Simple Mathematical, Dynamical Stochastic Models Capturing the Observed Diversity of the El Niño Southern Oscillation (ENSO)

Lecture 4: Stochastic Toolkits, Basic Numerics and Linear Stability Analysis of the ENSO Models

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Outline of this lecture

1. Solving of the coupled model via the method of lines

2. Linear stability analysis

3. Basic stochastic toolkits (simple SDE, additive v.s. multiplicative noise, Gaussian v.s. non-Gaussian PDFs, Markov jump process ...)


Review of the ENSO model

Atmosphere

\[- y v - \partial_x \theta = 0\]
\[y u - \partial_y \theta = 0\]
\[- (\partial_x u + \partial_y v) = E_q/(1 - \overline{Q})\]
\[u, v : \text{winds}\]
\[\theta : \text{temperature}\]
\[E_q = \alpha T : \text{latent heat}\]

Ocean

\[\partial_z U - c_1 Y V + c_1 \partial_x H = c_1 \tau_x\]
\[Y U + \partial_y H = 0\]
\[\partial_z H + c_1 (\partial_x U + \partial_y V) = 0\]
\[U, V : \text{ocean current}\]
\[H : \text{thermocline depth}\]
\[\tau_x = \gamma u : \text{wind stress}\]

SST

\[\partial_z T = -c_1 \zeta E_q + c_1 \eta H\]
\[T : \text{sea surface temperature}\]
\[\eta : \text{thermocline feedback}\]
\[(\eta \text{ stronger in eastern Pacific)}\]

✓ Deterministic, ✓ Linear and ? Stable
Meridional truncation \((x, y, \tau) \rightarrow (x, \tau)\)

**Original**

Atmosphere
\[-yv - \partial_x \theta = 0\]
\[yu - \partial_y \theta = 0\]
\[-(\partial_x u + \partial_y v) = E_q/(1 - \bar{Q})\]

Ocean
\[\partial_\tau U - c_1 Y V + c_1 \partial_x H = c_1 \tau_x\]
\[Y U + \partial_y H = 0\]
\[\partial_\tau H + c_1(\partial_x U + \partial_y V) = 0\]

SST
\[\partial_\tau T = -c_1 \zeta E_q + c_1 \eta H\]

**Truncated to \(\phi_0(y)\) and \(\psi_0(y)\)**

\[\partial_x K_A = -\chi_A E_q (2 - 2\bar{Q})^{-1}\]
\[-\partial_x R_A / 3 = -\chi_A E_q (3 - 3\bar{Q})^{-1}\]
\[\partial_\tau K_O + c_1 \partial_x K_O = \chi_O c_1 \gamma (K_A - R_A)/2\]
\[\partial_\tau R_O - (c_1 / 3) \partial_x R_O = -\chi_O c_1 \gamma (K_A - R_A)/3\]
\[\partial_\tau T = -c_1 \zeta E_q + c_1 \eta (K_O + R_O)\]

Reconstructed variables:

\[u = (K_A - R_A)\phi_0 + (R_A / \sqrt{2})\phi_2\]
\[\theta = -(K_A + R_A)\phi_0 - (R_A / \sqrt{2})\phi_2\]
\[U = (K_O - R_O)\psi_0 + (R_O / \sqrt{2})\psi_2\]
\[H = (K_O + R_O)\psi_0 + (R_O / \sqrt{2})\psi_2\]
Method of characteristics is applied to derive the meridional truncated equations.

A good reference to follow is the following book
Introduction to PDEs and Waves for the Atmosphere and Ocean, 2003, by Andrew J. Majda, Chapter 9

Other useful references for technical details are:
Stechmann and Majda, *Monthly Weather Review*, 2015, Appendix,
Biello and Majda, *Dynamics of Atmospheres and Oceans*, 2006, Appendix c.4

The details are also included in the Reserve Slides.
Numerical solver of the coupled model

We solve the meridionally truncated model.

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We solve the meridionally truncated model.
Atmosphere
\[ \partial_x K_A = -\chi_A E_q (2 - 2\tilde{Q})^{-1} \]
\[ - \partial_x R_A / 3 = -\chi_A E_q (3 - 3\tilde{Q})^{-1} \]
(B.C.)
\[ K_A(0, \tau) = K_A(L_A, \tau) \]
\[ R_A(0, \tau) = R_A(L_A, \tau) \]

Ocean
\[ \partial_\tau K_O + c_1 \partial_x K_O = \chi_O c_1 \gamma (K_A - R_A) / 2 \]
\[ \partial_\tau R_O - (c_1 / 3) \partial_x R_O = -\chi_O c_1 \gamma (K_A - R_A) / 3 \]
(B.C.)
\[ K_O(0, \tau) = r_W R_O(0, \tau) \]
\[ R_O(L_O, \tau) = r_E K_O(L_O, \tau) \]

SST
\[ \partial_\tau T = -c_1 \zeta E_q + c_1 \eta (K_O + R_O) \]
Solvability condition of the atmosphere model.

\[ \partial_x K_A = -\chi_A E_q (2 - 2\bar{Q})^{-1} \]
\[ - \partial_x R_A/3 = -\chi_A E_q (3 - 3\bar{Q})^{-1} \]
\[ K_A(0, \tau) = K_A(L_A, \tau) \]
\[ R_A(0, \tau) = R_A(L_A, \tau) \]

Intuition: With periodic boundary condition,

\[ \frac{1}{L_A} \int_0^{L_A} \partial_x K_A \, dx = K_A(L_A) - K_A(0) = 0, \]
\[ \frac{1}{L_A} \int_0^{L_A} \partial_x R_A \, dx = R_A(L_A) - R_A(0) = 0, \]

which implies

\[ \frac{1}{L_A} \int_0^{L_A} E_q \, dx = 0. \]
Solvability condition of the atmosphere model.

\[ + \partial_x K_A = -\chi_A (E_q - \langle E_q \rangle)(2 - 2\bar{Q})^{-1} \]
\[ - \partial_x R_A / 3 = -\chi_A (E_q - \langle E_q \rangle)(3 - 3\bar{Q})^{-1} \]

\[ K_A(0, \tau) = K_A(L_A, \tau) \]
\[ R_A(0, \tau) = R_A(L_A, \tau) \]

The absence of dissipation in the atmosphere imposes a solvability condition of a zero equatorial zonal mean of latent heating forcing (Majda and Klein, 2003; Stechmann and Ogrosky, 2014). This reads:

\[ \frac{1}{L_A} \int_0^{L_A} E_q \, dx = 0. \]

In particular,

\[ E_q - \langle E_q \rangle = \begin{cases} \alpha q T - \langle E_q \rangle & \text{for } x \in [0, L_O], \\ -\langle E_q \rangle & \text{for } x \in [L_O, L_A]. \end{cases} \]
Solvability condition of the atmosphere model.

\[
+ \partial_x K_A = -\chi_A (E_q - \langle E_q \rangle)(2 - 2\bar{Q})^{-1}
\]
\[
- \partial_x R_A / 3 = -\chi_A (E_q - \langle E_q \rangle)(3 - 3\bar{Q})^{-1}
\]
\[
K_A(0, \tau) = K_A(L_A, \tau)
\]
\[
R_A(0, \tau) = R_A(L_A, \tau)
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The absence of dissipation in the atmosphere imposes a **solvability condition** of a zero equatorial zonal mean of latent heating forcing (Majda and Klein, 2003; Stechmann and Ogrosky, 2014). This reads:

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E_q - \langle E_q \rangle = \begin{cases} 
\alpha_q T - \langle E_q \rangle & \text{for } x \in [0, L_O], \\
-\langle E_q \rangle & \text{for } x \in [L_O, L_A]. 
\end{cases}
\]

Note that if \(K_A\) is a solution, then \(K_A + \text{const}\) is also a solution.
Solvability condition of the atmosphere model.

\[ d_A K_A + \partial_x K_A = -\chi_A(E_q - \langle E_q \rangle)(2 - 2\bar{Q})^{-1} \]
\[ d_A R_A - \partial_x R_A / 3 = -\chi_A(E_q - \langle E_q \rangle)(3 - 3\bar{Q})^{-1} \]
\[ K_A(0, \tau) = K_A(L_A, \tau) \]
\[ R_A(0, \tau) = R_A(L_A, \tau) \]

The absence of dissipation in the atmosphere imposes a solvability condition of a zero equatorial zonal mean of latent heating forcing (Majda and Klein, 2003; Stechmann and Ogrosky, 2014). This reads:

\[ \frac{1}{L_A} \int_0^{L_A} E_q dx = 0. \]

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Note that if \( K_A \) is a solution, then \( K_A + \text{const} \) is also a solution.

