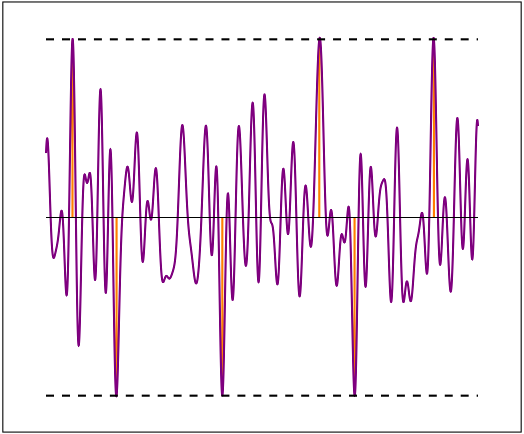
Closed-form solution $v_{\ell_2} = A_T \left(A_T^T A_T\right)^{-1} \operatorname{sign}(x_T)$

We need to prove that $q_{\ell_2} := A^T v_{\ell_2}$ satisfies

$$(q_{\ell_2})_T = \operatorname{sign}(x_T)$$

$$\left\|(q_{\ell_2})_{T^c}\right\|_\infty < 1$$

Random Fourier measurements



Tough stuff, we will prove the result for Gaussian measurements

Bounds on singular values of Gaussian submatrix

Fix a support T, $|T| \leq s$

For any unit-norm vector x with support T

$$1 - \epsilon \leq \frac{1}{\sqrt{m}} \left|\left|Ax\right|\right|_2^2 \leq 1 + \epsilon$$

with probability at least

$$1 - 2 \left(\frac{12}{\epsilon} \right)^s \exp \left(- \frac{m\epsilon^2}{32} \right)$$

Bounds on singular values of Gaussian submatrix

Setting $\epsilon = 1/2$ gives

$$1 - \frac{1}{2} \leq \frac{1}{\sqrt{m}} \left|\left|Ax\right|\right|_2 \leq 1 + \frac{1}{2}$$

with probability at least

$$1 - \exp\left(-\frac{Cm}{s}\right)$$

for some constant C

Bound on dual certificate

Minimum singular value of A_T

$$\sigma_{\min}(A_T) \geq \frac{\sqrt{m}}{2}$$

with probability $1 - \exp\left(-\frac{Cm}{s}\right)$

This implies $A_T^T A_T$ is invertible so

$$(q_{\ell_2})_T = A_T^T A_T \left(A_T^T A_T\right)^{-1} \operatorname{sign}(x_T) = \operatorname{sign}(x_T)$$

Bound on dual certificate

To bound $(q_{\ell_2})_{T^c}$, for each $i \in T^c$ we define

$$
\begin{aligned}
(q_{\ell_2})_i &= A_i^T A_T \left(A_T^T A_T\right)^{-1} \operatorname{sign}(x_T) \\
&= A_i^T w
\end{aligned}
$$

A_i and w are independent

By the bound on $\sigma_{\min}(A_T)$

$$
||w||_2 \leq \frac{||\operatorname{sign}(x_T)||_2}{\sigma_{\min}(A_T)} \leq 2\sqrt{\frac{s}{m}}
$$

with probability $1 - \exp\left(-\frac{Cm}{s}\right)$

Bound on dual certificate

Conditioned on w, $A_i^T w$ is Gaussian with mean 0 and variance $||w||_2^2$

$$\mathrm{P}\left(\left|A_i^T w\right| \geq 1 | w = w'\right) \leq \mathrm{P}\left(|u| > \frac{1}{||w'||_2}\right) \leq 2\exp\left(-\frac{1}{2\,||w'||_2^2}\right)$$

Where u has mean 0 and variance 1

For $\quad \mathcal{E} := \left\{||w||_2 \leq 2\sqrt{\frac{s}{m}}\right\} \quad$ this implies

$$\mathrm{P}\left(\left|A_i^T w\right| \geq 1 \middle| \mathcal{E}\right) \leq 2\exp\left(-\frac{m}{8s}\right)$$

Bound on dual certificate

Finally

$$\mathrm{P}\left(\left|A_i^T w\right| \geq 1\right) \leq \mathrm{P}\left(\left|A_i^T w\right| \geq 1 \middle| \mathcal{E}\right) + \mathrm{P}\left(\mathcal{E}^c\right)$$
$$\leq \exp\left(-\frac{Cm}{s}\right) + 2\exp\left(-\frac{m}{8s}\right)$$

