Simple Mathematical, Dynamical Stochastic Models
Capturing the Observed Diversity of the
El Niño-Southern Oscillation

Lecture 2: Background and simple ENSO models (I)
14 september 2014, Courant Institute
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Model with NonLinear Thermocline Feedback

We now have an ENSO model that reproduces a **basic ENSO oscillation** with regular El Nino and La Nina events. The model has a nonlinear thermocline feedback allowing constant **Hopf bifurcations** between stable and unstable regimes. Hereafter we analyze the dynamics of this basic ENSO oscillation.
Outline

1. Introduction to The El Nino Southern Oscillation
2. Simple Model Overview
3. Ocean Dynamics
4. SST Thermodynamics
5. Atmosphere Dynamics
6. Coupled Dynamics
7. Model Linear Solutions
8. Model with Nonlinear Thermocline Feedback
9. ENSO Oscillation Mechanisms
Simple ENSO Model

We derive hereafter a simple ENSO model:
-1) Includes atmosphere dynamics, ocean dynamics and an SST thermodynamic budget relevant to ENSO
-2) simple coupling between the ocean, atmosphere and SST:
-3) The model admits solutions consisting of equatorial waves in the ocean (Ko, Ro)

\begin{align*}
\text{Atmosphere} & \quad \begin{align*}
- yu - \partial_x \theta &= 0 \\
yu - \partial_y \theta &= 0 \\
-(\partial_x u + \partial_y v) &= E_q / (1 - Q)
\end{align*} \\
\text{Ocean} & \quad \begin{align*}
\partial_t U - c_1 Y V + c_1 \partial_x H &= c_1 \tau_x \\
Y U + \partial_Y H &= 0 \\
\partial_t H + c_1 (\partial_x U + \partial_Y V) &= 0
\end{align*} \\
\text{SST} & \quad \partial_t T = -c_1 \zeta E_q + c_1 \eta(x) H
\end{align*}

\begin{align*}
u, v & \quad \text{winds} \\
\theta & \quad \text{temperature} \\
E_q & = \alpha T \quad \text{latent heat} \\
U, V & \quad \text{currents} \\
H & \quad \text{thermocline depth} \\
\tau_x & = \gamma u \quad \text{wind stress} \\
T & \quad \text{sea surface temperature}
\end{align*}
Atmosphere Dynamics

1) We start with a **skeleton model** for the atmosphere (Majda and Stechmann, 2009). The starting skeleton model captures the Madden-Julian Oscillation and intraseasonal variability in general in the tropics. We will **modify it here to capture the ENSO**.

In the **starting skeleton model** there is interaction between:

(i) planetary dry dynamics: wind speed anomalies u,v, potential temperature anomalies $\theta$

(ii) planetary lower level moisture anomalies: q

(iii) planetary envelope of synoptic/convective activity: a

**Parameters:**

moisture gradient $\bar{Q}$, growth rate $\Gamma$, source of cooling $s^\theta$ and moistening $s^q$

**Units** (different from the ENSO model!):

x,y: 1500 km, t: 8 hours, u,v: 50 m/s, $\theta$, q: 15 K, $\bar{H}a$:10 K/day
The **starting skeleton model** has the following features:

\[
\begin{align*}
\partial_t u - yv - \partial_x \theta &= 0 \quad (i) \\
yu - \partial_y \theta &= 0 \\
\partial_t \theta - (\partial_x u + \partial_y v) &= Ha - s^\theta \\
\partial_t q + \overline{Q}(\partial_x u + \partial_y v) &= -Ha + s^q + E_q \quad (ii) \\
\partial_t a &= \Gamma qa \quad (iii)
\end{align*}
\]

The variable \(a\) is the **planetary envelope** of synoptic convection. It parametrizes the effect of small-scale convection on the planetary scale. An analogue for \(a\) in observation is **Outgoing Long Wave Radiations** (OLR). See Majda and Stechmann (2015).

The skeleton model has an associated vertical structure (first baroclinic mode). See book of Majda (2003). To reconstruct the flow with vertical structure use:

\[
\begin{align*}
u_R &= u\sqrt{2}\cos(z), \quad \theta_R = \theta\sqrt{2}\sin(z), \text{ etc...}
\end{align*}
\]
Atmosphere Dynamics

The **starting skeleton model** is derived as follows from the momentum and mass continuity budgets:

**Starting equations:**

\[
\begin{align*}
\partial_t u - yv &= -\partial_x p \\
yu &= -\partial_y p \\
0 &= -\partial_z p + \theta \\
\partial_x u + \partial_y v + \partial_z w &= 0 \\
\partial_t \theta + w &= Ha - s^\theta \\
\partial_t q - Qw &= -Ha + s^q \\
\partial_t a &= \Gamma qa
\end{align*}
\]

**Galerkin Projection on the basis of functions of the vertical baroclinic modes:**

\[
\begin{align*}
a(x,y,z,t) &= \sum a_n(x, y, t) F_n(z) \\
a_n &= \int a F_n dz \\
F_n(z) &= \sqrt{2}\cos(nz)
\end{align*}
\]

**Truncation to the first baroclinic mode:**

\[
\begin{align*}
\partial_t u_1 - yv_1 - \partial_x \theta_1 &= 0 \\
yu_1 - \partial_y \theta_1 &= 0 \\
\partial_t \theta_1 - (\partial_x u_1 + \partial_y v_1) &= Ha_1 - s^\theta_1 \\
\partial_t q + Q(\partial_x u_1 + \partial_y v_1) &= -Ha_1 + s^q_1 \\
\partial_t a_1 &= \Gamma qa_1
\end{align*}
\]