To guarantee the uniqueness of the solution, we add a tiny damping \( d_A \) into the model. With this tiny damping, the system has a unique solution with \( \langle K_A \rangle = \langle R_A \rangle = 0 \).

For numerical values, \( d_A = 10^{-8} \) while other coefficients are of order \( O(1) \).
Solving the coupled system using method of lines.

\[
\begin{align*}
\partial_{\tau} K_O + c_1 \partial_x K_O &= \frac{a}{2} (K_A - R_A), & K_O(0, \tau) &= r_W R_O(0, \tau), \\
\partial_{\tau} R_O - \frac{c_1}{3} \partial_x R_O &= -\frac{a}{3} (K_A - R_A), & R_O(L_O, \tau) &= r_E K_O(L_O, \tau), \\
\partial_{\tau} T &= -b T + c_1 \eta (K_O + R_O), \\
d_A K_A + \partial_x K_A &= \frac{m_1}{\alpha q} \left(1_{[0,L_O]} \alpha q \ T - \langle E_q \rangle \right), & K_A(0, \tau) &= K_A(L_A, \tau), \\
d_A R_A - \frac{1}{3} \partial_x R_A &= \frac{m_2}{\alpha q} \left(1_{[0,L_O]} \alpha q \ T - \langle E_q \rangle \right), & R_A(0, \tau) &= R_A(L_A, \tau),
\end{align*}
\]

where for notation simplicity we have defined

\[ a = \chi_O c_1 \gamma, \quad b = c_1 \zeta \alpha q, \quad m_1 = -\chi_A \alpha q / (2 - 2 \bar{Q}), \quad m_2 = -\chi_A \alpha q / (3 - 3 \bar{Q}), \]

Domain: \( K_O, R_O \) and \( T: [0, L_O] \), \( K_A \) and \( R_A: [0, L_A] \).

Define a vector \( \mathbf{u} \) that contains all the prognostic variables at discrete equal-partitioned grid points,

\[ \mathbf{u} = (K_O; R_O; T) = (K_{O,1}, \ldots K_{O,N_O}, R_{O,1}, \ldots R_{O,N_O}, T_1, \ldots T_{N_O})^*. \]

Therefore, we solve the following equation

\[
\frac{d \mathbf{u}}{d\tau} = \mathbf{M} \mathbf{u}.
\]

Recall: \( E_q = \alpha q T \)
\[
\frac{d}{d\tau} \begin{pmatrix} K_O \\ R_O \\ T \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix} \begin{pmatrix} K_O \\ R_O \\ T \end{pmatrix}
\]

Let's start with the ocean model

\[
\partial_\tau K_O + c_1 \partial_x K_O = \frac{a}{2} (K_A - R_A), \quad K_O(0, \tau) = r_W R_O(0, \tau),
\]

\[
\partial_\tau R_O - \frac{c_1}{3} \partial_x R_O = -\frac{a}{3} (K_A - R_A), \quad R_O(L_O, \tau) = r_E K_O(L_O, \tau),
\]

An upwind scheme is utilized for the spatial discretization,

\[
\partial_\tau K_{O,i} = -\frac{c_1}{\Delta x} (K_{O,i} - K_{O,i-1}) + \frac{a}{2} (K_{A,i} - R_{A,i}), \quad i = 1, \ldots, N_O,
\]

\[
\partial_\tau R_{O,i} = \frac{c_1}{3\Delta x} (R_{O,i+1} - R_{O,i}) - \frac{a}{3} (K_{A,i} - R_{A,i}), \quad i = 1, \ldots, N_O,
\]

where the boundary conditions are given by \( K_{O,0} = r_W R_{O,1} \) and \( R_{O,N_O+1} = r_E K_{O,N_O} \).

**Upwind scheme.**

\[
\frac{\partial u}{\partial \tau} + a \frac{\partial u}{\partial x} = 0 \implies \begin{cases} 
\frac{u_{i+1}^n - u_i^n}{\Delta \tau} + a \frac{u_i^n - u_{i-1}^n}{\Delta x} = 0 & \text{for } a > 0 \\
\frac{u_{i+1}^n - u_i^n}{\Delta \tau} + a \frac{u_{i+1}^n - u_i^n}{\Delta x} = 0 & \text{for } a < 0
\end{cases}
\]

\[
u = u_0(a\tau - x) \implies \tau = \frac{x}{a} + \text{const.}
\]

Stability: \( \left| \frac{a\Delta \tau}{\Delta x} \right| \leq 1 \).
\[ \frac{d}{d\tau} \begin{pmatrix} K_O \\ R_O \\ T \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix} \begin{pmatrix} K_O \\ R_O \\ T \end{pmatrix} \]

Let’s start with the ocean model

\[ \partial_\tau K_O + c_1 \partial_x K_O = \frac{a}{2} (K_A - R_A), \quad K_O(0, \tau) = r_WR_O(0, \tau), \]

\[ \partial_\tau R_O - \frac{c_1}{3} \partial_x R_O = -\frac{a}{3} (K_A - R_A), \quad R_O(L_O, \tau) = r_EK_O(L_O, \tau), \]

An upwind scheme is utilized for the spatial discretization,

\[ \partial_\tau K_O, i = -\frac{c_1}{\Delta x} (K_O, i - K_O, i-1) + \frac{a}{2} (K_A, i - R_A, i), \quad i = 1, \ldots, N_O, \]

\[ \partial_\tau R_O, i = \frac{c_1}{3\Delta x} (R_O, i+1 - R_O, i) - \frac{a}{3} (K_A, i - R_A, i), \quad i = 1, \ldots, N_O, \]

where the boundary conditions are given by \( K_O, 0 = r_WR_O, 1 \) and \( R_O, N_O+1 = r_EK_O, N_O \).

**Upwind scheme.**

\[ \frac{\partial u}{\partial \tau} + a \frac{\partial u}{\partial x} = 0 \implies \begin{cases} \frac{u_{i+1}^n - u_i^n}{a \Delta \tau} + a \frac{u_{i+1}^n - u_{i-1}^n}{a \Delta x} = 0 & \text{for } a > 0 \\ \frac{u_{i+1}^n - u_i^n}{a \Delta \tau} + a \frac{u_{i+1}^n - u_{i-1}^n}{a \Delta x} = 0 & \text{for } a < 0 \end{cases} \]

\[ u = u_0(a\tau - x) \implies \tau = \frac{x}{a} + \text{const.} \quad \text{Stability: } \left| \frac{a \Delta \tau}{\Delta x} \right| \leq 1. \]
\[ \frac{d}{d\tau} \begin{pmatrix} K_O \\ R_O \\ T \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix} \begin{pmatrix} K_O \\ R_O \\ T \end{pmatrix} \]

Let's start with the ocean model

\[ \partial_\tau K_O + c_1 \partial_x K_O = \frac{a}{2} (K_A - R_A), \quad K_O(0, \tau) = r_WR_O(0, \tau), \]
\[ \partial_\tau R_O - \frac{c_1}{3} \partial_x R_O = -\frac{a}{3} (K_A - R_A), \quad R_O(L_O, \tau) = r_EK_O(L_O, \tau), \]

An upwind scheme is utilized for the spatial discretization,

\[ \partial_\tau K_{O,i} = -\frac{c_1}{\Delta x} (K_{O,i} - K_{O,i-1}) + \frac{a}{2} (K_{A,i} - R_{A,i}), \quad i = 1, \ldots, N_O, \]
\[ \partial_\tau R_{O,i} = \frac{c_1}{3\Delta x} (R_{O,i+1} - R_{O,i}) - \frac{a}{3} (K_{A,i} - R_{A,i}), \quad i = 1, \ldots, N_O, \]

where the boundary conditions are given by \( K_{O,0} = r_WR_{O,1} \) and \( R_{O,N_O+1} = r_EK_{O,N_O} \).

