Beyond the Gaussian universality class

MSRI/Evans Talk

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Outline

Part 1: Random growth models

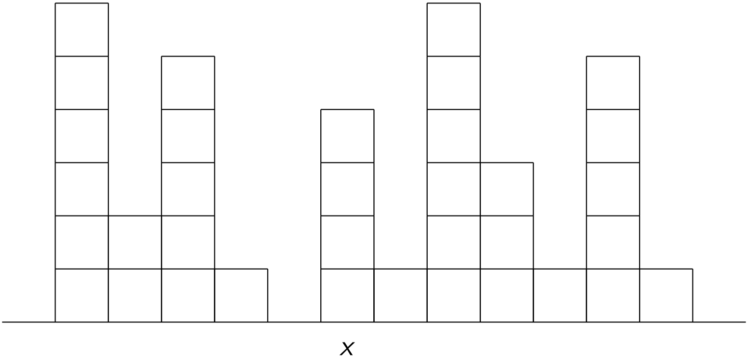
- Random deposition, ballistic deposition, corner growth
- Limit shape, fluctuations $\to$ KPZ/EW universality class
- KPZ equation: crossover between EW $\to$ KPZ

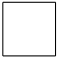
Part 2: Examples galore

- Interacting particle systems
- Polymers in disordered environments
- Permutations and Young diagrams
- Random matrix eigenvalues
- Non-intersecting paths and tilings

Tools: Probability, combinatorics, PDEs, stochastic analysis, orthogonal polynomials, asymptotic analysis, integrable systems, representation theory of (in)finite groups, algebraic geometry...

Trivial example: random deposition



 falls in each column at rate 1, independently

exp of rate λ: $P(w_{x,i} > s) = e^{-\lambda s}$ and mean $1/\lambda$

$h(t,x)$ = height above x at time t

Limit shape and fluctuations

$w_{x,i}$: waiting time for block i in column x to fall.
The $\{w_{x,i}\}$ are independent, identically distributed and

$$\{h(t,x) < n\} = \{\sum_{i=1}^{n} w_{x,i} > t\}$$

Shape: The LLN applies to each x column:

$$\lim_{t\to\infty} \frac{h(t,x)}{t} = 1 \qquad \text{almost surely}$$

Fluctuations: The CLT gives weak convergence to a family of standard Gaussians $N(x)$ **with no spatial correlation**:

$$\frac{h(t,x)-t}{\sqrt{t}} \Rightarrow N(x) \qquad \text{i.e, } P\left(\frac{h(t,x)-t}{\sqrt{t}} \le s\right) \to \int_{-\infty}^{s} \frac{e^{-r^2/2}}{\sqrt{2\pi}} dr$$

Gaussian universality class

What does it mean to be in the Gaussian universality class?

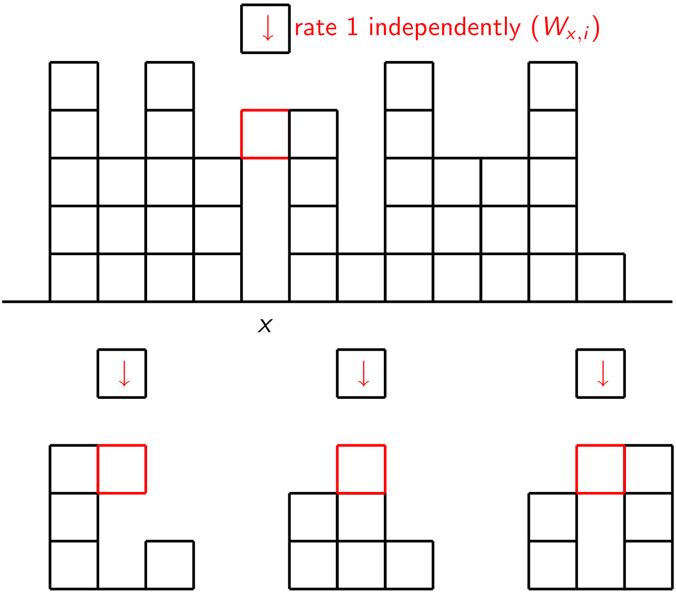
- Fluctuation scale: $t^{1/2}$ (i.e., 1/2 exponent).
- Fluctuation limit: Gaussian.
- Spatial correlation scale: t^0 (trivial 0 exponent).

Universality classes are identified by their fluctuation scale/type and spatial correlations scale.