(the subscript “1” is omitted afterwards)
Atmosphere Dynamics

To obtain the **ENSO model atmosphere**, the starting skeleton model is modified as follows:

1) Non-dimensionalize to units relevant to the ENSO (identical the ones of the ocean):

Starting skeleton model

\[
(\partial_t + d_A)u - yv - \partial_x \theta = 0 \\
yu - \partial_y \theta = 0 \\
(\partial_t + d_A)\theta - (\partial_x u + \partial_y v) = H a - s^\theta \\
(\partial_t + d_A)q + Q(\partial_x u + \partial_y v) = -Ha + s^q + E_q \\
\partial_t a = \Gamma q a
\]

Recall the units that match the scales of the ENSO:

<table>
<thead>
<tr>
<th>ENSO typical scales</th>
<th>time: 33 days</th>
</tr>
</thead>
<tbody>
<tr>
<td>x: 15000 km</td>
<td></td>
</tr>
<tr>
<td>y: 330 km</td>
<td></td>
</tr>
<tr>
<td>Circulation U: 0.25 m.s-1, H: 20m, etc</td>
<td></td>
</tr>
</tbody>
</table>

Change the model units from the ones of the starting skeleton model to the ones of the ENSO.

Starting skeleton model with **ENSO units**

\[
(\epsilon \partial_\tau + \epsilon d_A)u - yv - \partial_x \theta = 0 \\
yu - \partial_y \theta = 0 \\
(\epsilon \partial_\tau + \epsilon d_A)\theta - (\partial_x u + \partial_y v) = H a - s^\theta \\
(\epsilon \partial_\tau + \epsilon d_A)q + Q(\partial_x u + \partial_y v) = -Ha + s^q + E_q \\
\epsilon \partial_\tau a = \Gamma q a
\]

The skeleton model has now same units than the ocean!
2) We now simplify the system removing small terms

Starting skeleton model with **ENSO units**

\[
\begin{align*}
(\epsilon \partial_\tau + \epsilon d_A)u - yv - \partial_x \theta &= 0 \\
yu - \partial_y \theta &= 0 \\
(\epsilon \partial_\tau + \epsilon d_A)\theta - (\partial_x u + \partial_y v) &= \overline{Ha} - s^\theta \\
(\epsilon \partial_\tau + \epsilon d_A)q + Q(\partial_x u + \partial_y v) &= -\overline{Ha} + s^q + E_q \\
\epsilon \partial_\tau a &= \Gamma qa
\end{align*}
\]

We identify small non-dimensional terms. Here:

\[\epsilon = 0.1 \quad \text{the Froude number is small}
\]

other parameters are of order O(1)

Removing small terms we obtain the **ENSO atmosphere model**:

\[
\begin{align*}
yv \partial_x \theta &= 0 \\
yu - \partial_y \theta &= 0 \\
- (\partial_x u + \partial_y v) &= E_q / (1 - \overline{Q})
\end{align*}
\]

In particular, we have removed the **time derivatives** in the atmosphere. On ENSO timescales, the atmosphere is assumed to “**adjust instantly**” to the latent heat release Eq. We have also neglected dissipation. The ENSO atmosphere is a non-dissipative version of the Matsuno-Gill model.

Also note that convection is in direct balance with latent heat release from the ocean and other sources:

\[\overline{Ha} = (E_q + s^q - \overline{Q} s^\theta) / (1 - \overline{Q})\]

In addition, the zonal mean circulation is treated separately:

\[(1/L_A) \int_0^{L_A} E_q dx = 0.\]
Atmosphere Dynamics

The ENSO atmosphere reproduces a large-scale response to latent heating Eq, with delocalized winds. The wind anomalies are oriented towards the center of latent heat release.

ENSO model atmosphere

\[
\begin{align*}
-y \nu \partial_x \theta &= 0 \\
y u - \partial_y \theta &= 0 \\
-(\partial_x u + \partial_y v) &= E_q/(1 - Q)
\end{align*}
\]

Latent heat release from the ocean  Convection  Delocalized zonal winds response
Example during El Nino 1998. The increased SST in the eastern Pacific release latent heat $ Eq > 0 $, creating **delocalized** zonal winds anomalies in the western Pacific $ u > 0 $ oriented eastward.

Note however that all winds fluctuations (for example small bursts) are not necessarily related to SST variations!

**ENSO model atmosphere**

\[-yv \partial_x \theta = 0\]
\[yu - \partial_y \theta = 0\]
\[-(\partial_x u + \partial_y v) = E_q / (1 - \bar{Q})\]
Next, in order to find simple solutions for the ENSO atmosphere we project on the basis function of parabolic cylinder functions.

**Parabolic cylinder functions in the atmosphere:**

\[
\phi_m(y) = \frac{H_m e^{-y^2/2}}{\sqrt{2^m m! \sqrt{\pi}}}, \quad 0 \leq m
\]

\[
H_m(Y) = (-1)^m e^{Y^2} \frac{d^m}{dY^m} e^{-Y^2}
\]

Note the functions are identical to the ones of the ocean except for a different meridional scaling:

\[
\psi_m(y) = \phi_m(Y) \text{ where } Y = y/\sqrt{c}.
\]

The strategy for projecting is therefore identical!
Atmosphere Dynamics

Next, in order to find simple solutions for the ENSO atmosphere we project on the basis function of parabolic cylinder functions.