**Upwind scheme.**

\[ \frac{\partial u}{\partial \tau} + a \frac{\partial u}{\partial x} = 0 \implies \begin{cases} \frac{u_{i+1}^n - u_i^n}{\Delta \tau} + a \frac{u_i^n - u_{i-1}^n}{\Delta x} = 0 & \text{for } a > 0 \\ \frac{u_{i+1}^n - u_i^n}{\Delta \tau} + a \frac{u_{i+1}^n - u_i^n}{\Delta x} = 0 & \text{for } a < 0 \end{cases} \]

\[ u = u_0(a\tau - x) \implies \tau = \frac{x}{a} + \text{const.} \quad \text{Stability: } \left| \frac{a\Delta \tau}{\Delta x} \right| \leq 1. \]
\[
\frac{d}{dT} \begin{pmatrix}
K_O \\
R_O \\
T
\end{pmatrix} = \begin{pmatrix}
M_{11} & M_{12} & M_{13} \\
M_{21} & M_{22} & M_{23} \\
M_{31} & M_{32} & M_{33}
\end{pmatrix} \begin{pmatrix}
K_O \\
R_O \\
T
\end{pmatrix}
\]

\[
\partial_T K_{O,i} = -\frac{c_1}{\Delta x} (K_{O,i} - K_{O,i-1}) + \frac{a}{2} (K_{A,i} - R_{A,i}) ,
\]

\[
\partial_T R_{O,i} = \frac{c_1}{3\Delta x} (R_{O,i+1} - R_{O,i}) - \frac{a}{3} (K_{A,i} - R_{A,i}) ,
\]

for \(i = 1, \ldots, N_O\). The boundary conditions are given by

\[
K_{O,0} = r_W R_{O,1}, \quad \text{and} \quad R_{O,N_O+1} = r_E K_{O,N_O}. 
\]
\[
\frac{d}{dT} \begin{pmatrix} K_0 \\ R_0 \\ T \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{23} & M_{33} \end{pmatrix} \begin{pmatrix} K_0 \\ R_0 \\ T \end{pmatrix}
\]

Next, for the atmosphere model

\[d_A K_A + \partial_x K_A = m_1 (1_{[0, L_O]} \alpha_q T - \langle E_q \rangle) / \alpha_q, \quad K_A(0, \tau) = K_A(L_A, \tau),\]

\[d_A R_A - \partial_x R_A / 3 = m_2 (1_{[0, L_O]} \alpha_q T - \langle E_q \rangle) / \alpha_q, \quad R_A(0, \tau) = R_A(L_A, \tau),\]

The discretization results in

\[d_A K_{A,i} + \frac{1}{\Delta x} (K_{A,i+1} - K_{A,i}) = m_1 \left( T_i - \frac{1}{N_A} \sum_{j=1}^{N_O} T_j \right),\]

\[d_A R_{A,i} - \frac{1}{3 \Delta x} (R_{A,i+1} - R_{A,i}) = m_2 \left( T_i - \frac{1}{N_A} \sum_{j=1}^{N_O} T_j \right), \quad i = 1, \ldots, N_O,\]

and

\[d_A K_{A,i} + \frac{1}{\Delta x} (K_{A,i+1} - K_{A,i}) = m_1 \left( - \frac{1}{N_A} \sum_{j=1}^{N_O} T_j \right),\]

\[d_A R_{A,i} - \frac{1}{3 \Delta x} (R_{A,i+1} - R_{A,i}) = m_2 \left( - \frac{1}{N_A} \sum_{j=1}^{N_O} T_j \right), \quad i = N_O + 1, \ldots, N_A,\]

where we have used the fact \(E_q = \alpha_q T\). Note that the averaging is taken for the whole equator and therefore the factor \(1/N_A\) in front of the summation appears.
\[
\frac{d}{d\tau} \begin{pmatrix} K_0 \\ R_0 \\ T \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{23} & M_{33} \end{pmatrix} \begin{pmatrix} K_0 \\ R_0 \\ T \end{pmatrix}
\]

Rearranging the terms gives

\[
K_{A,i+1} + (d_A \Delta x - 1)K_{A,i} = \frac{N_A - 1}{N_A} \Delta x m_1 \left( T_i - \frac{1}{N_A - 1} \sum_{1 \leq j \leq N_O} T_j \right),
\]

\[
-R_{A,i+1} + (3d_A \Delta x + 1)R_{A,i} = 3 \frac{N_A - 1}{N_A} \Delta x m_2 \left( T_i - \frac{1}{N_A - 1} \sum_{1 \leq j \leq N_O} T_j \right), \quad i = 1, \ldots N_O
\]

and

\[
K_{A,i+1} + (d_A \Delta x - 1)K_{A,i} = \frac{N_A - 1}{N_A} \Delta x m_1 \left( - \frac{1}{N_A - 1} \sum_{1 \leq j \leq N_O} T_j \right),
\]

\[
-R_{A,i+1} + (3d_A \Delta x + 1)R_{A,i} = 3 \frac{N_A - 1}{N_A} \Delta x m_2 \left( - \frac{1}{N_A - 1} \sum_{1 \leq j \leq N_O} T_j \right), \quad i = N_O + 1, \ldots, N_A
\]

Thus, \(K_{A,i}\) and \(R_{A,i}\) for \(i = 1, \ldots N_O\) can be solved via the following linear systems,

\[
M_K \cdot K_A = \tilde{C}_1 B_K T,
\]

\[
M_R \cdot K_R = \tilde{C}_2 B_R T.
\]
\[
\frac{d}{d\tau} \begin{pmatrix} K_O \\ R_O \\ T \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{23} & M_{33} \end{pmatrix} \begin{pmatrix} K_O \\ R_O \\ T \end{pmatrix}
\]

\[
K_{A,i+1} + (d_A \Delta x - 1) K_{A,i} = \frac{N_A - 1}{N_A} \Delta x m_1 \left( T_i - \frac{1}{N_A - 1} \sum_{1 \leq j \leq N_O} T_j \right), \quad i = 1, \ldots, N_O,
\]

\[
K_{A,i+1} + (d_A \Delta x - 1) K_{A,i} = \frac{N_A - 1}{N_A} \Delta x m_1 \left( -\frac{1}{N_A - 1} \sum_{1 \leq j \leq N_O} T_j \right), \quad i = N_O + 1, \ldots, N_A,
\]

\[
M_K \cdot K_A = \tilde{C}_1 B_K T,
\]

\[
M_R \cdot K_R = \tilde{C}_2 B_R T,
\]

where

\[
\beta = d_A \Delta x, \quad \tilde{C}_1 = \frac{N_A - 1}{N_A} \Delta x m_1, \quad \tilde{C}_2 = 3 \frac{N_A - 1}{N_A} \Delta x m_2,
\]