If the number of measurements satisfies

$$m \geq C' s \log n$$

we have exact recovery with probability $1 - \frac{1}{n}$ by the union bound

Proof of bounds on singular values

Let $\mathcal{X}_T$ be the set of unit-norm vectors x with support T

Aim: Prove that for any $x \in \mathcal{X}_T$

$$(1-\epsilon) \leq \frac{1}{\sqrt{m}} \left|\left|Ax\right|\right|_2 \leq (1+\epsilon)$$

With probability $1 - 2\exp\left(-\frac{m\epsilon^2}{8}\right)$ for any fixed unit-norm vector v

$$(1-\epsilon) \leq \frac{1}{m} \left|\left|Av\right|\right|_2^2 \leq (1+\epsilon)$$

We apply this result on an ϵ-net of $\mathcal{X}_T$

ϵ-net and covering number

$\mathcal{N}_\epsilon \subseteq \mathcal{X}$ is an ϵ-net of $\mathcal{X}$ if for every $y \in \mathcal{X}$ there is $x \in \mathcal{N}_\epsilon$ such that

$$||x - y||_2 \leq \epsilon.$$

The covering number $\mathcal{N}(\mathcal{X}, \epsilon)$ of a set $\mathcal{X}$ at scale ϵ is the minimal cardinality of an ϵ-net of $\mathcal{X}$

The covering number of $\mathcal{X}_T$ is

$$\mathcal{N}(\mathcal{X}_T, \epsilon) \leq \left(\frac{3}{\epsilon}\right)^s$$

Proof of bounds on singular values

By the union bound and the bound for fixed vectors

$$\left| \frac{1}{m} \left|\left|Au\right|\right|_2^2 - 1 \right| > \frac{\epsilon}{2}$$

for some $u \in \mathcal{N}\left(\mathcal{X}, \epsilon/4\right)$ with probability at most

$$2\left(\frac{12}{\epsilon}\right)^s \exp\left(-\frac{m\epsilon^2}{32}\right)$$

Proof of bounds on singular values

Assume that for all $u \in \mathcal{N}\left(\mathcal{X}, \epsilon/4\right)$

$$1 - \frac{\epsilon}{2} \leq \frac{1}{\sqrt{m}} \left|\left|Au\right|\right|_2 \leq 1 + \frac{\epsilon}{2}$$

Define α as the smallest number such that for all $x \in \mathcal{X}_T$

$$\frac{1}{\sqrt{m}} \left|\left|Ax\right|\right|_2 \leq 1 + \alpha$$

For any $x \in \mathcal{X}_T$, there is a $u \in \mathcal{N}\left(\mathcal{X}, \epsilon/4\right)$ such that

$$\begin{aligned} \frac{1}{\sqrt{m}} \left|\left|Ax\right|\right|_2 &\leq \frac{1}{\sqrt{m}} \left(\left|\left|Au\right|\right|_2 + \left|\left|A\left(x-u\right)\right|\right|_2\right) \\ &\leq 1 + \frac{\epsilon}{2} + \frac{\left(1+\alpha\right)\epsilon}{4} \end{aligned}$$

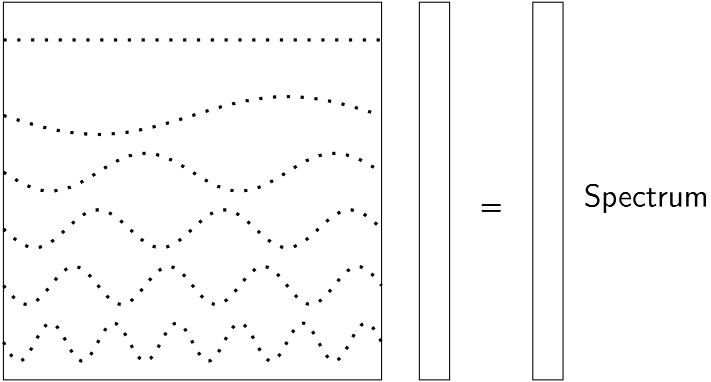
Proof of bounds on singular values

We conclude

$$\alpha \leq \frac{3\epsilon}{4-\epsilon} \leq \epsilon$$

$$\begin{aligned}\frac{1}{\sqrt{m}} \left|\left|Ax\right|\right|_2 &\geq \frac{1}{\sqrt{m}} \left(\left|\left|Au\right|\right|_2 - \left|\left|A\left(x-u\right)\right|\right|_2\right) \\ &\geq 1 - \frac{\epsilon}{2} - \frac{\left(1+\epsilon\right)\epsilon}{4} \\ &\geq 1 - \epsilon\end{aligned}$$

Is the problem well posed?