1) Start with the ENSO model atmosphere

\[-y v \cdot \partial_x \theta = 0\]
\[y u - \partial_y \theta = 0\]
\[-(\partial_x u + \partial_y v) = E_q/(1 - \overline{Q})\]

2) Project the system onto the first Parabolic Cylinder functions:

\[\{u, v, \theta, E_q\}(x, y, \tau) = \{u, v, \theta, \chi_A E_q\}(x, \tau)\phi_o(y)\]

3) Find the expressions for the atmospheric equatorial waves:

\[\partial_x K_A = -\chi_A E_q(2 - 2\overline{Q})^{-1}\]
\[-\partial_x R_A/3 = -\chi_A E_q(3 - 3\overline{Q})^{-1}\]

With reconstruction

\[u = (K_A - R_A)\phi_0 + (R_A/\sqrt{2})\phi_2\]
\[\theta = -(K_A + R_A)\phi_0 - (R_A/\sqrt{2})\phi_2\]

The atmosphere equatorial Kelvin wave \(K_A\) and Rossby wave \(R_A\) are derived in fashion identical to the ones of the ocean. However, their adjust instantly (no propagations) and there are periodic boundary conditions around the equatorial belt.

4) In addition to this, the zonal mean circulation is treated separately simply as:

\[(1/L_A) \int_0^{L_A} E_q dx = 0\]
Atmosphere Dynamics

The atmosphere model with equatorial waves adjusting instantly is in very good agreement with observations (see Stechmann and Ogrosky 2015). Note that there is no dissipation in the atmosphere. Here OLR is used to determine. \( \overline{H_a} \)

**ENSO model atmosphere**

\[
\begin{align*}
-yv \cdot \partial_x \theta &= 0 \\
yu - \partial_y \theta &= 0 \\
-(\partial_x u + \partial_y v) &= E_q/(1 - Q) \\
\overline{H_a} &= (E_q + s^q - \overline{Q} s^\theta)/(1 - \overline{Q})
\end{align*}
\]

**Atmosphere equatorial waves solutions**

\[
\begin{align*}
\partial_x K_A &= -\chi_A E_q (2 - 2Q)^{-1} \\
-\partial_x R_A/3 &= -\chi_A E_q (3 - 3Q)^{-1}
\end{align*}
\]

Kelvin wave during El Nino or La Nina (Stechmann Ogrosky 2015), from model and observations
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9. ENSO Oscillation Mechanisms
We derive hereafter a simple ENSO model:
-1) Includes atmosphere dynamics, ocean dynamics and an SST thermodynamic budget relevant to ENSO
-2) simple coupling between the ocean, atmosphere and SST:
-3) The model admits solutions consisting of equatorial waves in the ocean and atmosphere (Ko, Ro, Ka, Ra)

Atmosphere
\[-y v - \partial_x \theta = 0\]
\[y u - \partial_y \theta = 0\]
\[-(\partial_x u + \partial_y v) = E_q / (1 - Q)\]
\[\theta: \text{temperature}\]
\[u, v: \text{winds}\]
\[E_q = \alpha T: \text{latent heat}\]

Ocean
\[\partial_t U - c_1 Y V + c_1 \partial_x H = c_1 \tau_x\]
\[Y U + \partial_Y H = 0\]
\[\partial_t H + c_1 (\partial_x U + \partial_Y V) = 0\]
\[U, V: \text{currents}\]
\[H: \text{thermocline depth}\]
\[\tau_x = \gamma u: \text{wind stress}\]

SST
\[\partial_t T = -c_1 \zeta E_q + c_1 \eta(x) H\]
\[T: \text{sea surface temperature}\]
Coupled Dynamics

We now express flux exchanges between the atmosphere and ocean to couple them:

1) **Wind stresses** on the ocean are related to wind speed using an empirical bulk formula:

\[ \tau_x = \gamma u \]

With the bulk coefficient: \( \gamma = \rho_A c_D \bar{u} \)

\( \rho_A \): air density at sea level
\( c_D \): drag coefficient
\( \bar{u} \): estimate of mean wind speed amplitude, around \( 5 \text{ m.s}^{-1} \)

2) **Latent heat release** from the ocean to the atmosphere is related to SST using an empirical formula:

\[ E_q = \alpha_q T \]

For increased \( \text{SST} \geq 0 \) there is increased release of latent heat towards the atmosphere. Note that this also appears now as a dissipation term in the SST budget:

\[ \partial \tau T / c_1 = -\zeta E_q + \eta H \]

Note that this relation derives from an empirical formula for total latent heat release related to lower level moisture:

\[ \bar{E}_q + E_q = q_{SAT}(T + \bar{T}) - \beta_q (q + \bar{q}) \quad \text{with} \quad q_{SAT}(T + \bar{T}) = q_c \exp(q_e(T + \bar{T})) \]

That is linearized and simplified. In particular the latent heat parameter reads: \( \alpha_q(T) = q_c q_e \exp(q_e \bar{T}) / \tau_q \)
Simple ENSO Model

We have now completed the derivation of the simple ENSO model:

1) Includes atmosphere dynamics, ocean dynamics and an SST thermodynamic budget relevant to ENSO

2) Simple coupling between the ocean, atmosphere and SST

3) The model admits solutions consisting of equatorial waves in the ocean and atmosphere (Ko, Ro, Ka, Ra):

Atmosphere

\[-y v - \partial_x \theta = 0\]
\[y u - \partial_y \theta = 0\]
\[-(\partial_x u + \partial_y v) = E_q / (1 - Q)\]

\[u, v: \text{ winds}\]
\[\theta: \text{ temperature}\]
\[E_q = \alpha T: \text{ latent heat}\]

Ocean

\[\partial_t U - c_1 Y V + c_1 \partial_x H = c_1 \tau_x\]
\[Y U + \partial_Y H = 0\]
\[\partial_t H + c_1 (\partial_x U + \partial_Y V) = 0\]

\[U, V: \text{ currents}\]
\[H: \text{ thermocline depth}\]
\[\tau_x = \gamma u: \text{ wind stress}\]

SST

\[\partial_t T = -c_1 \zeta E_q + c_1 \eta(x) H\]

\[T: \text{ sea surface temperature}\]
Coupled Dynamics

For coupling equatorial waves solutions however a few additional modifications must be taken into account.

The ocean and atmosphere use different parabolic cylinder functions for equatorial waves, therefore to couple both we use projection coefficients from one basis to the other:

$$\chi_A = \int_{-\infty}^{+\infty} \phi_o(y)\psi_0(y)\,dy$$
$$\chi_O = \int_{-\infty}^{+\infty} \phi_o(Y)\psi_0(Y)\,dY$$

Those projection coefficient appear in the complete model for its equatorial wave solutions:

<table>
<thead>
<tr>
<th>atmosphere</th>
<th>ocean</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\partial_x K_A = -\chi_A E_q (2 - 2Q)^{-1}$</td>
<td>$\partial_\tau K_O + c_1 \partial_x K_O = \chi_O c_1 \tau_x / 2$</td>
</tr>
<tr>
<td>$- \partial_x R_A / 3 = -\chi_A E_q (3 - 3Q)^{-1}$</td>
<td>$\partial_\tau R_O - (c_1 / 3) \partial_x R_O = -\chi_O c_1 \tau_x / 3$</td>
</tr>
<tr>
<td>$K_A(0, \tau) = K_A(L_A, \tau)$</td>
<td>$K_O(0, t) = r_W R_O(0, t)$</td>
</tr>
<tr>
<td>$R_A(0, \tau) = R_A(L_A, \tau)$</td>
<td>$R_O(L_O, t) = r_E K_O(L_O, t)$</td>
</tr>
</tbody>
</table>

SST

$$\partial_\tau T / c_1 = -\zeta E_q + \eta (K_O + R_O)$$
$$E_q = \alpha_q T$$
$$\tau_x = \gamma (K_A - R_A)$$

Atmosphere projected on $\phi_0$
Except Eq projected on $\psi_0$

Ocean and SST projected on $\psi_0$
Except $\tau_x$ projected on $\phi_0$
For completeness we recall the **complete ENSO model for the equatorial wave solutions:**

**ENSO model for equatorial wave solutions**

\[
\begin{align*}
ant\text{mosphere} & \quad \partial_x K_A = -\chi_A E_q (2 - 2Q)^{-1} \\
& \quad - \partial_x R_A / 3 = -\chi_A E_q (3 - 3Q)^{-1} \\
& \quad K_A(0, \tau) = K_A(L_A, \tau) \\
& \quad R_A(0, \tau) = R_A(L_A, \tau) \\
ocean & \quad \partial_\tau K_O + c_1 \partial_x K_O = \chi_O c_1 \tau_x / 2 \\
& \quad \partial_\tau R_O - (c_1/3) \partial_x R_O = -\chi_O c_1 \tau_x / 3 \\
& \quad K_O(0, t) = r_W R_O(0, t) \\
& \quad R_O(L_O, t) = r_E K_O(L_O, t) \\
\text{SST} & \quad \partial_\tau T / c_1 = -\zeta E_q + \eta(K_O + R_O) \\
& \quad E_q = \alpha_q T \\
\text{couplings} & \quad \tau_x = \gamma(K_A - R_A)
\end{align*}
\]

Reconstructions of the variables:

\[
\begin{align*}
\text{u} = & (K_A - R_A)\phi_0 + (R_A / \sqrt{2})\phi_2 \\
\theta = & -(K_A + R_A)\phi_0 - (R_A / \sqrt{2})\phi_2 \\
U = & (K_O - R_O)\psi_0 + (R_O / \sqrt{2})\psi_2 \\
H = & (K_O + R_O)\psi_0 + (R_O / \sqrt{2})\psi_2
\end{align*}
\]
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We have derived a simple ENSO model:

1) Includes **atmosphere dynamics**, **ocean dynamics** and an **SST thermodynamic budget** relevant to ENSO
2) simple **coupling** between the ocean, atmosphere and SST:
3) The model admits solutions consisting of **equatorial waves** in the ocean and atmosphere (Ko, Ro, Ka, Ra)

**Atmosphere**

\[-yv - \partial_x \theta = 0\]

\[yu - \partial_y \theta = 0\]

\[-(\partial_x u + \partial_y v) = \frac{E_q}{1 - Q}\]

\[u, v: \text{ winds} \]

\[\theta: \text{ temperature} \]

\[E_q = \alpha T: \text{ latent heat} \]