\[
M_K = \begin{pmatrix}
-1 + \beta & 1 \\
1 & -1 + \beta \\
& \ddots & \ddots & 1 \\
& & 1 & -1 + \beta \\
& & & 1 & -1 + \beta \\
& & & & 1 & -1 + \beta \\
\end{pmatrix}_{N_A \times N_A}, \quad K_A = \begin{pmatrix}
K_{A,1} \\
\vdots \\
K_{A,N_O} \\
\vdots \\
K_{A,N_A - 1} \\
K_{A,N_A} \\
\end{pmatrix}_{N_A \times 1}
\]
\[
\frac{d}{d\tau} \begin{pmatrix} K_0 \\ R_0 \\ T \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{23} & M_{33} \end{pmatrix} \begin{pmatrix} K_0 \\ R_0 \\ T \end{pmatrix}
\]

\[
-R_{A,i+1} + (3d_A\Delta x + 1)R_{A,i} = 3\frac{N_A - 1}{N_A} \Delta x m_2 \left( T_i - \frac{1}{N_A - 1} \sum_{1 \leq j \leq N_O}^{i \neq i} T_j \right), \quad i = 1, \ldots, N_O
\]

\[
-R_{A,i+1} + (3d_A\Delta x + 1)R_{A,i} = 3\frac{N_A - 1}{N_A} \Delta x m_2 \left( -\frac{1}{N_A - 1} \sum_{1 \leq j \leq N_O} T_j \right), \quad i = N_O + 1, \ldots, N_A
\]

\[
M_K \cdot K_A = \tilde{C}_1 B_K T,
\]

\[
M_R \cdot K_R = \tilde{C}_2 B_R T,
\]

where

\[
\beta = d_A\Delta x, \quad \tilde{C}_1 = \frac{N_A - 1}{N_A} \Delta x m_1, \quad \tilde{C}_2 = 3\frac{N_A - 1}{N_A} \Delta x m_2,
\]

\[
M_R = \begin{pmatrix}
1 + 3\beta & -1 & \cdots & -1 \\
\vdots & 1 + 3\beta & \ddots & \vdots \\
-1 & \cdots & 1 + 3\beta & -1 \\
-1 & \cdots & \cdots & 1 + 3\beta \\
\end{pmatrix}
\quad \text{and} \quad
R_A = \begin{pmatrix}
R_{A,1} \\
R_{A,2} \\
\vdots \\
R_{A,N_O} \\
R_{A,N_A-1} \\
R_{A,N_A} \\
\end{pmatrix}
\]
\[
\frac{d}{dT} \begin{pmatrix} K_O \\ R_O \\ T \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix} \begin{pmatrix} K_O \\ R_O \\ T \end{pmatrix}
\]

\[K_{A,i+1} + (d_A \Delta x - 1) K_{A,i} = \frac{N_A - 1}{N_A} \Delta x m_1 \left( T_i - \frac{1}{N_A - 1} \sum_{1 \leq j \leq N_O} T_j \right), \quad i = 1, \ldots, N_O,\]

\[K_{A,i+1} + (d_A \Delta x - 1) K_{A,i} = \frac{N_A - 1}{N_A} \Delta x m_1 \left( - \frac{1}{N_A - 1} \sum_{1 \leq j \leq N_O} T_j \right), \quad i = N_O + 1, \ldots, N_A,\]

\[
M_K \cdot K_A = \tilde{C}_1 B_K T,
\]

\[
M_R \cdot K_R = \tilde{C}_2 B_R T,
\]

where

\[
B_K = B_R = \begin{pmatrix}
-\frac{1}{N_A - 1} & -\frac{1}{N_A - 1} & \cdots & -\frac{1}{N_A - 1} \\
\frac{1}{N_A - 1} & 1 & \cdots & 1 \\
\frac{1}{N_A - 1} & 1 & \cdots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
\frac{1}{N_A - 1} & 1 & \cdots & 1 \\
\frac{1}{N_A - 1} & 1 & \cdots & 1 \\
\frac{1}{N_A - 1} & 1 & \cdots & 1
\end{pmatrix}_{N_A \times N_O}, \quad T = \begin{pmatrix} T_1 \\ \vdots \\ T_{N_O} \end{pmatrix}_{N_O \times 1}
\]

Clearly, \( K_A \) and \( R_A \) can be expressed by \( T \).
\[
\frac{d}{d\tau} \begin{pmatrix} K_O \\ R_O \\ T \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{23} & M_{33} \end{pmatrix} \begin{pmatrix} K_O \\ R_O \\ T \end{pmatrix}
\]

\[M_K \cdot K_A = \tilde{C}_1 B_K T,\]
\[M_R \cdot K_R = \tilde{C}_2 B_R T.\]

Recall the ocean model,
\[
\partial_{\tau} K_{O,i} = -\frac{c_1}{\Delta x} (K_{O,i} - K_{O,i-1}) + \frac{a}{2} (K_{A,i} - R_{A,i}),
\]
\[
\partial_{\tau} R_{O,i} = \frac{c_1}{3\Delta x} (R_{O,i+1} - R_{O,i}) - \frac{a}{3} (K_{A,i} - R_{A,i}),
\]

Since \(K_A\) and \(R_A\) can be expressed by \(T\), we have
\[
M_{13} = \frac{a}{2} \left[ \tilde{C}_1 M_K^{-1} B_K - \tilde{C}_2 M_R^{-1} B_R \right] \bigg|_{\text{Row } \{1:N_O\}},
\]
\[
M_{23} = -\frac{a}{3} \left[ \tilde{C}_1 M_K^{-1} B_K - \tilde{C}_2 M_R^{-1} B_R \right] \bigg|_{\text{Row } \{1:N_O\}},
\]

where the original matrices on the right hand side should be of size \(N_A \times N_O\) but only the first \(N_O\) rows are utilized to form \(M_{13}\) and \(M_{23}\) which correspond to \(K_{A,1}, \ldots, K_{A,N_O}\) and \(R_{A,1}, \ldots, R_{A,N_O}\) within the Pacific band.
\[
\frac{d}{d\tau} \begin{pmatrix} K_0 \\ R_0 \\ T \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{23} & M_{33} \end{pmatrix} \begin{pmatrix} K_0 \\ R_0 \\ T \end{pmatrix}
\]

Finally, let's consider the SST model,

\[
\partial_\tau T = -bT + c_1 \eta(K_O + R_O),
\]

the discrete form of which is straightforward,

\[
\partial_\tau T_i = -bT_i + c_1 \eta(K_{O,i} + R_{O,i}).
\]

The matrices \(M_{31}, M_{32}\) and \(M_{33}\) are all diagonal and their \((i, i)\)-th diagonal entries are given by

\[
(M_{31})_{ii} = (M_{32})_{ii} = c_1 \eta(x_i), \quad (M_{33})_{ii} = -b.
\]
Linear stability analysis

Recall that

\[ u = (K_0; R_0; T) = (K_{O,1}, \ldots K_{O,N_O}, R_{O,1}, \ldots R_{O,N_O}, T_1, \ldots T_{N_O})^*, \]

and

\[ \frac{du}{d\tau} = Mu. \]

Applying the eigen-decomposition of \( u \) gives

\[ \frac{du}{d\tau} = L\Lambda L^{-1}u, \]

where \( \Lambda \) is a diagonal matrix with diagonal components being \( \lambda_i, i = 1, \ldots, 3N_O \).

Define \( v = L^{-1}u \). Then we have

\[ \frac{dv}{d\tau} = \Lambda v, \quad \text{or componentwise} \quad \frac{dv_i}{d\tau} = \lambda_i v_i. \]

Once \( v \) is solved, \( u \) can be easily recovered by

\[ u = \sum_{i=1}^{3N_O} L_i v_i, \]

where \( L_i \) is the \( i \)-th column of \( L \).

