On the self-similarity assumption in dynamic models for large eddy simulations

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The consistency between dynamic models for large eddy simulations and their underlying self-similarity assumption is discussed. The interpretation of the resolved field is shown to be fully determined by the choice of the test-filter. Consequences for comparison to direct numerical simulations and experimental results are presented. © 1997 American Institute of Physics.

and does not depend explicitly on $G_1$ and $G_2$ (details may be found in Refs. 5–8). For simplifying the numerical scheme, both $\tau_{ij}$ and $T_{ij}$ are modeled in the same way with the same $C$ (although the DP might be implemented with different models at grid and test levels). This choice is justified by the SSA, itself motivated by the fact that the flow is supposed to exhibit a well developed inertial range in which self-similarity arguments are valid. However, this justification holds only if the filters $G_1$ and $G_2$ are themselves self-similar, i.e., if

$$G_1(x) = \delta^{-d} G_2\left(\frac{x}{\delta}\right).$$

(6)

Hence, $G_1$ and $G_2$ must have identical shapes and may only differ by their characteristic width. Indeed, the model coefficient depends on the filter shape.\(^{10}\) Unfortunately, in the DP nothing ensures that the filters $G_1$ and $G_2$ have the same shape. Another difficulty comes from the definition of $\delta$. Indeed, $\Delta_1$ and $\Delta_2$ are usually not known and must be guessed. For example, $\Delta_1$ is often identified with the mesh size which implies that the LES are under-resolved\(^{11}\) while $\Delta_2$ is approximated by $\Delta_n$, which is only true for the sharp Fourier cutoff.

We now present a simple re-interpretation of the filters that solves the aforementioned difficulties of the present formulation of the DP independently of the type of the test filter. The discussion is presented for a one-dimensional filter but it is valid in $d$-dimensions. Let us introduce an infinite set of self-similar filters in the sense of the definition (6) \(\{G_n = G(\ell_n)\}\) defined by

$$G_n(x) = \alpha^{-n} K\left(\frac{x}{\alpha^{\ell_n}}\right),$$

(7)

where $n$ is an integer and $K(x) = K(-x)$ is the filter kernel. The characteristic width of $G_n$ is $\ell_n = \alpha^{\ell_n} \ell_0$. Here, $\ell_0$ is an arbitrary length and $\alpha > 1$ is a parameter. Clearly, the filters $G_n$ defined by (7) are all self-similar. Let us now consider a second set \(\{G_\ast_n = G(\ell_n^\ast)\}\) defined by

$$G_\ast_n = G_n \times G_n \times \cdots \times G_n.$$  

(8)

For positive kernel $K$, the relation between $G_\ast_n$ and $G_n$ can be derived by analogy with probability distribution functions (PDF). Indeed, the filters are normalized in order to ensure $G_n \ast w = w$ for $w$ constant. Let us denote $x_n$ a stochastic process for which the PDF is $G_n$. The first moment $\langle x_n \rangle$ vanishes because $G_n(x) = G_n(-x)$. The second moment may then be related to the filter width ($\ell_n^\ast \approx \ell_n^2$) where

$$\ell_n^2 = \int x^2 G_n(x).$$

(9)

Remarkably, $G_\ast_n$ as defined by (8) corresponds to the PDF of $x_n^\ast = x_n + x_{n-1} + \cdots + x_{-\infty}$ if all the $x_i$ are independent stochastic processes.\(^{12}\) In that case, the second moment of $x_n^\ast$ is given by $(\ell_n^\ast)^2 = \sigma_n^2 + \sigma_{n-1}^2 + \cdots + \sigma_{-\infty}^2$. Using the property that the $\sigma_n^2$, like the $\ell_n$, follow a geometrical law, one obtains: $(\ell_n^\ast)^2 = \sigma_n^2 \alpha^2 (\alpha^2 - 1)$ and, consequently,

$$\ell_n^\ast = \frac{\alpha}{\sqrt{\alpha^2 - 1}} \ell_n.$$  

(10)

Of course, this result is only valid if the kernel $K(x)$ is positive and has a finite second moment.\(^{13}\) From the relation (10), we conclude that filter widths $\ell_n^\ast$ follow the same geometrical law as the $\ell_n$, $\ell_n^\ast = \alpha^{\ell_n - 1} \ell_n$. Moreover, the shape of $G_\ast_n$ does not depend on $n$ since $G_\ast_n$ is obtained by an infinite number of convolutions. This is easily seen in Fourier space, where the convolutions reduce to simple products. Hence, the $G_\ast_n$ also constitute a set of self-similar filters.