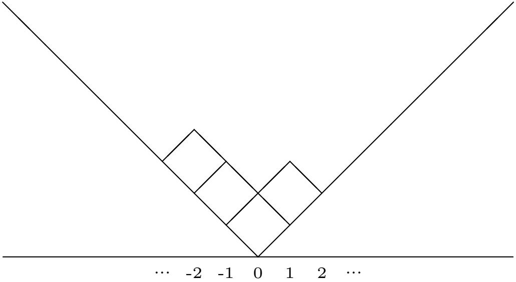
Varying the distribution of the $w_{x,i}$ does not change the universality class for random deposition.

Are there simple growth models which are NOT in this class?

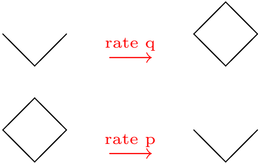
Ballistic deposition: sticky blocks



Corner growth model: a solvable model



$q - p = \gamma \in [0, 1]$
$q + p = 1$



The height function $h(t, x)$ is given by the growth interface at time t above position x. So initially:

$$h(0, x) = |x|$$

What is the limit shape? What is the scale/type of fluctuations and the spatial correlation structure? What is the universality class?

Simulation

http://www-wt.iam.uni-bonn.de/∼ferrari/animations/LargeTASEPAnimation.html

Corner growth is in the KPZ universality class

Limit shape given by model dependent solutions to PDEs:
Let $h_\gamma(t,x)$ be the height function with asymmetry $\gamma = q - p \geq 0$.

Theorem (Johansson '99/ Tracy-Widom '08): For $\gamma > 0$

$$\lim_{t\to\infty} \mathbb{P}\left(\frac{h_\gamma(\frac{t}{\gamma},0) - \frac{1}{2}t}{2^{-1/3}t^{1/3}} \geq -s \right) = F_{\mathrm{GUE}}(s).$$

KPZ class: $t^{1/3}$ fluctuation scaling and F_{GUE} limit; $t^{2/3}$ correlation length scaling (3 : 2 : 1 scaling exponent ratio).

EW class: $t^{1/4}$ fluctuation scaling and Gaussian limit (for $\gamma = 0$); $t^{1/2}$ correlation length scaling.

Question: What happens between $\gamma > 0$ and $\gamma = 0$?

KPZ equation and the KPZ universality class

Kardar-Parisi-Zhang '86 goal: define continuum random function $\mathcal{H}(T, X)$ to describe large class of growth models height functions

$$\text{(KPZ):} \quad \partial_T \mathcal{H} = \overset{\text{smoothing}}{\tfrac{1}{2}\partial_X^2 \mathcal{H}} - \overset{\text{lateral growth}}{\tfrac{1}{2}(\partial_X \mathcal{H})^2} + \overset{\text{noise}}{\dot{\mathscr{W}}}$$

Recall: A growth model is in the **KPZ universality class** if its height function $h(t, x)$ behaves like

$$h(t, x) \approx C(t, x) + t^{1/3}\xi(x)$$

where C is deterministic (LLN) and ξ has certain universal forms depending on the initial conditions. (e.g., corner growth)

Question: How is corner growth related to KPZ equation?

Question: Is the KPZ equation in the KPZ universality class?

Challenge: Make sense of the KPZ equation

Unfortunately KPZ equation is ill-posed (due to non-linearity).

Stochastic Heat Equation (SHE) with multiplicative noise:

$$\text{(SHE):} \quad \partial_T \mathcal{Z} = \frac{1}{2}\partial_X^2 \mathcal{Z} - \mathcal{Z}\dot{\mathscr{W}}, \qquad \mathcal{Z}(T=0,X) = \delta_{X=0}$$

is well-defined ($\dot{\mathscr{W}}$ is space-time Gaussian white noise).

Define: **Hopf-Cole solution to KPZ** as:

$$\text{(Hopf-Cole):} \quad \overset{\text{KPZ}}{\mathcal{H}(T,X)} = -\log \overset{\text{SHE}}{\mathcal{Z}(T,X)}$$

Answers! (Amir-C.-Quastel)

Question: How is corner growth related to KPZ equation?
Question: What happens between $\gamma > 0$ and $\gamma = 0$?

