Error Bounds and Error Estimates for Nonlinear Eigenvalue Problems

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Outline

1. One big idea.
2. One little idea.
3. One illustrative example.
$A : \mathbb{C} \to \mathbb{C}^{n \times n}$ analytic in $\Omega$, usually a Laplace or $z$-transform

$$\Lambda (A) := \{ z \in \mathbb{C} : A(z) \text{ singular} \}$$

$$\Lambda_\epsilon (A) := \{ z \in \mathbb{C} : \| A(z)^{-1} \| \geq \epsilon^{-1} \}$$

- $\Lambda (A)$ and $\Lambda_\epsilon (A)$ describe asymptotics, transients of some linear differential or difference equation.
- Lots of function theoretic proofs from analyzing ordinary eigenvalue problems carry over without change.
Counting eigenvalues

If $A$ nonsingular on $\Gamma$, analytic inside, count eigs inside by

$$W_\Gamma(\det(A)) = \frac{1}{2\pi i} \int_\Gamma \frac{d}{dz} \ln \det(A(z)) \, dz$$

$$= \text{tr} \left( \frac{1}{2\pi i} \int_\Gamma A(z)^{-1} A'(z) \, dz \right)$$

Suppose $E$ also analytic inside $\Gamma$. By continuity,

$$W_\Gamma(\det(A)) = W_\Gamma(\det(A + sE))$$

for $s$ in neighborhood of 0 such that $A + sE$ remains nonsingular on $\Gamma$. 
Idea 1

Winding number counts give continuity of eigenvalues \( \Rightarrow \) Should consider eigenvalues of \( A + sE \) for \( 0 \leq s \leq 1 \):

Analyticity of \( A \) and \( E \) +
Matrix nonsingularity test for \( A + sE = \)
Inclusion region for \( \Lambda(A + E) \) +
Eigenvalue counts for connected components of region
Example: Matrix Rouché

\[ \|A^{-1}(z)E(z)\| < 1 \text{ on } \Gamma \iff \text{same eigenvalue count in } \Gamma \]

Proof:
\[ \|A^{-1}(z)E(z)\| < 1 \implies A(z) + sE(z) \text{ invertible for } 0 \leq s \leq 1. \]

(Gohberg and Sigal proved a more general version in 1971.)
Example: Nonlinear Gershgorin

Define

\[ G_i = \left\{ z : |a_{ii}(z)| < \sum_{j \neq i} |a_{ij}(z)| \right\} \]

Then

1. \( \Lambda(A) \subset \bigcup_i G_i \)
2. Connected component \( \bigcup_{i=1}^m G_i \) contains \( m \) eigs
   (if bounded and disjoint from \( \partial \Omega \))

Proof: Write \( A = D + F \) where \( D = \text{diag}(A) \).
\( D + sF \) is diagonally dominant (so invertible) off \( \bigcup_i G_i \).
Example: Pseudospectral containment

Define \( D = \{ z : \| E(z) \| < \epsilon \} \). Then

1. \( \Lambda(A + E) \subset \Lambda_\epsilon(A) \cup D^c \)

2. A bounded component of \( \Lambda_\epsilon(A) \) strictly inside \( D \) contains the same number of eigs of \( A \) and \( A + E \).
Can use the usual proof to get first-order changes to isolated nonlinear eigenvalues. Let $E$ be a function perturbing $A$. If $A(\lambda)v = 0$ and $w^*A(\lambda) = 0$, then

\[
0 = \delta(w^*A(\lambda)v) \\
= w^*A(\lambda)\delta v + (\delta w)^*A(\lambda)v + w^*\delta(A(\lambda))v \\
= w^*\delta(A(\lambda))v \\
= w^*(E(\lambda) + A'(\lambda)\delta\lambda)v
\]

So nonlinear eigenvalue changes like

\[
\delta\lambda = \frac{w^*E(\lambda)v}{w^*A'(\lambda)v}
\]
Example: Lattice Schrödinger

Consider the discrete analogue to Schrödinger’s equation:

\[ H\psi = (-T + V)\psi = E\psi \]

where

\[ (H\psi)_k = -\psi_{k-1} + 2\psi_k - \psi_{k+1} + V_k\psi_k. \]

Assume \( V_k = 0 \) for \( k \leq 0 \) and \( k \geq L \). May be complex.