**Ocean**

\[\partial_t U - c_1 Y V + c_1 \partial_x H = c_1 \tau_x\]

\[Y U + \partial_Y H = 0\]

\[\partial_t H + c_1 (\partial_x U + \partial_Y V) = 0\]

\[U, V: \text{ currents} \]

\[H: \text{ thermocline depth} \]

\[\tau_x = \gamma u: \text{ wind stress} \]

**SST**

\[\partial_t T = -c_1 \zeta E_q + c_1 \eta(x) H\]

\[T: \text{ sea surface temperature} \]
Model Linear Solutions

Hereafter, as a first step we compute the **linear solutions** of the ENSO model:

1) First, the simple ENSO model is **deterministic and linear**:

<table>
<thead>
<tr>
<th>Atmosphere</th>
<th>$-yv - \partial_x \theta = 0$</th>
<th>$u, v$: winds</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$yu - \partial_y \theta = 0$</td>
<td>$\theta$: temperature</td>
</tr>
<tr>
<td></td>
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<td></td>
<td>$\partial_t H + c_1 (\partial_x U + \partial_Y V) = 0$</td>
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<td>SST</td>
<td>$\partial_t T = -c_1 \zeta E_q + c_1 \eta(x) H$</td>
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</tr>
</tbody>
</table>

and can be expressed in the more general **matrix form**:

$$\partial_\tau X = MX$$

$X = \{K_O, R_O, T\}$ is a vector describing the entire state of the system (at all locations $x,y$)

$M$ is the evolution matrix
Model Linear Solutions

Hereafter, as a first step we compute the **linear solutions** of the ENSO model:

2) The ENSO model in matrix form: \( \partial_\tau X = MX \)

admits linear solutions of the form: \( X = A \exp(\lambda t) \) that satisfy \( \partial_\tau X = \lambda X \)

The eigenvalues \( \lambda \) are complex and computed by solving \( \det(M - \lambda I) = 0 \) (see next lectures)

3) The first linear solution \( X = A \exp(\lambda t) \) is analyzed.

We replace \( \lambda = \omega i + g \) and express the linear solution as:

\[
X = A \exp(i\omega t) \exp(gt)
\]

Oscillatory part of the solution: \( \omega \) is the oscillation frequency

Growth/decay \( \exp(gt) \) with time. The \( g \) is the growth/decay rate.

\( X^O = A \exp(i\omega) \)

4) The first linear solutions captures the basic ENSO oscillation. However \( g < 0 \) **therefore the solution decays over time**. Other linear solutions have much weaker \( g < 0 \) and therefore are secondary (they decay much quicker)..
Recall the linear solution of the ENSO model.

\[ X = A \exp(i\omega t) \exp(gt) \]

Oscillatory part of the solution: 
- \( \omega \) is the oscillation frequency
- \( A \) is the eigenvector (complex)

Growth/decay \( \exp(gt) \) with time. The \( g \) is the growth/decay rate.

5) We now recompute the linear solution for a **different parameter values** (here \( \eta_E \)).
Varying model parameters we can find both regimes where the first linear solution is either **stable** \( (g<0) \) or **unstable** \( (g>0) \). The frequency and eigenvectors are also modified.

Going from a stable to unstable regime or inversely is called a **Hopf Bifurcation**. Hereafter we will make this bifurcation happen in the model.
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Model with NonLinear Thermocline Feedback

- Hereafter we will **modify the model slightly** in order to reproduce a **maintained** ENSO oscillation.

- In order to obtain this regime we will combine two linear solutions of the model (one stable, one unstable) into a **Hopf bifurcation** by introducing a simple **nonlinear thermocline feedback**.

- The modification is for this lecture only (next lectures will consider different modifications based on winds).

**Model Solutions With Nonlinear Thermocline Feedback**

<table>
<thead>
<tr>
<th>Zonal winds</th>
<th>Zonal currents</th>
<th>Thermocline depth</th>
<th>SST</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u$</td>
<td>$U$</td>
<td>$H$</td>
<td>$T$</td>
</tr>
</tbody>
</table>

Averages of SST in eastern and western Pacific
Model with NonLinear Thermocline Feedback

We now include in the ENSO model a nonlinear thermocline feedback allowing **Hopf bifurcations** (for this lecture only)

\[
\begin{align*}
\text{Atmosphere} & \quad -yu - \partial_x \theta = 0 \\
& \quad yu - \partial_y \theta = 0 \\
& \quad -(\partial_x u + \partial_y v) = E_q/(1 - Q) \\
\text{Ocean} & \quad \partial_t U - c_1 YV + c_1 \partial_x H = c_1 \tau_x \\
& \quad YU + \partial_Y H = 0 \\
& \quad \partial_t H + c_1 (\partial_x U + \partial_Y V) = 0 \\
\text{SST} & \quad \partial_r T = -c_1 \zeta E_q + c_1 \eta(x, H)H \\
\end{align*}
\]

\(u, v\): winds
\(\theta\): temperature
\(E_q = \alpha T\): latent heat
\(U, V\): currents
\(H\): thermocline depth
\(\tau_x = \gamma u\): wind stress

