V63-0120-004: Discrete Mathematics
Midterm 2

Solutions

[3 points] Compute $gcd(238, 182)$, using the Euclidean algorithm.

$238 = 1.182 + 56$, $182 = 3.56 + 14$, $56 = 4.14 + 0$, so $gcd(238, 182) = 14$

[2+1 pts] Express the decimal number 83 in binary. Express the binary number 1011011 in decimal.

$83 = 64 + 16 + 2 + 1$, so the binary expression is 1010011.
The binary number 1011011 represents $1 + 2 + 8 + 16 + 64 = 91$.

[1 pt] How many hexadecimal digits are needed to represent $1024 = 2^{10}$?

It takes 11 binary digits to represent this number. In hexadecimal, blocks of 4 bits are replaced by one hexadecimal digit. So it takes 3 hexadecimal digits.

[2+2 pts] Using iterated squaring, compute $7^8 \mod 11$, then $7^{11} \mod 11$.

$7^2 \mod 11 = 49 \mod 11 = 5$,
$7^4 \mod 11 = 5^2 \mod 11 = 3$,
$7^8 \mod 11 = 3^2 \mod 11 = 9$.
$7^{11} \mod 11 = 7^8 7^2 7 \mod 11 = 9.5.7 \mod 11 = 9(5.7 \mod 11) \mod 11 = 9.2 \mod 11 = 7$.

[2 pts] Using the Euclidean division, or by trial and error, find the inverse s of 8 modulo 11 satisfying $0 < s < 11$.

$s = 7$, since $7.8 \mod 11 = 56 \mod 11 = 1$.

[2 pts] Let m be a positive integer, $X = \{0, \ldots, m-1\}$, and the function $f : X \to X$ is defined by

$$f(x) = (x + 4) \mod m$$

for all $x \in X$. What is f^{-1} ?

Let $g : X \to X$ be defined by $g(x) = x - 4 \mod m$. Then $g(f(x)) = f(g(x)) = x$ for all $x \in X$. So f is a bijection and $g = f^{-1}$.

[1 pt] In a group of boxes, boxes are red, green, blue or yellow; square, round or rectangular; light or heavy. There is exactly one box of each possible type. Are there more square boxes or red boxes ?

There are $4.1.2 = 8$ square boxes and $1.3.2 = 6$ red boxes.

[1+2+1+2 pts] The 26 letters $ABC \dots Z$ are used to form strings of length n (n is some given positive integer).
(a) how many strings can be formed ?
(b) how many strings containing the letter A can be formed ?
(c) how many strings can be formed if we do not allow repetitions ?
(d) how many strings sorted in alphabetical order can be formed (repetitions allowed)?

(a) 26^n
(b) one can form 25^n strings without using A, so the number of strings containing A is $26^n - 25^n$.
(c) $P(26, n)$ if $n \leq 26$, 0 is $n > 26$.
(d) This is the same problem as picking n letters among 26 with replacement and no order. So the number is $C(n + 26 - 1, 26 - 1) = C(n + 25, 25)$.

[1+1 pts] How may 2-combinations of the set $\{a, b, c, d\}$ are there ? List them.

There are $C(4, 2) = 6$ 2-combinations of $\{a, b, c, d\}$:
$\{a, b\}$, $\{a, c\}$, $\{a, d\}$, $\{b, c\}$, $\{b, d\}$, $\{c, d\}$.

[1+1 pts] In how many ways can one select a committee of six from a group of 14 persons ? In how many ways can one select a committee of six (3 men, 3 women) from a group of 14 persons (6 men, 8 women)?

$C(14, 6)$ and $C(6, 3)C(8, 3)$ respectively.

[2 pts] How many bridge hands (13 cards selected from a 52-card deck) contain five of one suit, three of another suit, three of another suit, and two of another suit ?

One can choose: (i) the suit from which to draw 5 cards; (ii) the suit from which to draw 2 cards; (iii) the cards. So one gets:

$$4.3.C(13, 5)C(13, 2)C(13, 3)C(13, 3)$$

[1 pt] Three dice (red, green, blue) are rolled. What is the probability of getting three distinct numbers ?

There are 6^3 possible outcomes; 6.5.4 of these outcomes give distinct numbers. So the probability is $\frac{6.5.4}{6^3} = \frac{5}{9}$.

[2 pts] One is dealt a five-card poker hand. What is the probability of getting five consecutive cards ? (the order is $234 \dots QKA$)

The total number of poker hands is $C(52, 5)$. If one gets five consecutive cards, the lowest card of the hand can be $2, 3 \ldots 10$ (9 possibilities). Knowing this, one has 4^5 possibilities (choosing one suit for each card). So the probability is:

$$\frac{9.4^5}{C(52,5)} = \frac{192}{54145}$$

[2+2 pts] The Red Sox win with probability .8 when it is raining and with probability .6 if it is dry. It rains 30% of the time. What is the probability that it is raining, given a win of the Red Sox ? Are the events “win” and “rain” mutually exclusive ? independent ?

Let W be the event “win” and R the event “rain”. From Bayes’ formula:

$$P(R|W) = \frac{P(W|R)P(R)}{P(W|R)P(R) + P(W|\overline{R})P(\overline{R})} = \frac{.8 * .3}{.8 * .3 + .6 * (1 - .3)} = \frac{4}{11}$$

Also, $P(R \cap W) = P(W|R)P(R) = .24 > 0$, so the events W and R are not mutually exclusive. And they are not independent since $P(R|W) \neq P(R)$ (if they were independent, $P(R|W) = P(R \cap W)/P(W) = P(R)P(W)/P(W) = P(R)$).

[2 pts] Using Pascal’s triangle or otherwise, expand $(a+b)^6$.

One can draw the first 6 rows of Pascal’s triangle, and read binomial coefficients from here. The sixth row is $1, 6, 15, 20, 15, 6, 1$, so the expansion is:

$$(a+b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$$

[3 pts] When one expands $(x+y+z)^7$, one gets a sum of monomials:

$$(x+y+z)^7 = x^7 + \cdots + 210x^2y^3z^2 + \cdots + 35y^4z^3 + \cdots + z^7$$

How many of these monomials are there ? [e.g. in the expansion $(x+y+z)^2 = x^2 + 2xy + 2xz + y^2 + 2yz + z^2$, one gets 6 monomials]

This is the same problem as choosing 7 items among 3 ($\{x, y, z\}$) with replacement and no order. So the number of monomials is $C(7+3-1, 3-1) = C(9,2) = 36$.