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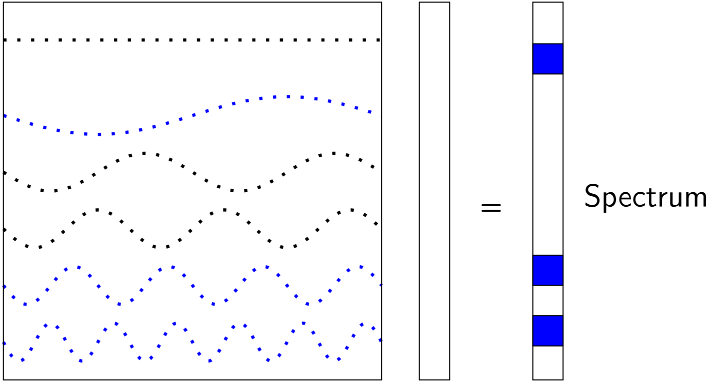

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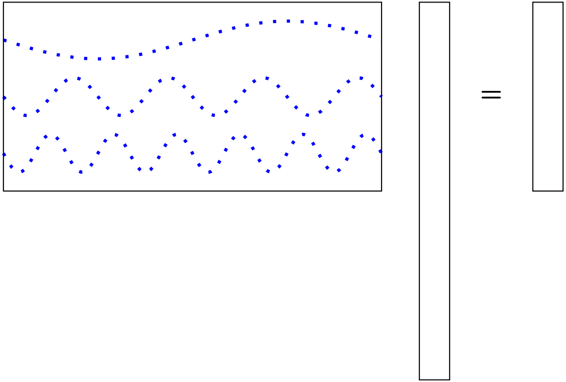

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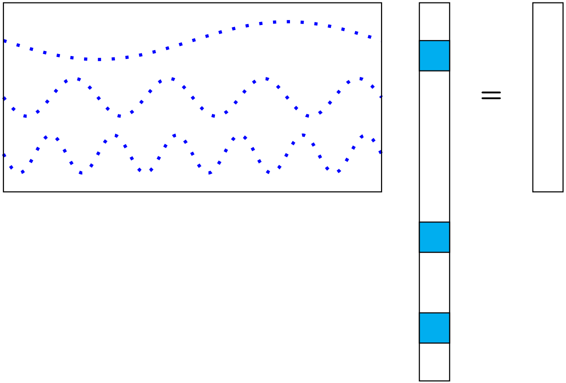
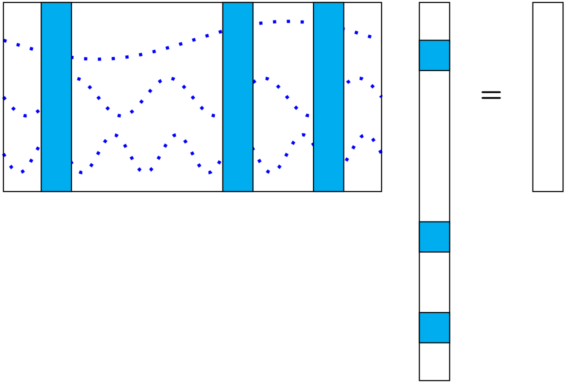


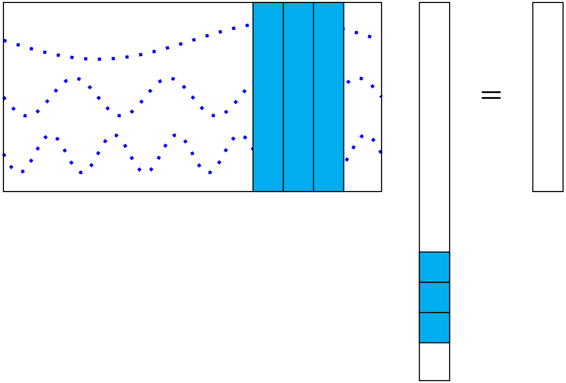
Aim: Study effect of measurement operator on sparse vectors

Is the problem well posed?