**Property:** If there is a \( v_j \) such that \( \|L_j v_j\| \gg \|L_i v_i\| \) for all \( i \neq j \), then

\[ u \approx L_j v_j. \]
The leading two modes are the ENSO modes.

All the eigenvalues $\lambda_j$ have negative real part $r_j < 0$. The coupled system is stable.

With $N_A = 64$ and $N_O = 28$, the frequency of the two ENSO modes is 0.22 year$^{-1}$ (4.5 years) and decay rate is 0.55 year$^{-1}$ (1.8 years).

The frequency and decay rate of the two ENSO modes are robust with respect to the number of $N_A$ and $N_O$. 

Linear solution:

\[
\frac{du}{d\tau} = Mu \rightarrow \frac{dv}{d\tau} = \Lambda v
\]

\[
\rightarrow v_j = v_j(0)e^{\lambda_j \tau} \quad \lambda_j = r_j + i\omega_j
\]
Robustness results with respect to the number of $N_A$ and $N_O$. 

(a) $N_A = 64$
(b) $N_A = 128$
(c) $N_A = 256$
(d) $N_A = 512$
Atmosphere:
\[- y v - \partial_x \theta = 0 \]
\[ y u - \partial_y \theta = 0 \]
\[- (\partial_x u + \partial_y v) = E_q/(1 - \bar{Q}) \]

Ocean:
\[ \partial_{\tau} U - c_1 Y V + c_1 \partial_x H = c_1 \tau_x \]
\[ Y U + \partial_Y H = 0 \]
\[ \partial_{\tau} H + c_1 (\partial_x U + \partial_Y V) = 0 \]

SST:
\[ \partial_{\tau} T = -c_1 \zeta E_q + c_1 \eta H \]

Linear solution with \( N_A = 64 \) and \( N_O = 28 \), where the decay rate is set to be zero for illustration purpose.
Summary of the model and model solutions:

▶ deterministic, linear and stable
▶ success of triggering regular ENSO cycles
▶ realistic interactions between atmosphere, ocean and SST

What lacks in the model solution ...

▶ stochasticity and irregularity

Next lecture: **stochastic wind bursts** will be incorporated into the model that triggers realistic ENSO cycles and capturing the statistical features in the eastern Pacific.

To this end, we now discuss the basic stochastic toolkits that are needed in the model.
Stochastic toolkits

- Probability density function (PDF), Gaussian and non-Gaussian features.
- Stochastic differential equations (SDEs), Gaussian and non-Gaussian dynamics, additive and multiplicative noise.
- Markov jump process.
1. Probability density function, Gaussian and non-Gaussian statistics.

A probability density function (PDF) $p(x)$ of a continuous random variable is a function whose value at any given point in the sample space can be interpreted as providing a relative likelihood that the value of the random variable would equal that sample.

$$\mathbb{P}(a \leq X \leq b) = \int_a^b p(x)dx$$

1. Gaussian distribution $\mathcal{N}(\mu, \sigma^2)$, completely determined by its mean $\mu$ and variance $\sigma^2$,

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma^2} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

2. Non-Gaussian distribution ...
Moments.

Consider 1-D case for simplicity. The $n$-th moment and central moment of $X$ are defined respectively by

$$
\mu^n = E[X^n] = \int_{-\infty}^{\infty} x^n p(x) dx, \quad n \geq 1,
$$

$$
\tilde{\mu}^n = E[(X - \mu)^n] = \int_{-\infty}^{\infty} (x - \mu)^n p(x) dx, \quad n \geq 2.
$$

<table>
<thead>
<tr>
<th>Statistics</th>
<th>Description</th>
<th>Moment</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>the central tendency</td>
<td>1-st moment</td>
<td>$\mu$</td>
</tr>
<tr>
<td>Variance</td>
<td>spreading out</td>
<td>2-nd central moment</td>
<td>$\sigma^2 = \tilde{\mu}^2$</td>
</tr>
<tr>
<td>Skewness</td>
<td>the asymmetry</td>
<td>3-rd normalized central moment</td>
<td>$\tilde{\mu}^3 / \sigma^3$</td>
</tr>
<tr>
<td>Kurtosis</td>
<td>the “peakedness”</td>
<td>4-th normalized central moment</td>
<td>$\tilde{\mu}^4 / \sigma^4$</td>
</tr>
</tbody>
</table>
For a Gaussian distribution, skewness = 0 and kurtosis = 3.
2. Stochastic process and stochastic differential equations.

Stochastic process.

- A stochastic process is a collection of random variables, representing the evolution of some system of random values over time.

Wiener process (Brownian motion).
Real-valued stochastic process $W(t)$ such that

- $W(0) = 0$ a.s. (with probability one).
- $W(t)$ is continuous with probability one, but a.s. nowhere differentiable.
- $W(t)$ has independent increments with distribution $W(t) - W(s) \sim \mathcal{N}(0, t - s)$ for $0 \leq s < t$. Particularly, $W(t) \sim \mathcal{N}(0, t)$.

Properties: for each time $t \geq 0$,

- $E(W(t)) = 0$,
- $E(W^2(t)) = t$.

See Gardiner’s book for more details.
Stochastic differential equations (SDEs).

\[ du(t) = A(u(t), t)dt + B(u(t), t)dW(t) \]

▶ Path-wise solution:

\[ u(t) = u(0) + \int_0^t A(u(s), s)ds + \int_0^t B(u(s), s)dW(s) \]

\[ \text{Stieltjes integral} \quad \text{Ito integral} \]

Ito stochastic integral is defined as

\[ \int_0^t B(u(s), s)dW(s) := \text{m.s. lim}_{n \to \infty} \sum_{j=1}^n B(u(t_{j-1}), t_{j-1})(W(t_j) - W(t_{j-1})) \]

where \( \text{m.s. lim}_{n \to \infty} u_n = u \to \text{lim}_{n \to \infty} \langle (u_n - u)^2 \rangle = 0 \).

The simplest numerical scheme is the so-called Euler-Maruyama scheme (analogy to the forward Euler in the deterministic case).

▶ Solution of ensembles. The PDF of \( u(t) \) satisfies the so-called Fokker-Planck equation:

\[ \frac{\partial p(u, t)}{\partial t} = -\frac{\partial}{\partial u} \left( A(u(t), t)p(u, t) \right) + \frac{1}{2} \frac{\partial^2}{\partial u^2} \left( B^2(u(t), t)p(u, t) \right). \]

\[ \text{Drift} \quad \text{Diffusion} \]

Practically, the statistically steady state PDF can be computed by collecting all the sample points from a long trajectory when the system has ergodicity.
A simple example:
Linear SDE with additive noise (Ornstein-Uhlenbeck process)
(See Gardiner’s book.)

\[ du = -a(u - \mu)dt + \sigma dW_t \]

Path-wise solution: Changing variable \( v(u, t) = ue^{at} \) leads to

\[
\begin{align*}
    dv &= du \cdot e^{at} + au \cdot e^{at} dt \\
    &= ( -a(u - \mu)dt + \sigma dW_t ) \cdot e^{at} - au \cdot e^{at} dt \\
    &= (a\mu dt + \sigma dW_t) \cdot e^{at}.
\end{align*}
\]

Therefore,

\[
    v - v_0 = \int_0^t a\mu e^{as} ds + \sigma \int_0^t e^{as} dW_s
\]

\[
    = \mu(e^{at} - 1) + \sigma \int_0^t e^{as} dW_s
\]

Replacing \( v \) by \( ue^{at} \) leads to

\[
    ue^{at} - u_0 e^{a0} = \mu(e^{at} - 1) + \sigma \int_0^t e^{as} dW_s
\]

\[
    \rightarrow u - u_0 e^{-at} = \mu(e^{at} - 1)e^{-at} + \sigma e^{-at} \int_0^t e^{as} dW_s
\]

\[
    \rightarrow u = u_0 e^{-at} + \mu(1 - e^{-at}) + \sigma \int_0^t e^{-a(t-s)} dW_s
\]
\[ du = -a(u - \mu)dt + \sigma dW_s \]