Scaling: Time/space/fluctuations as usual; asymmetry weakly as

weak asymmetry — standard KPZ class scaling with 3 : 2 : 1 exponent ratio

$$\gamma = \epsilon, \qquad t = \epsilon^{-3} T, \qquad x = \epsilon^{-2} X, \qquad \text{fluc.} = \epsilon^{-1}$$

Theorem (Amir, C., Quastel): The Hopf-Cole transformed discrete process

log correction — KPZ scaled height fluctuations

$$Z_\epsilon(T, X) = \frac{\epsilon^{-1}}{2} \exp\left\{ -\frac{h_\gamma(\frac{t}{\gamma}, x) - \frac{1}{2}t}{\epsilon^{-1}} \right\}$$

converges to the solution to the SHE $\mathcal{Z}$ with delta initial data.

The KPZ equation sits between symmetric and asymmetric cases

The crossover distributions

Question: What happens between $\gamma > 0$ and $\gamma = 0$?
Question: Is the KPZ equation in the KPZ universality class?

Theorem (Amir, C.,Quastel): KPZ equation solution and weakly asymmetric corner growth one-point marginal distributions are asymptotically given by the **crossover distributions**:

$C(T,X)$ constant

$$P(\mathcal{H}(T,X) - C(T,X) \geq -s) = F_T(s)$$

Sasamoto and Spohn find these as well!

$$F_T(s) = \int \frac{d\mu}{\mu} e^{-\mu} \det(I - K_\sigma)$$

Corollaries: Show crossover between $\gamma > 0$ and $\gamma = 0$

$$F_T(T^{1/3}s) \xrightarrow{T\to\infty} F_{\mathrm{GUE}}(2^{1/3}s), \qquad F_T(cT^{1/4}s) \xrightarrow{T\to 0} G(s)$$

Punchline: The KPZ equation represents a crossover between the symmetric EW class and the asymmetric KPZ class.

Part 1 review

- Growth models (random/ballistic deposition).
- Solvable corner growth model.
- KPZ (asymmetric) and EW (symmetric) universality classes.
- Continuum KPZ equation and Hopf-Cole solution (SHE)
- The KPZ equation is the crossover between asymmetry and symmetry and its one-point function shows this explicitly.

Filling out the KPZ universality class

The Gaussian central limit theory applies to many very different types of probabilistic models. How general is the KPZ class?

We will consider the following examples:

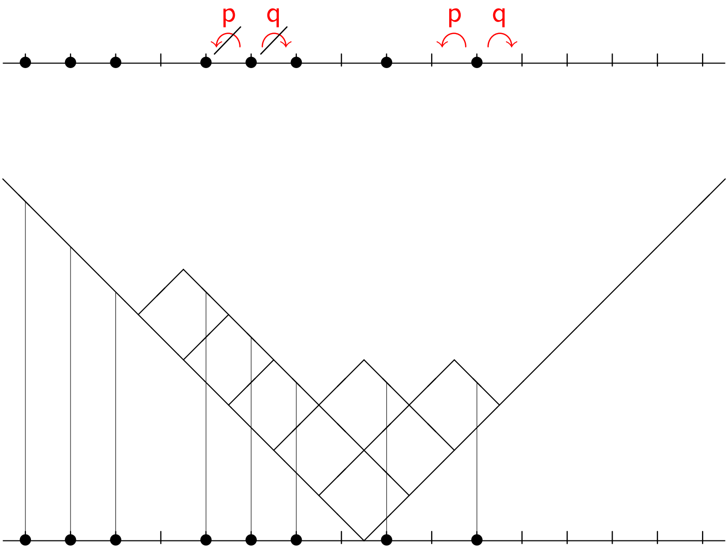
- Interacting particle systems
- Polymers in disordered environments
- Permutations and Young diagrams
- Random matrix eigenvalues
- Non-intersecting paths and tilings

Universality for these problems means that changing the parameters (within reason) does not affect the limiting fluctuations.

A common thread runs through these models (for particular choices of parameters) hence the fluctuations should all be determined by the KPZ universality class.

Simple exclusion process

Continuous time, discrete space Markov process:



More general interacting particle systems

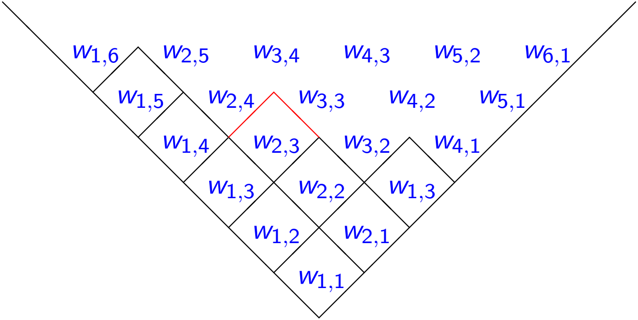
The integrated density of particle systems gives a growth model height function. How general is the KPZ/EW behavior?