Equivalently, is the submatrix always well conditioned?

Is the problem well posed?



Equivalently, is the submatrix always well conditioned?

Restricted isometry property (RIP)

For any s-sparse vector x

$$(1-\epsilon_s)\,||x||_2 \leq \frac{1}{\sqrt{m}}\,||Ax||_2 \leq (1+\epsilon_s)\,||x||_2$$

with probability at least $1-2\exp(-C_1 n)$ if

$$m \geq \frac{C_1 s}{\epsilon_s^2}\log\left(\frac{n}{s}\right)$$

$2s$-RIP implies that for any s-sparse signals x_1, x_2

$$||y_2 - y_1||_2 \geq (1-\epsilon_{2s})\,||x_2 - x_1||_2$$

Robustness

Noisy data

$$y = Ax + z \qquad \text{where } ||z||_2 \leq \epsilon_0$$

Relaxed problem

$$\begin{aligned} &\text{minimize} && ||\tilde{x}||_1 \\ &\text{subject to} && ||A\tilde{x} - y||_2 \leq \epsilon_0 \end{aligned}$$

Robustness

If x is s-sparse under the RIP, solution $\hat{x}$ satisfies

$$||\hat{x} - x||_2 \leq C_0 \, \epsilon_0$$

If x is not sparse

$$||\hat{x} - x||_2 \leq C_0 \, \epsilon_0 + C_1 \frac{||x - x_s||_1}{\sqrt{s}}$$

x_s contains s entries of x with largest magnitude

Proof of the RIP

For a fixed support T,

$$(1-\epsilon)\,||x||_2 \leq \frac{1}{\sqrt{m}}\,||Ax||_2 \leq (1+\epsilon)\,||x||_2$$

for any x with support T with probability at least

$$1-2\left(\frac{12}{\epsilon}\right)^s \exp\left(-\frac{m\epsilon^2}{32}\right)$$

Proof of the RIP

Number of possible supports

$$\binom{n}{s} \leq \left(\frac{en}{s}\right)^s$$

By the union bound the result holds with probability at least

$$\begin{aligned}
&1 - 2\left(\frac{en}{s}\right)^s \left(\frac{12}{\epsilon}\right)^s \exp\left(-\frac{m\epsilon^2}{32}\right) \\
&= 1 - \exp\left(\log 2 + s + s\log\left(\frac{n}{s}\right) + s\log\left(\frac{12}{\epsilon}\right) - \frac{m\epsilon^2}{2}\right) \\
&\leq 1 - \frac{C_2}{n}
\end{aligned}$$

as long as $\quad m \geq \frac{C_1 s}{\epsilon^2} \log\left(\frac{n}{s}\right)$

Sampling problem

Aim: Estimate bandlimited signal $g \in \mathbb{L}_2([0,1])$

$$g(t) := \sum_{k=-f}^{f} c_k \exp(i2\pi kt)$$

from samples

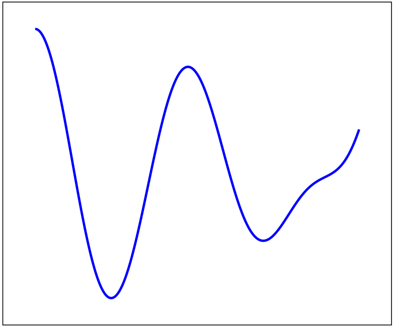
$$g(0),\ g\left(\frac{1}{n}\right),\ g\left(\frac{2}{n}\right),\ \ldots,\ g\left(\frac{n-1}{n}\right)$$

Questions:

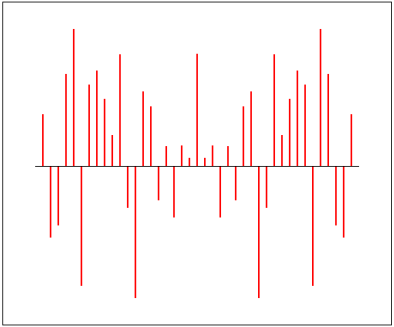
1. At what rate do we need to sample?
2. How do we recover the signal from the samples?