Solution: 
\[ u(t) = u_0 e^{-at} + \mu (1 - e^{-at}) + \sigma \int_0^t e^{-a(t-s)} dW_s \]

When \( t \) is large, \( e^{-at} \approx 0 \) and

\[ u(t) \approx \mu + \sigma \int_0^t e^{-a(t-s)} dW_s. \]

Let's make a discrete approximation. Take \( t - s = \Delta t, 2\Delta t, \ldots, n\Delta t, \ldots \) and the corresponding discretizations of the Brownian motion are \( \Delta W_1, \Delta W_2, \ldots, \Delta W_n, \ldots \), which are independent increments by definition. Therefore,

\[ u(t) \approx \mu + \sigma \left( e^{-a\Delta t} \Delta W_1 + e^{-2a\Delta t} \Delta W_2 + e^{-3a\Delta t} \Delta W_3 + e^{-4a\Delta t} \Delta W_4 + \ldots \right) \]

If \( a \) is large, then \( u(t) \) essentially only depends on a few \( \Delta W_i \) with small \( i \) and the system has a short “memory”.
Linear SDE with additive noise

\[ du = -a(u - \mu)dt + \sigma dW_t \]

- \( a \): damping. 1/\( a \) is the so-called decorrelation time that quantifies the memory of the system.
- \( \mu \): mean state of the signal.
- \( \sigma \): noise amplitude.

The equilibrium PDF of \( u \) is a Gaussian distribution \( \mathcal{N}(\mu, \sigma^2/(2a)) \).

Note that if the system is nonlinear or the noise is non-Gaussian, then the PDF can be non-Gaussian.
Examples of dynamics with non-Gaussian statistics.

(a) Non-Gaussian noise (multiplicative noise/state-dependent noise)

\[ du = -a(u - \mu) dt + \sigma u dW_t + \sigma_a dW_t^a, \]

with \( a = 2, \mu = 2, \sigma_a = 0.01 \) and \( \sigma = 0.6 \).
(The exact solution can be found in Gardiner's book)

(b) Nonlinear dynamics

\[ du = -a(u - bu^2 + cu^3 - \mu) dt + \sigma dW_t \]

with \( a = 0.4, b = 1, c = 1, \mu = 2 \) and \( \sigma = 0.8 \).

Fat tail is often associated with extreme events!
3. Markov jump process.

Two-state Markov jump process. Two assumptions:

1. Markov property: \( P(X_t = y|X_r, 0 \leq r \leq \tau) = P(X_t = y|X_\tau) \),
2. Time homogeneity: \( P(X_t = y|X_\tau = x) = P(X_{t-\tau} = y|X_0 = x) \).

Switching rate:

\[ \nu : s_{st} \to s_{un}, \]
\[ \mu : s_{un} \to s_{st}. \]

Switching times:

\[ P(T_{st} \leq t) = 1 - e^{\nu t}, \]
\[ P(T_{un} \leq t) = 1 - e^{\mu t}. \]

Expectation:

\[ E[X] = \frac{\mu s_{st} + \nu s_{un}}{\nu + \mu}. \]

These rates define the following local transition probabilities for small \( \Delta t \)

\[ P(X_{t+\Delta t} = s_{un}|X_t = s_{st}) = \nu \Delta t + o(\Delta t), \]
\[ P(X_{t+\Delta t} = s_{st}|X_t = s_{un}) = \mu \Delta t + o(\Delta t), \]
\[ P(X_{t+\Delta t} = s_{st}|X_t = s_{st}) = 1 - \nu \Delta t + o(\Delta t), \]
\[ P(X_{t+\Delta t} = s_{un}|X_t = s_{un}) = 1 - \mu \Delta t + o(\Delta t), \]
Finite state Markov jump process

- Define finite state set $S = \{s_i\}_{i \in \mathcal{N}}, \quad \mathcal{N} = \{1, 2, \ldots, N\}$. Non-trivial: at least one of the $s_i$ is different from the rest.

- A finite state Markov jump process is a random process that undergoes transitions from one state to another on a state space.

- It must possess the Markov property ("memorylessness"): the probability distribution of the next state depends only on the current state and not on the sequence of events that preceded it.

- Doesn’t satisfy the Fokker-Planck equation but satisfies so-called Master equation.

- Has discontinuous paths.
Example 1. Model with additive noise

\[ du = -d_u u dt + v dt \]
\[ dv = -d_v v dt + \sigma_v dW_t, \quad \text{with} \quad d_u = 0.4, d_v = 2. \]

1. Additive noise. \( \sigma_v \equiv 3 \)
Example 2. Combining Markov jump process with multiplicative noise.

\[ du = -d_u u dt + v dt \]
\[ dv = -d_v v dt + \sigma_v dW_t, \quad \text{with} \quad d_u = 0.4, d_v = 2. \]

2. Multiplicative noise. \( \sigma_v = \sigma_v(u) \). Transitions between the following two states depend on \( u \).

\[ \sigma_T = \begin{cases} 3.0 & \text{State 1} \\ 0.2 & \text{State 0} \end{cases} \]

\[ du = (-d_u + i\omega_u)u dt + v dt \]
\[ dv = -d_v v dt + \sigma_v dW_t. \]

Let's rewrite

\[ u = u_{Re} + iu_{Im} := H_W + iT_E, \]

and define \( \tau_x := v \). These variables mimic the following physical scenario:

- \( H_W \): Heat content or thermocline in the western Pacific
- \( T_E \): SST in the eastern Pacific
- \( \tau_x \) stochastic wind stress in the western Pacific

Using the new notations, the above system becomes

\[ dH_W = (-d_u H_W - \omega_u T_E)dt + \tau_x dt \]
\[ dT_E = (-d_u T_E + \omega_u H_W)dt \]
\[ d\tau_x = -d_{\tau_x} \tau_x dt + \sigma_{\tau_x}(H_W)dW_t \]

where we let \( \sigma_{\tau_x} \) depend on \( H_W \).

Parameters: \( d_u = 0.2, d_{\tau_x} = 2, \omega_u = 0.4 \).

The memory time of \( \tau_x \) is much shorter than that of \( T_E \) and \( H_W \).
\[ dH_W = (-d_u H_W - \omega_u T_E) dt + \tau_x dt \]
\[ dT_E = (-d_u T_E + \omega_u H_W) dt \]
\[ d\tau_x = -d_{\tau_x} \tau_x dt + \sigma_{\tau_x}(H_W) dW_t \]

Multiplicative noise. \( \sigma_{\tau_x} = \sigma_{\tau_x}(H_W) \). Transitions between the following two states depend on \( H_W \).

\[ \sigma_{\tau_x} = \begin{cases} 
3.0 & \text{State 1} \\
0.2 & \text{State 0} 
\end{cases} \]
Next lecture: **stochastic wind bursts** will be incorporated into the model that triggers realistic ENSO cycles and capturing the statistical features in the eastern Pacific.
Thank you
Reserve Slides
# Sketch of the derivations

<table>
<thead>
<tr>
<th>Original</th>
<th>Truncated to $\psi_0(y)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ocean $\partial_\tau U - c_1 YV + c_1 \partial_x H = c_1 \tau_x$</td>
<td>$\partial_\tau K_O + c_1 \partial_x K_O = \chi_O c_1 \tau_x / 2$</td>
</tr>
<tr>
<td>$YU + \partial_Y H = 0$</td>
<td>$\partial_\tau R_O - (c_1 / 3) \partial_x R_O = -\chi_O c_1 \tau_x / 3$</td>
</tr>
<tr>
<td>$\partial_\tau H + c_1 (\partial_x U + \partial_Y V) = 0$</td>
<td></td>
</tr>
</tbody>
</table>

## Step 1. Introduce Riemann invariants: $q = (U + H) / \sqrt{2}$ and $r = (U - H) / \sqrt{2}$,

\[
\partial_\tau q + c_1 \partial_x q + \frac{c_1}{\sqrt{2}} (\partial_Y V - VY) = \frac{c_1}{\sqrt{2}} \tau_x,
\]

\[
\partial_\tau r - c_1 \partial_x r - \frac{c_1}{\sqrt{2}} (\partial_Y V + VY) = \frac{c_1}{\sqrt{2}} \tau_x.
\]

## Step 2. Make use of the so-called raising and lowering operators to simplify the system

\[
L_+ := \frac{1}{\sqrt{2}} (\partial_Y + Y), \quad L_- := \frac{1}{\sqrt{2}} (\partial_Y - Y).
\]

They have the properties $L_+ \psi_m = \sqrt{m} \psi_{m-1}$ and $L_- \psi_m = -\sqrt{m + 1} \psi_{m+1}$.