Open problems in universality

- **Non-nearest neighbor jump distributions?** e.g., try to jump back four at rate $1/3$ and forward three at rate $2/3$.
- **Location dependent, or local environment dependent jump rates?** e.g., a particle's jump right is suppress by a particle there, or a particle four sites to the left.
- **Weaker forms of exclusion?** e.g., multiple particles per site, but a lower probability of jumping to sites with more particles.
- **Zero/finite range process?** e.g., the more particles behind you, the faster / slower you move.

Last passage percolation

Keep track of the times $L(x,y)$ when box (x,y) is grown.
$w_{i,j}$ = time for box (i,j) to grow (once it can)



$$L(x,y) = \max(L(x-1,y), L(x,y-1)) + w_{x,y}$$

Deterministic — KPZ fluctuations

$$L(tx,ty) \sim t(\sqrt{x}+\sqrt{y})^2 + t^{1/3}\xi(x,y)$$

What about general independent identically distributed random waiting times $w_{i,j}$? Are the fluctuations still $t^{1/3}$?

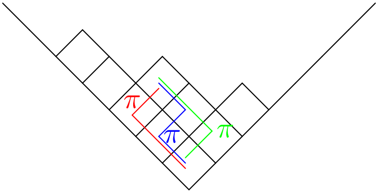
Discrete polymer ground state

Iterating the recursion for $L(x, y)$ gives:

$$L(x,y) = \max_{\pi:(1,1)\to(x,y)} T(\pi)$$

π is a directed path (only $\nearrow$ and $\nwarrow$ steps); $-T(\pi)$ is π's energy

$$T(\pi) = \sum_{(i,j)\in\pi} w_{i,j}.$$



$$T(\color{red}{\pi}) = w_{1,1} + w_{1,2} + w_{1,3} + w_{2,3}$$

$$T(\color{blue}{\pi}) = w_{1,1} + w_{1,2} + w_{2,2} + w_{2,3}$$

$$T(\color{green}{\pi}) = w_{1,1} + w_{2,1} + w_{2,2} + w_{2,3}$$

$$L(2,3) = \max(T(\color{red}{\pi}), T(\color{blue}{\pi}), T(\color{green}{\pi}))$$

Polymer universality

A **polymer measure** assigns a Boltzmann weight to a collection Π of directed ($\nearrow$ and $\nwarrow$) paths in a disordered environment $\{w_{i,j}\}$.

Probability

$$P(\pi) = \frac{1}{Z_\beta} \exp\{\beta T(\pi)\},$$

Partition function

$$Z_\beta = \sum_{\pi \in \Pi} \exp\{\beta T(\pi)\}$$

β is *inverse temperature* and if $\Pi = \{\pi : (1,1) \to (x,y)\}$ then

$$F_\beta(x,y) := \frac{\log Z_\beta}{\beta} \to L(x,y) \text{ as } \beta \to \infty$$

Therefore expect that as t goes to infinity

Deterministic — KPZ fluctuations

$$F_\beta(tx, ty) \sim tC(x,y) + t^{1/3}\xi(x,y)$$

Prove this holds for all $\beta > 0$ and all waiting distributions.

Ulam's problem and random partitions

For a permutation σ, $L(\sigma) =$ longest increasing subsequence:

e.g. for $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 5 & 2 & 8 & 1 & 3 & 4 & 7 & 9 & 6 \end{pmatrix}$, $L(\sigma) = 5$.

Theorem (Baik, Deift, Johansson): For uniformly $\sigma \in S_{N^2}$

$$\lim_{N\to\infty} P\left(\frac{L_{N^2} - 2N}{N^{1/3}} \le s\right) = F_{\mathrm{GUE}}(s)$$

Robinson Schensted (Knuth) correspondence:
Haar measure on $S_N \rightsquigarrow$ Plancherel measure on partitions λ of N.

$$P(\lambda) = \frac{\dim(\lambda)^2}{N!}$$

Other natural measures from **representation theory** arise in connection with growth models and polymer models as well (e.g., Schur measures, q-deformed Plancherel,...)