\(T\): sea surface temperature

Thermocline feedback is here nonlinear
1) We now combine both a stable and unstable regime of the ENSO model. For this we modify the thermocline feedback with a **simple nonlinear switch**. In the SST equation of the ENSO model:

\[
\frac{\partial_T T}{c_1} = -\zeta E_q + \eta H
\]

the thermocline parameter reads:

\[
\eta(x) = \eta_W + \frac{\eta_W + \eta_E}{2} \tanh(7.5(x - \frac{L_O}{2}) + 1)
\]

modified with nonlinearity:

\[
\eta_E = \begin{cases} 
\eta_1 & \text{for } H_E \leq H_0 \quad \text{Stable regime} \\
\eta_0 & \text{for } H_E > H_0 \quad \text{Unstable regime}
\end{cases}
\]

- \(H_E\): Average of thermocline depth anomalies \(H\) in eastern Pacific
- \(H_O\): Threshold for switching between stable/unstable regimes

2) The ENSO model now experiences **Hopf Bifurcations** constantly as it switches back and forth between the stable and unstable regime.

This is a simple way to obtain a **maintained ENSO oscillation** without growth or decay of solutions.
We now have an ENSO model that reproduces a **basic ENSO oscillation** with regular El Nino and La Nina events. The model has a nonlinear thermocline feedback allowing constant **Hopf bifurcations** between stable and unstable regimes. Hereafter we analyze the dynamics of this basic ENSO oscillation.
A similar yet more complex nonlinear thermocline feedback is also used in the Cane-Zebiak model (1987), along with other nonlinearities allowing Hopf bifurcations. The model historically made the first accurate El Nino predictions in the 1980s.

\[
\frac{\partial T}{\partial t} = -\mathbf{u}_1 \cdot \nabla (\bar{T} + T) - \bar{\mathbf{u}}_1 \cdot \nabla T - \left\{ M(\bar{w}_s + w_s) - M(\bar{w}) \right\} \\
\times \bar{T}_z - M(\bar{w}_s + w_s) \frac{T - T_e}{H_1} - \alpha_s T, \quad (A11)
\]

\[T_e = \gamma T_{\text{sub}} + (1 - \gamma) T.\]

\[T_{\text{sub}} = \begin{cases} 
T_1 \{ \tanh[b_1(\bar{h} + h)] - \tanh(b_1 \bar{h}) \}, & h > 0 \\
T_2 \{ \tanh[b_2(\bar{h} - h)] - \tanh(b_2 \bar{h}) \}, & h < 0,
\end{cases}\]

\(\bar{h}(x)\) is the prescribed mean upper layer depth.

(SST budget in the Cane-Zebiak model:)

(Zebiak and Cane 1987)
Outline

1. Introduction to The El Nino Southern Oscillation

2. Simple Model Overview

3. Ocean Dynamics

4. SST Thermodynamics

5. Atmosphere Dynamics

6. Coupled Dynamics

7. Model Linear Solutions

8. Model with Nonlinear Thermocline Feedback

9. ENSO Oscillation Mechanisms
ENSO Oscillation Mechanisms

We now review **basic ENSO Oscillation Mechanisms**, that allow the **transitions between El Nina and La Nina**.

We will illustrate this with the ENSO model with Nonlinear Thermocline Feedback, however **the illustration is the same in the simpler ENSO model**.
We first recall **couplings between the ocean and atmosphere** in the model during El Nino.

1) Increased SST $T>0$ in the east releases latent heat $E_q$ and forces an **instant and delocalized zonal winds** response $u>0$ in the center (and conversely).

### Atmosphere

\[
\begin{align*}
-y v \cdot \partial_x \theta &= 0 \\
y u - \partial_y \theta &= 0 \\
-(\partial_x u + \partial_y v) &= E_q/(1 - Q)
\end{align*}
\]

### SST

\[
\frac{\partial T}{c_1} = -\zeta E_q + \eta H
\]

### Couplings

\[
E_q = \alpha_q T \quad \tau_x = \gamma u
\]

### Ocean

\[
\begin{align*}
\partial_x U &= -\frac{c_1}{H} Y V + c_1 \partial_x H - c_1 \tau_x \\
Y U + \partial_y H &= 0 \\
\tau_x H + c_1 \{ \partial_x U + \partial_y V \} &= 0
\end{align*}
\]
1) Increased SST $T>0$ in the east releases latent heat $E_q$ and forces an \textbf{instant and delocalized zonal winds} response $u>0$ in the center (and conversely).

\begin{align*}
\text{atmosphere} \\
-y v \cdot \partial_x \theta &= 0 \\
y u - \partial_y \theta &= 0 \\
-(\partial_x u + \partial_y v) &= E_q / (1 - Q)
\end{align*}

\begin{align*}
\text{SST} & \quad \partial \tau T / c_1 = -\zeta E_q + \eta H \\
\text{Couplings} & \quad E_q = \alpha_q T \\
& \quad \tau_x = \gamma u
\end{align*}

\begin{align*}
\text{Ocean} \\
\partial \tau U - c_1 Y V + c_1 \partial_x H &= c_1 \tau_x \\
Y U + \partial_Y H &= 0 \\
\partial \tau H + c_1 [\partial_x U + \partial_Y V] &= 0
\end{align*}
2) Positive winds $u>0$ force **ocean equatorial waves** (both upwelling and downwelling) that first increase the thermocline depth $H>0$ in the east then decrease it. This occurs through the **downwelling Kelvin waves** that reach the eastern Pacific first, and later the **upwelling Rossby waves** that reflect into upwelling Kelvin waves.