Write $\tau_x = \sum_{m=0}^{\infty} \tau_{x,m} \psi_m$. Project the $q$ equation onto $\psi_0$ leads to the dynamics of ocean Kelvin wave.

## Step 3. According to $YU + \partial_Y H = 0$, introduce $\Omega := L_+ q + L_- r$. With some manipulations, the dynamics of ocean Rossby wave associated with $\Omega$ is reached.
Meridional truncation — Derivations (See Majda 2003; Ch. 9)

<table>
<thead>
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**Riemann invariants**: mathematical transformations made on a system of conservation equations to make them more easily solvable.

Let’s start with

$$\partial_\tau U + c_1 \partial_x H = \ldots,$$

$$\partial_\tau U + c_1 \partial_x U = \ldots,$$

which leads to two eigenvalues $\lambda_q, \lambda_r$ and two Riemann invariants $q, r$,

$$\lambda_q = c_1, \quad q = (U + H) / \sqrt{2},$$

$$\lambda_r = - c_1, \quad r = (U - H) / \sqrt{2}.$$  

Adding and subtracting the equation of $U$ and $H$ leads to the equations for $q$ and $r$,

$$\partial_\tau q + c_1 \partial_x q + \frac{c_1}{\sqrt{2}} \left( \partial_y V - VY \right) = \frac{c_1}{\sqrt{2}} \tau_x,$$

$$\partial_\tau r - c_1 \partial_x r - \frac{c_1}{\sqrt{2}} \left( \partial_y V + VY \right) = \frac{c_1}{\sqrt{2}} \tau_x.$$
\[
\partial_\tau q + c_1 \partial_x q + \frac{c_1}{\sqrt{2}} (\partial_Y V - VY) = \frac{c_1}{\sqrt{2}} \tau_x, \quad q = \frac{U + H}{\sqrt{2}}
\]
\[
\partial_\tau r - c_1 \partial_x r - \frac{c_1}{\sqrt{2}} (\partial_Y V + VY) = \frac{c_1}{\sqrt{2}} \tau_x, \quad r = \frac{U - H}{\sqrt{2}}
\]

We introduce the raising and lowering operators,
\[
L_+ := \frac{1}{\sqrt{2}} (\partial_Y + Y), \quad L_- := \frac{1}{\sqrt{2}} (\partial_Y - Y).
\]

Therefore,
\[
\partial_\tau q + c_1 \partial_x q + c_1 L_- V = \frac{c_1}{\sqrt{2}} \tau_x,
\]
\[
\partial_\tau r - c_1 \partial_x r - c_1 L_+ V = \frac{c_1}{\sqrt{2}} \tau_x.
\]

Now let’s recall the parabolic cylinder functions
\[
\psi_0(y) = \pi^{-1/4} e^{-\frac{y^2}{2}}, \quad \psi_1(y) = \pi^{-1/4} 2^{1/2} Ye^{-\frac{y^2}{2}},
\]
\[
\psi_2(y) = \pi^{-1/4} 2^{-1/2} (2Y^2 - 1)e^{-\frac{y^2}{2}}, \quad \ldots
\]

where \(\psi_i\) and \(\phi_j\) are orthogonal with each other for \(i \neq j\).

We have the following properties
\[
L_+ \psi_m = \sqrt{m} \psi_{m-1} \quad \text{and} \quad L_- \psi_m = -\sqrt{m+1} \psi_{m+1}.
\]
\[
\partial_\tau q + c_1 \partial_x q + c_1 L_- V = \frac{c_1}{\sqrt{2}} \tau_x, \quad L_- \psi_m = -\sqrt{m+1} \psi_{m+1},
\]

\[
\partial_\tau r - c_1 \partial_x r - c_1 L_+ V = \frac{c_1}{\sqrt{2}} \tau_x, \quad L_+ \psi_m = \sqrt{m} \psi_{m-1}.
\]

Write all the variables into the form

\[
q = \sum_{m=0}^{\infty} q_m \psi_m, \quad r = \sum_{m=0}^{\infty} r_m \psi_m, \quad V = \sum_{m=0}^{\infty} V_m \psi_m, \quad \tau_x = \sum_{m=0}^{\infty} \tau_{x,m} \psi_m, \ldots
\]

Notably,

\[
L_- V = \sum_{m=0}^{\infty} -\sqrt{m+1} V_m \psi_{m+1}.
\]

Therefore, project the \( \tau_x \) equation onto \( \psi_0 \) and the equation of \( q \) gives

\[
\partial_\tau q_0 + c_1 \partial_x q_0 = \frac{c_1}{\sqrt{2}} \tau_{x,0}.
\]

Let's define \( K_O = \frac{1}{\sqrt{2}} q_0 \) and then we have

\[
\text{Ocean Kelvin wave:} \quad \partial_\tau K_O + c_1 \partial_x K_O = \frac{c_1}{2} \tau_{x,0}.
\]
\[
\begin{align*}
\partial_\tau U - c_1 YV + c_1 \partial_\chi H &= c_1 \tau_x, \\
YU + \partial_\gamma H &= 0, \\
\partial_\tau H + c_1 (\partial_\chi U + \partial_\gamma V) &= 0
\end{align*}
\]

\[
L_+ := \frac{1}{\sqrt{2}} (\partial_\gamma + Y) \quad q = (U + H)/\sqrt{2},
\]

\[
L_- := \frac{1}{\sqrt{2}} (\partial_\gamma - Y) \quad r = (U - H)/\sqrt{2}.
\]

\[
\begin{align*}
(\ast) \quad \partial_\tau q + c_1 \partial_\chi q + c_1 L_- V &= \frac{c_1}{\sqrt{2}} \tau_x, \\
(\ast) \quad \partial_\tau r - c_1 \partial_\chi r - c_1 L_+ V &= \frac{c_1}{\sqrt{2}} \tau_x
\end{align*}
\]

\[
\begin{align*}
q &= (U + H)/\sqrt{2}, & r &= (U - H)/\sqrt{2}, \\
\partial_\gamma q &= \frac{\sqrt{2}}{2} (U + H), & \partial_\gamma r &= \frac{\sqrt{2}}{2} (U - H).
\end{align*}
\]

\[
\begin{align*}
U &= (q + r)/\sqrt{2}, & H &= (q - r)/\sqrt{2} \\
YU + \partial_\gamma H &= 0 \quad \rightarrow \quad Y(q + r)/\sqrt{2} + \partial_\gamma (q - r)/\sqrt{2} = 0 \quad \rightarrow \quad L_+ q - L_- r = 0.
\end{align*}
\]

This suggests to look at

\[
\Omega := L_+ q + L_- r.
\]

Why look at this \( \Omega \)? Two reasons:

1. If \( \Omega \) is solved, then \( q \) and \( r \) can be obtained easily.

2. \( L_+ q + L_- r \) and \( L_+ q - L_- r \) appear simultaneously when \( L_\pm \) are operated on the two equations with \((\ast)\). Making use of \( L_+ q - L_- r = 0 \), we may have a chance to solve \( \Omega \).
\[ \partial_\tau q + c_1 \partial_x q + c_1 L_- V = \frac{c_1}{\sqrt{2}} \tau, \quad L_- \psi_m = -\sqrt{m+1}\psi_{m+1}, \]
\[ \partial_\tau r - c_1 \partial_x r - c_1 L_+ V = \frac{c_1}{\sqrt{2}} \tau, \quad L_+ \psi_m = \sqrt{m}\psi_{m-1}. \]
\[ \Omega := L_+ q + L_- r \quad \text{and} \quad 0 = L_+ q - L_- r. \]

Operating \( L_+ \) and \( L_- \) to the two equations yields
\[ L_+ \partial_\tau q + c_1 L_+ \partial_x q + c_1 L_+ L_- V = \frac{c_1}{\sqrt{2}} L_+ \tau, \]
\[ L_- \partial_\tau r - c_1 L_- \partial_x r - c_1 L_- L_+ V = \frac{c_1}{\sqrt{2}} L_- \tau. \]

Taking the summation and difference result in
\[ \partial_\tau \Omega + c_1 (L_+ L_- - L_- L_+) V = \frac{c_1}{\sqrt{2}} (L_+ + L_-) \tau, \]
\[ c_1 \partial_x \Omega + c_1 (L_+ L_- + L_- L_+) V = \frac{c_1}{\sqrt{2}} (L_+ - L_-) \tau. \]

Note that
\[ L_+ L_- \psi_m = L_+ (-\sqrt{m+1}\psi_{m+1}) = -\sqrt{m+1}(\sqrt{m+1}\psi_m) = (m+1)\psi_m, \]
\[ L_- L_+ \psi_m = L_- (\sqrt{m}\psi_{m-1}) = \sqrt{m}( -\sqrt{m}\psi_m) = m\psi_m. \]
\[ \partial_\tau \Omega + c_1 (L_+ L_- - L_- L_+) V = \frac{c_1}{\sqrt{2}} (L_+ + L_-) \tau_x, \]
\[ c_1 \partial_x \Omega + c_1 (L_+ L_- + L_- L_+) V = \frac{c_1}{\sqrt{2}} (L_+ - L_-) \tau_x, \]
\[ L_+ L_- \psi_m = (m + 1) \psi_m, \quad \text{and} \quad L_- L_+ \psi_m = m \psi_m. \]

We write the variables into the following form:

\[ V = \sum_{m=0}^{\infty} V_m \psi_m, \quad \Omega = \sum_{m=0}^{\infty} \Omega_m \psi_m, \quad \tau_x = \sum_{m=0}^{\infty} \tau_x, m \psi_m. \]

Inserting \( V \) and \( \Omega \) into the above equations and projecting onto \( \psi_m \) lead to

\[ \partial_\tau \Omega_m + c_1 V_m = \frac{c_1}{\sqrt{2}} (\sqrt{m + 1} \tau_{x,m+1} - \sqrt{m} \tau_{x,m-1}) \]
\[ c_1 \partial_x \Omega_m + c_1 (2m + 1) V_m = \frac{c_1}{\sqrt{2}} (\sqrt{m + 1} \tau_{x,m+1} + \sqrt{m} \tau_{x,m-1}) \]

Eliminating the \( V_m \) terms yields

\[ \partial_\tau \Omega_m - \frac{c_1}{2m + 1} \partial_x \Omega_m = \]
\[ \frac{c_1}{\sqrt{2}} \left( (\sqrt{m + 1} \tau_{x,m+1} - \sqrt{m} \tau_{x,m-1}) - \frac{(\sqrt{m + 1} \tau_{x,m+1} + \sqrt{m} \tau_{x,m-1})}{2m + 1} \right) \]
\[
\partial_\tau \Omega_m - \frac{c_1}{2m+1} \partial_x \Omega_m = \\
\frac{c_1}{\sqrt{2}} \left( (\sqrt{m+1}\tau_{x,m+1} - \sqrt{m}\tau_{x,m-1}) - \frac{(\sqrt{m+1}\tau_{x,m+1} + \sqrt{m}\tau_{x,m-1})}{2m+1} \right)
\]

Project the wind stress forcing \( \tau_x \) onto \( \psi_0 \). The only relevant component of \( \Omega_m \) is that with \( m = 1 \). Define \( R_O = \frac{1}{2\sqrt{2}} \Omega_1 \). Then we have

\[
\text{Ocean Rossby wave:} \quad \partial_\tau R_O - \frac{c_1}{3} \partial_x R_O = -\frac{c_1}{3} \tau_{x,0}.
\]

Remark: Rigorously speaking, the Rossby wave in the above equation is the first Rossby wave. If more meridional bases of \( \tau_x \) are included, then the second, third ... Rossby waves \( \Omega_m, m = 2, 3, \ldots \) will be triggered.
Therefore, we reach ocean Kelvin and Rossby waves.

Ocean Kelvin wave: \[ \partial_t K_O + c_1 \partial_x K_O = \frac{c_1}{2} \tau_{x,0}. \]

Ocean Rossby wave: \[ \partial_t R_O - \frac{c_1}{3} \partial_x R_O = -\frac{c_1}{3} \tau_{x,0}. \]

Next goal is to reconstruct \( U \) and \( H \) from \( K_O \) and \( R_O \). Recall that

\( q = (U + H)/\sqrt{2}, \quad r = (U - H)/\sqrt{2}, \)

\( K_O = \frac{1}{\sqrt{2}} q_0, \quad R_O = \frac{1}{2\sqrt{2}} \Omega_1, \quad \Omega = L_+ q + L_- r, \quad 0 = L_+ q - L_- r. \)

We again make use of the properties of raising and lowering operators

\[ L_- \psi_m = -\sqrt{m+1} \psi_{m+1}, \quad L_+ \psi_m = \sqrt{m} \psi_{m-1}. \]

such that

\( \Omega_1 \psi_1 = L_+ (q_2 \psi_2) + L_- (r_0 \psi_0) = \sqrt{2} q_2 \psi_1 - r_0 \psi_1. \)

\[ \rightarrow \quad \sqrt{2} q_2 - r_0 = \Omega_1. \]

In addition, we have

\[ L_+ q - L_- r = 0 \quad \rightarrow \quad \sqrt{2} q_2 + r_0 = 0. \]

Therefore,

\[ q_2 = \frac{\Omega_1}{2\sqrt{2}} = R_O \quad \text{and} \quad r_0 = -\frac{\Omega_1}{2} = -\sqrt{2} R_O. \]
\[ q = (U + H)/\sqrt{2}, \quad r = (U - H)/\sqrt{2}, \]
\[ q_0 = \sqrt{2}K_O, \quad q_2 = R_O, \quad r_0 = -\sqrt{2}R_O. \]

\[ U = \frac{1}{\sqrt{2}} (q + r) = \frac{1}{\sqrt{2}} (q_0 \psi_0 + r_0 \psi_0 + q_2 \psi_2) = \frac{1}{\sqrt{2}} \left( \sqrt{2}K_O - \sqrt{2}R_O \right) \psi_0 + \frac{1}{\sqrt{2}} R_O \psi_2 \]

Therefore,

\[ U = \frac{1}{\sqrt{2}} (q + r) = (K_O - R_O) \psi_0 + \frac{R_O}{\sqrt{2}} \psi_2, \]
\[ H = \frac{1}{\sqrt{2}} (q - r) = (K_O + R_O) \psi_0 + \frac{R_O}{\sqrt{2}} \psi_2. \]