Continuation of Sparse Eigendecompositions

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CSE 07, 23 Feb 07
Basic setting

Have a $C^k$ function

$$A : [0, 1] \rightarrow \mathbb{R}^{n \times n}$$

Want to compute eigenvectors $v(s)$ and values $\lambda(s)$ for $A(s)$. More generally, want an invariant subspace basis $V(s)$.

Applications:

1. Resonant system design
2. Bifurcation analysis
Example: Cantilever tuning
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Example: Belousov-Zhabotinski reaction

www.pojman.com/NLCD-movies/NLCD-movies.html
Reaction-diffusion models

\[
\frac{\partial u}{\partial t} = D \nabla^2 u + F(u; s)
\]

Describes many systems:

- Chemical reactions (like the B-Z reaction)
- Signals in nerves
- Ecological systems
- Phase transitions

See *Chemical Oscillations, Waves, and Turbulence* (Kuramoto).
Stability analysis

Linearize about an equilibrium branch $u_0(s)$:

$$\frac{\partial}{\partial t} \delta u = \left( D\nabla^2 + F_u(u_0(s); s) \right) \delta u = A(s) \delta u$$

- Stable if eigenvalues of $A(s)$ have negative real part
- When stability changes, have a *bifurcation*
- Complex eigs cross imaginary axis $\rightarrow$ oscillations, a *Hopf bifurcation*
Hopf bifurcation in the Brusselator

$L = 0.3$
Subspaces and stability analysis

- Diagnose stability from a small subspace (slow dynamics)
- Idea: Continue invariant subspace along with the solution
- Problem: Switching subspaces
- Problem: Missing information
CIS algorithm

Compute a continuous block Schur form

\[ Q(s)^T A(s) Q(s) = \begin{bmatrix} T_{11}(s) & T_{12}(s) \\ 0 & T_{22}(s) \end{bmatrix} \]

Algorithm phases:
- Initialize
- Predict
- Correct
- Normalize
- Adapt
Initialization

- Compute rightmost part of the spectrum
- Include all unstable eigenvalues + a few stable ones
- Keep eigenvalue clusters together (prevent artificially short steps)
Prediction

Normalize to tangent plane:

\[ \bar{Q}_1(s) = Q_1(s) \left( Q_1(s_k)^T Q_1(s) \right)^{-1} \]

Predict \( \bar{Q}_1(s_{k+1}) \) by polynomial fitting through \( \bar{Q}_1(s_k), \bar{Q}_1(s_{k-1}), \ldots \).

Suggests projection space should include computed spaces from previous few steps.
Solve the nonlinear equations

\[ AQ_1 - Q_1T_{11} = 0 \]
\[ (Q_1^{\text{prev}})^T Q_1 - I = 0 \]

- Linearization is a bordered Sylvester equation
- Newton \( \approx \) block RQ iteration
- Modified Newton \( \approx \) subspace iteration
- Or extract from a Krylov subspace

Then normalize to minimize Frobenius change in \( Q_1(s) \).
May need to adjust space if

- Real parts of continued eigenvalues overlap the rest of the spectrum (generic possibilities shown)
- Eigenvalues cross imaginary axis (bifurcation)
Testing continuity

How to ensure proposed basis spans the right subspace?

- Check rate of Newton convergence
- Check angles between subspaces
- Check distance between eigenvalues

Anything less heuristic?
Perturbation approaches

$A(s)$ and $Q_1(s)$ trace some paths in matrix spaces.
Perturbation approaches

Can determine that if $A(s)$ stays in some region, $Q_1(s)$ is well defined and stays in some other region.
Perturbation approaches

Can we test with smaller regions?
Checking the subspace

Recall:

$$\text{sep}(B, C) = \|S^{-1}\|^{-1}, \text{ where } S(X) = BX - XC$$

If for \( s \in [0, h] \),

$$\text{sep}(A_{11}, A_{22})^2 > 4\|A_{12}\|\|A_{21}\|$$

Then have a unique \( C^k \) invariant subspace associated with \( A_{11} \).

So if \( A_{21}(0) = 0 \) then \( Q_1(0) = [I; 0] \) extends to \( Q_1(s) \)
Checking the subspace

Have block Schur $Q^T A(s) Q = T$ at $s_k$ and $s_{k+1} = s_k + h$.

- Geodesic interpolation $U(s)$ between $Q(s_k)$ and $Q(s_{k+1})$.
- Similarity: $\hat{T}(s) = U(s)^T A(s) U(s)$.

If $\theta_{\text{max}}$ = largest angle between $Q(s_k)$ and $Q(s_{k+1})$,

$$\|\dot{T}\| \leq 2\theta_{\text{max}}\|A\| + \|\dot{A}\|.$$

Then get interpolation bounds:

- $\text{sep}(\hat{T}_{11}, \hat{T}_{22}) = O(1)$
- $\hat{T}_{12} = O(1)$
- $\hat{T}_{21} = O(h^2)$
Checking the subspace

Check that for all \( s \in [0, h] \),

\[
\text{sep}(\hat{T}_{11}, \hat{T}_{22})^2 > 4 \| \hat{T}_{12} \| \| \hat{T}_{21} \|
\]

So test based on:
- Conditioning of subspace (spectral separation)
- Measure of non-normality (\( \| \hat{T}_{12} \| \))
- Residual from interpolating (\( \| \hat{T}_{21} \| \))
Believe we can compute

\[ Q(s)^T A(s) Q(s) = \begin{bmatrix} T_{11}(s) & T_{12}(s) \\ 0 & T_{22}(s) \end{bmatrix} \]

What if the space isn’t rich enough \((T_{22} \text{ unstable})\)?

Want conditions for \(T_{22}\) stable.

- Do not want another nonsymmetric eigenproblem
- Only need sufficient conditions

Use the fact that we expect rapid decay in most modes.
Stability of $T_{22}$

Recall: if $\dot{u} = T_{22}u$ then

$$\frac{d}{dt} \|u\|^2 = 2u^T T_{22}u.$$ 

Define spectral abscissa

$$\omega(T_{22}) = \max_v \{ v^T T_{22} v \} = \lambda_{\text{max}}(H(T_{22}))$$

Finding $\omega(T_{22})$ is a symmetric exterior eigenvalue problem!

$\implies$ estimate with Lanczos.
Bound applied to a 2D Brusselator
Conclusions

- Continuing eigendecompositions is useful for resonator design and for bifurcation analysis
- Basic algorithm: predictor-corrector + Krylov subspaces
- Tests to make sure computed subspace is good enough
- Ongoing software work (CL-MATCONT extension)