\[
\begin{align*}
\text{atmosphere} & \\
-yy - \partial_x \theta &= 0 \\
yu - \partial_y \theta &= 0 \\
-(\partial_x u + \partial_y v) &= E_q/(1 - Q)
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\[
\begin{align*}
\text{SST} & \\
\partial_x T/c_1 &= -\zeta E_q + \eta H
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\[
\begin{align*}
\text{atmosphere} & \quad -y v \cdot \partial_x \theta = 0 \\
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& \quad - (\partial_x u + \partial_y v) = E_q/(1 - Q)
\end{align*}
\]

\[
\begin{align*}
\text{SST} & \quad \partial_x T/c_1 = -\zeta E_q + \eta H \\
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\text{Ocean} & \quad \partial_x U - c_1 Y V + c_1 \partial_x H = c_1 \tau_x \\
& \quad Y U + \partial_Y H = 0 \\
& \quad \partial_x H + c_1 [\partial_x U + \partial_Y V] = 0
\end{align*}
\]
3) A deep thermocline \( H > 0 \) in the east increases the SST through the thermocline feedback. Recall the thermocline feedback parameter \( \eta \) is maximal in the eastern Pacific.

\[
\begin{align*}
\text{atmosphere} & \\
\text{-} y v \cdot \partial_x \theta &= 0 \\
y u - \partial_y \theta &= 0 \\
-(\partial_x u + \partial_y v) &= \frac{E_q}{1 - Q}
\end{align*}
\]

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\begin{align*}
\text{SST} & \\
\partial_x T / c_1 &= -\zeta E_q + \eta H
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\partial_x U - c_1 Y V + c_1 \partial_x H &= -c_1 \tau_x \\
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\partial_x H + c_1 (\partial_x U + \partial_Y V) &= 0
\end{align*}
\]
ENSO Oscillation Mechanisms

3) A deep thermocline $H > 0$ in the east increases the SST through the **thermocline feedback**.

\[
\begin{align*}
\text{atmosphere} & \quad -y v \cdot \partial_x \theta = 0 \\
y u - \partial_y \theta & = 0 \\
- (\partial_x u + \partial_y v) & = E_q/(1 - Q)
\end{align*}
\]

\[
\text{SST} \quad \frac{\partial \tau}{c_1} = -\zeta E_q + \eta H
\]

\[
\text{Couplings} \quad E_q = \alpha_q T \quad \tau_x = \gamma u
\]

\[
\text{Ocean} \quad \begin{align*}
\partial_x U - c_1 Y V + c_1 \partial_x H & = c_1 \tau_x \\
Y U + \partial_Y H & = 0 \\
\partial_x H + c_1 \left[ \partial_x U + \partial_Y V \right] & = 0
\end{align*}
\]
For the equatorial Pacific to oscillate between El Nino and La Nina states, **two feedbacks are necessary:**
1) A **positive feedback** that reinforces initial anomalies and allows the growth of Nino (or Nina) events.
2) A **negative delayed feedback** that eventually reverses the conditions.

### Atmosphere
\[-y v \cdot \partial_x \theta = 0\]
\[y u - \partial_y \theta = 0\]
\[-(\partial_x u + \partial_y v) = E_q / (1 - Q)\]

### SST
\[\partial_t T / c_1 = -\zeta E_q + \eta H\]

### Couplings
\[E_q = \alpha_q T\]
\[\tau_x = \gamma u\]

### Ocean
\[\partial_t U - c_1 Y V + c_1 \partial_x H - c_1 \tau_x = 0\]
\[Y U + \partial_Y H = 0\]
\[\partial_x H + c_1 (\partial_x U + \partial_Y V) = 0\]
The **positive feedback** reinforces SST anomalies $T>0$ in the eastern Pacific through zonal winds $u>0$ that force first a rapid ocean response $H>0$ reaching the eastern Pacific. This is also called the **Bjerknes feedback** after its discoverer. This allows El Niño or La Nina events to develop.

\[
\begin{align*}
\text{atmosphere} & \quad -y \cdot \partial_x \theta = 0 \\
& \quad yu - \partial_y \theta = 0 \\
& \quad -(\partial_x u + \partial_y v) = E_q/(1 - Q) \\
\text{SST} & \quad \partial_T / c_1 = -\zeta E_q + \eta H \\
\text{Couplings} & \quad E_q = \alpha_q T, \quad \tau_x = \gamma u \\
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**atmosphere**

\[-y v \partial_x \theta = 0\]
\[yu - \partial_y \theta = 0\]
\[-(\partial_x u + \partial_y v) = \frac{E_q}{1 - Q}\]

**SST**

\[
\frac{\partial_T}{c_1} = -\zeta E_q + \eta H
\]

**Couplings**

\[E_q = \alpha_q T\]
\[\tau_x = \gamma u\]

**Ocean**

\[
\partial_T U - c_1 Y V + c_1 \partial_x H = c_1 \tau_x
\]
\[Y U + \partial_Y H = 0\]
\[
\partial_T H + c_1 (\partial_x U + \partial_Y V) = 0
\]

**Normal**

- Zonal SST gradient (warmer waters in west)
- Trade winds
- Thermocline
- Zonal gradient

