Continuation of Invariant Subspaces of Sparse Parameter-Dependent Matrices

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Outline

- Why parameter-dependent eigenproblems?
- The sparse CIS procedure
- Conclusions and future work
Example: Gap pull-in

\[ b = \frac{\beta}{g-u} \]
Example: Gap pull-in

Nonlinear governing equation:

\[ mu'' + \frac{\beta}{g-u} u' + ku - \frac{\alpha V^2}{2(g-u)^2} = 0 \]

Linearized at equilibrium:

\[ ku - \frac{\alpha V^2}{2(g-u)^2} = 0: \]

\[ m(\delta u)'' + \frac{\beta}{g-u} (\delta u)' + k \left( 1 - \frac{2u}{g-u} \right) (\delta x) = 0 \]
Example: Gap pull-in
Example: Gap pull-in
Example: Cantilever tuning
Example: Cantilever tuning
Given the ODE

\[ \frac{du}{dt} = f(u, \alpha) \]

write a branch of equilibria \((u(s), \alpha(s))\) so that 
\(f(u(s), \alpha(s)) = 0\). Analyze stability from eigenvalues of 

\[ A(s) = D_u f(u(s), \alpha(s)). \]

For differentiable \(A : [0, 1] \to \mathbb{R}^{n \times n} : \)

- Eigenvalues are continuous \(\lambda_i(s)\)
- \(\lambda_i(s)\) differentiable when distinct
- For a distinct set, there is a differentiable subspace
Related work

- Moving frames on solution manifolds (Rheinboldt)
- Real-analytic SVD computation (Bunse-Gerstner, Byers, Mehrmann, Nichols)
- Real-analytic null space computations and DAEs (Kunkel, Mehrmann)
- Smooth matrix decompositions (Dieci, Eirola)
- Bifurcation analysis (Thummler, Beyn, Kless; Friedman, Dieci, Demmel)
- Perturbation theory (Kato; Stewart; Demmel)
Compute a continuous block Schur form

\[ A(s) = \begin{bmatrix} Q_1(s) & Q_2(s) \end{bmatrix} \begin{bmatrix} T_{11}(s) & T_{12}(s) \\ 0 & T_{22}(s) \end{bmatrix} \begin{bmatrix} Q_1(s) & Q_2(s) \end{bmatrix}^T \]

Components:
- Choose initial invariant subspace
- Continue one step (predictor/corrector)
- Normalize the solution
- Adapt space and step size to improve convergence, resolve features of interest
Sparse case

- Compute only $Q_1(s)$
- Choose from a projection space $\mathcal{V}$
  - Changes from step to step
  - Should contain desired space + a little bit
  - Often Krylov or block Krylov
  - May use information from an outer iteration
  - May patch together bases from several steps
Compute rightmost part of the spectrum
Include all unstable eigenvalues + a few stable ones
Keep eigenvalue clusters together (prevent artificially short steps)
Prediction

Normalize to tangent plane:

\[ \tilde{Q}_1(s) = \frac{1}{Q_1(s) \left( Q_1(s_k)^T Q_1(s) \right)^{-1}} \]

Predict \( \tilde{Q}_1(s_{k+1}) \) by polynomial fitting through \( \tilde{Q}_1(s_k), \tilde{Q}_1(s_{k-1}), \ldots \)

Suggests projection space should include computed spaces from previous few steps.
Two variants of Newton:

- Iterate on full system (Beyn, Thummler, Kless); linearization is a bordered Sylvester equation
- Eliminate $T_{11}$ to get an algebraic Riccatti equation (Demmel); linearization is an ordinary Sylvester equation

Sparse case: restrict $\text{span}(\hat{Q}) \subset \mathcal{V}$ and apply Galerkin
- Can eliminate to get a Riccati equation again
- Some changes if $\text{span}(Q(s)) \not\subset \mathcal{V}$
Seek $R(\bar{Q}_{1}(s_{k+1}), \bar{T}_{11}(s_{k+1})) = 0$, where

$$R = \begin{bmatrix} A(s_{k+1})\bar{Q}(s_{k+1}) - \bar{Q}_{1}(s_{k+1} + 1)\bar{T}_{11}(s_{k+1}) \\ Q_{1}(s_{k})^{T}\bar{Q}_{1}(s_{k+1}) - I \end{bmatrix}$$

Or write $\bar{A}(s) = Q(s_{k})^{T}A(s)Q(s_{k})$ so that

$$\begin{bmatrix} \bar{A}_{11}(s) & \bar{A}_{12}(s) \\ \bar{A}_{21}(s) & \bar{A}_{22}(s) \end{bmatrix} \begin{bmatrix} I \\ Y(s) \end{bmatrix} = \begin{bmatrix} I \\ Y(s) \end{bmatrix} \bar{T}_{11}(s)$$

and eliminate $\bar{T}_{11}(s)$ and solve $F(Y) = 0$ at $s_{k+1}$, where

$$F(Y) = \bar{A}_{22}Y - Y\bar{A}_{11} + \bar{A}_{21} - Y\bar{A}_{12}Y$$
Acceptance criteria

- Diagnostics quantities: canonical angles, number of Newton iterations
- Start from standard perturbation result:

\[
\kappa := \frac{||\bar{A}_{12}||_2 ||F(Y_0)||_F}{\text{sep}(\bar{A}_{11} + \bar{A}_{12}Y_0, \bar{A}_{22} - Y_0\bar{A}_{12})^2}
\]

- Unique smallest-norm solution \( Y \) when \( \kappa < 1/4 \)
- Newton converges from \( Y_0 \) when \( \kappa < 1/12 \)
- Forcing \( \kappa < 1/12 \) at \( s_{k+1} \) is too severe
- Can we do better?
$A(s)$ and $Y(s)$ trace some paths in matrix spaces.
Can determine that if $A(s)$ stays in some region, $Y(s)$ is well defined and stays in some other region.
Perturbation approaches

Can we test with smaller regions?
Test for a connecting space

Given $Y^\text{end}$, is there a differentiable solution $Y(s)$ between such that $Y(s_k) = 0$ and $Y(s_{k+1}) = Y^\text{end}$?

Test by interpolation:

- Stewart’s perturbation result for fixed matrices extends to parameter-dependent matrices
- Linearly interpolate $\bar{A}(s)$ between $s_k$ and $s_{k+1}$
- Linearly interpolate starting guess $Y_0(s)$ between $Y(s_k) = 0$ and $Y(s_{k+1})$
- Test if $\kappa(s) < 1/4$ uniformly
May need to adjust space if

- Real parts of continued eigenvalues overlap the rest of the spectrum (generic possibilities shown)
- Eigenvalues cross imaginary axis (bifurcation)
Global control

- Attempt a step from $s_k$ to $s_{k+1}$
- If convergence failure,
  - Change the step size and retry, or
  - Choose a different space at $s_k$ and retry
- Check for interesting features (bifurcation or overlap)
  - If several occur, cut step to resolve them
  - If one occurs, may reinitialize at $s_{k+1}$
Use in MATCONT-L

- Integrated sparse CIS into the MATCONT bifurcation analysis tool (Dhooge, Govaerts, Kuznetsov, Mestrom, Riet)
- Replaced test functions on $A(s)$ with functions on $T_{11}(s)$
- Changes to MATCONT routines for step-size control, test function evaluation, and location of bifurcations
Example: Arch snap-through
Conclusions and future work

Discussed invariant subspace continuation and some applications for bifurcation analysis

Future work:
- Smarter projection space formation
- Structure preservation for second order systems
- Eigencontinuation for PDAE discretizations