Want to relate the spectrum for two variants:

1. Non-negative integers: \( \psi_0 = 0 \) and \( \psi \in l^2 \)
2. Bounded: \( \psi_k = 0 \) for \( k = 0 \) and \( k \geq L + N \).
Example: Lattice Schrödinger

For $V_1 = 0.1i$ and $V_2 = 4$, $N = 20$. 
Example: Lattice Schrödinger

For $V_1 = 0.1i$ and $V_2 = 4$, $N = 200$. 
Write $H$ in either case as

$$H = \begin{bmatrix} -T_{11} + V_{11} & -e_Le_1^T \\ -e_Le_1^T & -T_{22} \end{bmatrix}$$

Then $\Lambda(H) \cap \Lambda(-T_{22})^c = \Lambda(S)$, where

$$S(z) = (-T_{11} + V_{11}) - zI - \left( e_1^T (-T_{22} - zI)^{-1} e_1 \right) e_Le_L^T$$

Write $S^{(N)}(z)$ and $S^{(\infty)}(z)$ for bounded and unbounded cases.
For $z \notin [0, 4]$, choose $\xi^2 - (2 - z)\xi + 1 = 0$, $|\xi| < 1$. Then

$$S^{(\infty)}(z) = (-T_{11} + V_{11}) - zI - \xi e_L e_L^T$$
$$S^{(N)}(z) = (-T_{11} + V_{11}) - zI - \xi \left( \frac{1 - \xi^{2N}}{1 - \xi^{2(N+1)}} \right) e_L e_L^T$$

Convenient to write $z = 2 - \xi - \xi^{-1}$, use $\xi$ as primary variable.
Error bounds

Find \( \| S^{(\infty)} - S^{(N)} \| \leq \epsilon \) if

\[
|\xi| < \left(1 + \frac{\log(3\epsilon^{-1})}{2N + 1}\right)^{-1} = 1 - O\left(\frac{\log(\epsilon^{-1})}{N}\right).
\]

Therefore, eigenvalues in bounded case (in \( \xi \) plane) either

1. Are within \( O(\log(\epsilon^{-1})/N) \) of circle (continuous spectrum)
2. Are in \( \Lambda_{\epsilon}(S^{(\infty)}) \).

Get exponential convergence to discrete spectrum, linear convergence to continuous spectrum.
If $S^{(\infty)}$ has an isolated eigenvalue at $\gamma$, then $S^{(N)}$ asymptotically has eigenvalues $\gamma^{(N)} \to \gamma$ with

$$\gamma^{(N)} - \gamma = \gamma^{2N} \frac{w^* e_L e_L^T v_L}{(1 - \gamma^2)w^* v - w^* e_L e_L^T v} + O(\gamma^{2N+1})$$

where $S^{(\infty)}(\gamma)v = 0$ and $w^* S^{(\infty)}(\gamma) = 0$. 
Similar applications

- Resonance calculations, error analysis, and some asymptotics for (continuum) 1D Schrödinger problems (joint with M. Zworski)
- Error analysis of resonance calculations via radiation boundary conditions.
- Linear stability analysis for traveling waves.
- Bounds on distance to instability via subspace projections.
- Estimates of damping in MEMS resonators.
Conclusions

- For analytic NEPs, get analogues to standard perturbation bounds (Rouché, Gershgorin, pseudospectral)
- Also get first-order perturbation theory
- Get interesting problems via approximation of spectral Schur complements
- Get interesting questions from audience?