Sampling problem

Signal

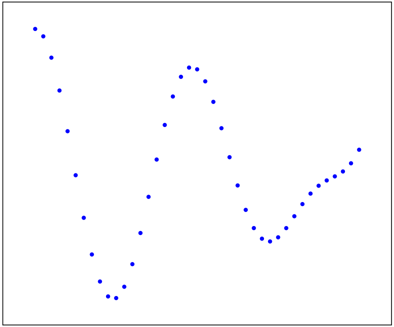


Spectrum

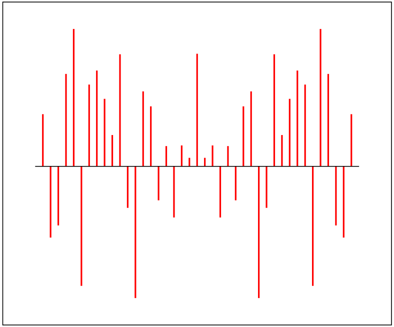


Sampling problem

Data



Spectrum



Notation

$$a_{-f:f}\left(t\right)_k := \begin{bmatrix} \exp\left(-i2\pi\left(-f\right)t\right) \\ \exp\left(-i2\pi\left(-f+1\right)t\right) \\ \cdots \\ \exp\left(-i2\pi\left(f-1\right)t\right) \\ \exp\left(-i2\pi ft\right) \end{bmatrix}$$

$$g\left(t\right) := \sum_{k=-f}^{f} c_k \exp\left(i2\pi kt\right) = a_{-f:f}\left(t\right)^* c$$

Data

n equations, $2f+1$ unknowns

$$F^* c = \begin{bmatrix} a_{-f:f}\left(0\right)^* \\ a_{-f:f}\left(\frac{1}{n}\right)^* \\ a_{-f:f}\left(\frac{2}{n}\right)^* \\ \cdots \\ a_{-f:f}\left(\frac{n-1}{n}\right)^* \end{bmatrix} c = \begin{bmatrix} g\left(0\right) \\ g\left(\frac{1}{n}\right) \\ g\left(\frac{2}{n}\right) \\ \cdots \\ g\left(\frac{n-1}{n}\right) \end{bmatrix}$$

Sampling rate $n \geq 2f+1$

Recovery

If $n = 2f + 1$, the vectors

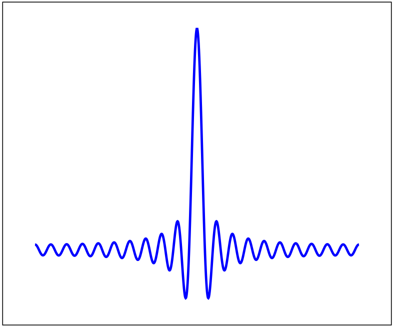
$$\frac{1}{\sqrt{n}} a_{-f:f}(0), \; \frac{1}{\sqrt{n}} a_{-f:f}\left(\frac{1}{n}\right), \ldots, \frac{1}{\sqrt{n}} a_{-f:f}\left(\frac{n-1}{n}\right)$$

form an orthonormal basis, so

$$c = \frac{1}{n} F F^* c = \frac{1}{n} F \begin{bmatrix} g(0) \\ g\left(\frac{1}{n}\right) \\ g\left(\frac{2}{n}\right) \\ \cdots \\ g\left(\frac{n-1}{n}\right) \end{bmatrix} = \frac{1}{n} \sum_{j=0}^{n} g\left(\frac{j}{n}\right) a_{-f:f}\left(\frac{j}{n}\right)$$

Periodized sinc or Dirichlet kernel

$$D(t) := \frac{1}{n} \sum_{k=-f}^{f} e^{-i2\pi kt} = \frac{\sin(\pi n t)}{n \sin(\pi t)}$$



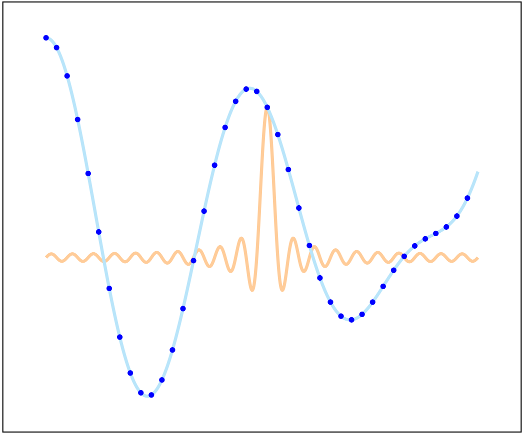
Recovery

Interpolation with weighted sincs!