Starting from these two sets of self-similar filters, it is now easy to give a self-similar formulation of the DP: Let us suppose that the test-filter and the grid filter are defined by:

$$G_1(\Delta_n) = G_n(\ell_n),$$

(11)

$$G_1(\Delta_1) = G_{n-1}(\ell_{n-1}).$$

(12)

As a direct consequence of (7) and (8), the ‘‘grid+test’’ filter is given by $G_2(\Delta_2) = G_\ast_n(\ell_n^\ast)$. Hence, with the definitions (11) and (12) for the test and grid filters, the filters $G_1$ and $G_2$ are automatically self-similar. Also, for any test-filter $G_1$ and any value of $\delta$ in the DP, the grid filter can be constructed explicitly:

$$G_1 = G_1(\Delta_1/\delta) \ast G_1(\Delta_1/\delta^2) \ast \cdots \ast G_1(\Delta_1/\delta^n).$$

(13)

In some cases, $G_1$ may be computed analytically:

$$G_1 = \begin{cases} G(\ell_{n-1}) & \text{sharp Fourier filter}, \\ G\left(\sqrt{\frac{\alpha^2}{\alpha^2 - 1}} \ell_{n-1}\right) & \text{Gaussian filter}. \end{cases}$$

(14)

For more complicated filters, $G_1$ can be evaluated numerically by iterating $G_1$. Typical profiles of $G_1$ for the top-hat filter $G_1$ are shown in Fig. 1.

The formulation of the DP in terms of the sets (7 and 8) and of the definitions (11 and 12) has several major advantages. First, the dynamic model is compatible with the SSA. It is now fully justified to use the same model for $\tau_{ij}$ and $T_{ij}$ independently of the test filter $G_1$ used to transform $\hat{u}_i$ into $\hat{u}_i$. Second, this new interpretation of the grid and test level velocities does not modified the implementation of the
the grid filter is now fully determined by the quantities obtained with the usual interpretation of the one-dimensional grid filter built as the triple product of one-dimensional filters. The unresolved energy is given by the first term only in the relation where \( G \) is a top-hat filter, \( \delta = 2 \) and \( \Delta k_{max} / \delta = 0.5 \) (dotted line); (3) the same quantity is used in the relation (13) (dashed line).

The application of \( G_1 \) to DNS data illustrated in Fig. 2 for isotropic turbulence where the unresolved energy clearly depends on the interpretation of the grid filter. In that case, the three-dimensional filter expressed in Fourier space is usually built as the triple product of one-dimensional filters \( G^3(k) = G^1(k_x)G^1(k_y)G^1(k_z) \). For isotropic turbulence, the energy spectra in the LES and DNS are related following the relation \( E_{LES}(k) = q(k)E_{DNS}(k) \) where the factor \( q(k) \) is defined by

\[
q(k) = \frac{1}{4\pi} \int d\Omega (G^1(k_x)G^1(k_y)G^1(k_z))^2
\]

and \( d\Omega \) represents the integration over the sphere of radius \( k \). The unresolved energy is given by \( E_{DNS} - E_{LES} \).

Also, we remark that the self-similar formulation of the DP is similar to the application of the renormalization group to the Navier–Stokes equation. Each filter \( G_1 \) may be regarded as one step in the small scale elimination used in the renormalization group procedure and the test filter used in the DP corresponds to the last iteration. The existence of a fixed point in the renormalization equation is assumed by using the same \( C \) at level \( n \) \( (G_2) \) and \( n-1 \) \( (G_1) \). Also, the scaling exponent appearing in the renormalization group are anticipated by using the scaling derived by Smagorinsky using Kolmogorov-like arguments. Hence, the self-similar formulation of the DP plays in the context of LES the same role as the renormalization group in the context of statistical theories of turbulence.

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12 W. Feller, An Introduction to Probability Theory and Its Applications, Volume II (Wiley, New York, 1971). The analog between the filters and the PDF should not lead to the conclusion that the \( \mathcal{S} \) are always Gaussian filters. Although they represent the PDF of a stochastic variable given by the sum of an infinite number of independent processes, the central limit theorem does not apply because the PDF of \( x_n \) is usually dominated by a small number of variables \( x_i \) corresponding to \( n = n, n-1, \ldots \).
13 Both these conditions are satisfied for most of the filters with the notable exception of the Fourier cutoff. In that case, however, it is easy to show that \( \mathcal{S} = \mathcal{S} \).