Largest eigenvalue of Gaussian random matrix ensembles

The F_{GUE} distribution was first discovered by Tracy and Widom in the context of the **G**aussian **U**nitary **E**nsemble of random matrix theory:

$$H_N = [H_{ij}]_{i,j=1}^N \text{ where } h_{ij} = \overline{h_{ji}} \sim \begin{cases} N(0,1/2) + iN(0,1/2) & i \neq j \\ N(0,1) & i = j \end{cases}$$

All the $N(0,\sigma^2)$ are independent Gaussian.
Theorem (Tracy, Widom): Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$ be the eigenvalues of H_N.

$$\lim_{N\to\infty} P\left(\frac{\lambda_1 - 2N}{N^{1/3}} \leq s\right) = F_{\mathrm{GUE}}(s).$$

There is deep and not fully understood relationship between the combinatorial problems and these random matrix problems.

Universality of the top eigenvalue

The GUE is both a **Wigner** and **invariant** ensemble.

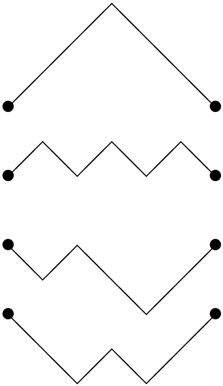
Wigner: H_{ij} $(1 \leq i < j \leq N)$ are i.i.d. complex (μ_C) and H_{ii} are i.i.d. real (μ_R). e.g., if μ are Gaussian we get GUE.
Universality: Soshnikov (given assumptions on μ_C and μ_R). Uses the moment method and combinatorics.

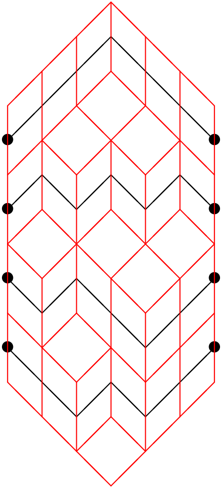
Invariant: H is chosen w.r.t. an eigenvalue potential V

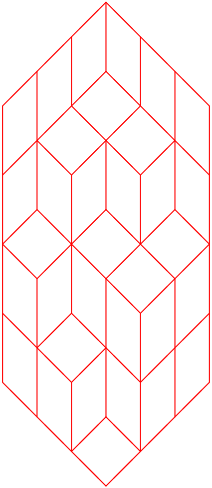
$$dP_N(H) = \frac{1}{Z_N} \exp\{-\mathrm{tr}(V(H))\} dH$$

where V is a polynomial of even degree and positive leading coefficient. e.g., if $V(H) = H^2$ we get GUE.
Universality: Deift, Kriecherbaur, McLaughlin, Venakides and Zhou. Uses orthogonal polynomials and Riemann Hilbert problem steepest descent.

Non-intersecting random walks and tilings





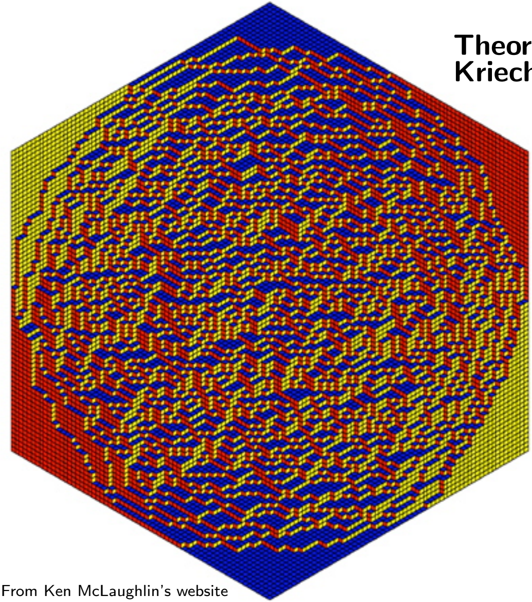


N lattice random walks conditioned not to intersect

uniform tiling of hexagon by rhombi

Tools: Karlin-McGregor determinant
Schur functions, representation theory...

Asymptotics of random rhombus tiling of the hexagon



From Ken McLaughlin's website

Theorem (Johansson/ Baik, Kriecherbauer, McLaughlin, Miller):

As N goes to infinity

top line fluctuates as $N^{1/3}$

with F_{GUE} limit dist

and has correlation scale $N^{2/3}$

Universality?

Can tile other shapes,

use non-uniform measure.

Replace hardcore repulsion?

Conclusion: An ever-broadening universality class

Gaussian universality class: weak dependence and roughly linear observable.

KPZ / EW universality class:

- Local spatial dependence drastically changes fluctuations.
- Continuum object (KPZ equation) at the crossover.
- Many other models show same fluctuations.
- Mysteries and open questions abound in this field.
- Topics here touch on almost every field of math and are studied in physics and biology too.