$$
\begin{aligned}
g(t) &= a_{-f:f}(t)^* c \\
&= \frac{1}{n}\sum_{j=0}^{n} g\left(\frac{j}{n}\right) a_{-f:f}(t)^* a_{-f:f}\left(\frac{j}{n}\right) \\
&= \sum_{j=0}^{n} g\left(\frac{j}{n}\right) D\left(t-\frac{j}{n}\right)
\end{aligned}
$$

$$
\begin{aligned}
D\left(t-\frac{j}{n}\right) &= \frac{1}{n}\sum_{k=-f}^{f} e^{-i2\pi k\left(t-\frac{j}{n}\right)} \\
&= \frac{1}{n} a_{-f:f}(t)^* a_{-f:f}\left(\frac{j}{n}\right)
\end{aligned}
$$

Recovery



Nyquist-Shannon sampling theorem

Condition: Sampling rate $\geq$ twice the highest frequency

Recovery: Interpolation with sinc kernel

Just linear algebra!

Sparse spectrum

Aim: Estimate a signal $g \in \mathbb{L}_2([0,1])$ with a sparse spectrum

$$g(t) := \sum_{k \in \mathcal{S}} c_k \exp(i2\pi kt)$$

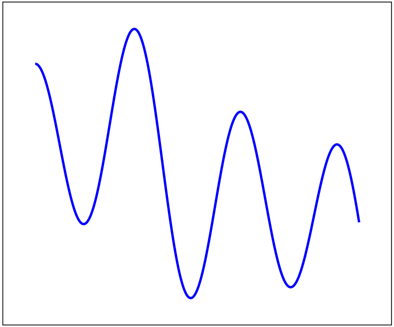
- Maximum frequency: f
- Number of sinusoids: s

How many measurements do we need?

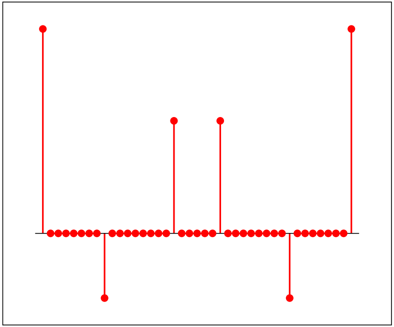
- Nyquist-Shannon: $2f + 1$
- Compressed sensing: $\mathcal{O}(s \log(2f + 1))$

Sparse spectrum

Signal



Spectrum



Linear estimation

n equations, $2f+1$ unknowns

$$F^* c = \begin{bmatrix} a_{-f:f}\left(0\right)^* \\ a_{-f:f}\left(\frac{1}{n}\right)^* \\ a_{-f:f}\left(\frac{2}{n}\right)^* \\ \cdots \\ a_{-f:f}\left(\frac{n-1}{n}\right)^* \end{bmatrix} c = \begin{bmatrix} g\left(0\right) \\ g\left(\frac{1}{n}\right) \\ g\left(\frac{2}{n}\right) \\ \cdots \\ g\left(\frac{n-1}{n}\right) \end{bmatrix}$$

Measurements: $2f+1$

Recovery: Sinc interpolation

Compressive sampling

m equations, $2f+1$ unknowns

$$F^*c = \begin{bmatrix} \cancel{a_{-f:f}(0)^*} \\ a_{-f:f}\left(\frac{1}{n}\right)^* \\ \cancel{a_{-f:f}\left(\frac{2}{n}\right)^*} \\ \cdots \\ a_{-f:f}\left(\frac{n-1}{n}\right)^* \end{bmatrix} c = \begin{bmatrix} \cancel{g(0)} \\ g\left(\frac{1}{n}\right) \\ \cancel{g\left(\frac{2}{n}\right)} \\ \cdots \\ g\left(\frac{n-1}{n}\right) \end{bmatrix}$$

Measurements: $m \geq C\, s \log(2f+1)$ (random undersampling)

Recovery: ℓ_1-norm minimization to compute c

Recovery