**El Niño**

- Zonal SST gradient weakens
- Trade winds weaken
- Thermocline
- Gradient weakens
- Weak circulation
The **positive feedback** also reinforces SST anomalies $T<0$ in the eastern Pacific through zonal winds $u<0$ that force first a rapide ocean response $H<0$ reaching the eastern Pacific. This is also called the **Bjerknes feedback** after its discoverer. This allows El Nino or La Nina events to develop.

\[
\begin{align*}
\text{atmosphere} & : -y v \cdot \partial_x \theta = 0 \\
y u - \partial_y \theta = 0 \\
- (\partial_x u + \partial_y v) &= E_q / (1 - Q)
\end{align*}
\]

\[
\text{SST} \quad \frac{\partial T}{c_1} = -\zeta E_q + \eta H
\]

\[
\text{Couplings} \quad E_q = \alpha_q T \quad \tau_x = \gamma u
\]

\[
\text{Ocean} \quad \partial_x U - c_1 Y V + c_1 \partial_x H = c_1 \tau_x \\
Y U + \partial_Y H = 0 \\
\partial_x H + c_1 \left( \partial_x U + \partial_Y V \right) = 0
\]

**Normal**

- Zonal SST gradient (warmer waters in west)
- Trade winds

**La Niña**

- SST gradient increases
- Trade winds increase
- Strong circulation

**Thermocline**

- Zonal gradient
- Australia
- Peru
The **delayed negative feedback** eventually decreases SST anomalies $T>0$ in the eastern Pacific through an opposite ocean response $H<0$ that eventually reaches the eastern Pacific. The delayed negative feedback comes from the slow adjustment of the ocean.

<table>
<thead>
<tr>
<th>Atmosphere</th>
<th>SST</th>
<th>Ocean</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-y v \cdot \partial_x \theta = 0$</td>
<td>$\partial_T T/c_1 = -\zeta E_q + \eta H$</td>
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<td>$\partial_x H + c_1 [\partial_x U + \partial_Y V] = 0$</td>
</tr>
</tbody>
</table>

![Diagram](image)

**Diagram:**
- **Delayed negative feedback** indicated by the rectangular box.
The **delayed negative feedback** in the ocean can be explained by wave propagations. This is the **delayed oscillator theory** for ENSO (Schopf and Suarez 1988).

Zonal winds \( u > 0 \) during El Nino create **downwelling Kelvin** waves that quickly reaches the eastern Pacific (positive feedback), but the delayed negative feedback comes from the **reflected upwelling Rossby** waves that reach the eastern Pacific later.

(Schopf and Suarez 1988)
The **delayed negative feedback** in the ocean can also be explained by meridional advection of heat content. This is the **recharge-discharge theory** for ENSO (Jin 1997).

- In between El Nino and La Nina, the thermocline is shallow overall in the Pacific: this is the **recharged state**

- In between La Nina and El Nina, the thermocline is deep overall in the Pacific: this is the **discharged state**

The recharge and discharge of heat content is due to the slow ocean adjustment, in particular **meridional advection** $v$ (towards the pole during El Nino and towards the equator during La Nina). This is implicit in the ENSO model.
The delayed negative feedback in the ocean can also be explained by meridional advection of heat content. This is the recharge-discharge theory for ENSO (Jin 1997).

- To illustrate the recharge-discharge mechanism, consider for example the **WWV index** in observations.

**Niño3.4-SST**: average of SST anomalies in east (170W-120W, 5N-5S)

**WWV**: average of thermocline anomalies depth over basin (120E-90W, 5N-5S)

Timeseries of **Niño3.4-SST** (C) and **WWV** ($10^{14}$m$^3$)

(NOAA)
The ENSO oscillation proceeds through 4 phases: Recharged, El Niño, Discharged and La Nina conditions in that order.

- Nino 3.4 SST and WWV are in quadrature.
- Increased WWV is a precursor for El Niño
- Decreased WWV is a precursor for La Nina
The recharge and discharge of heat content is due to the slow ocean adjustment, in particular *meridional advection* $v$ (towards the pole during El Niño and towards the equator during La Niña). This is implicit in the ENSO model.
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8. Model with Nonlinear Thermocline Feedback
9. ENSO Oscillation Mechanisms
Overview of next lectures

1) The present ENSO model will be **modified in next lectures** to capture more complex features of the ENSO.

2) We will cover more complex features such as the irregularity and intermittency of ENSO, its diversity including rare extreme El Nino events, and its synchronization to the seasonal cycle.

Atmosphere

\[-yv - \partial_x \theta = 0\]
\[yu - \partial_y \theta = 0\]
\[-(\partial_x u + \partial_y v) = \frac{E_q}{1 - Q}\]

**Equations:**
- $u, v$: winds
- $\theta$: temperature
- $E_q = \alpha T$: latent heat

Ocean

\[\partial_t U - c_1 Y V + c_1 \partial_x H = c_1 \tau_x\]
\[Y U + \partial_Y H = 0\]
\[\partial_t H + c_1 (\partial_x U + \partial_Y V) = 0\]

**Equations:**
- $U, V$: currents
- $H$: thermocline depth
- $\tau_x = \gamma u$: wind stress

SST

\[\partial_t T = -c_1 \zeta E_q + c_1 \eta(x) H\]

**Equations:**
- $T$: sea surface temperature

**Addition of stochastic wind bursts**

**Nonlinear thermocline feedback removed**