THE EXTREMAL POINT PROCESS OF BRANCHING BROWNIAN MOTION IN $\mathbb{R}^d$

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Abstract. We consider a branching Brownian motion in $\mathbb{R}^d$ with $d \geq 1$ in which the position $X_t^{(u)} \in \mathbb{R}^d$ of a particle u at time t can be encoded by its direction $\theta_t^{(u)} \in \mathbb{S}^{d-1}$ and its distance $R_t^{(u)}$ to 0. We prove that the *extremal point process* $\sum \delta_{\theta_t^{(u)}, R_t^{(u)} - m_t^{(d)}}$ (where the sum is over all particles alive at time t and $m_t^{(d)}$ is an explicit centring term) converges in distribution to a randomly shifted decorated Poisson point process on $\mathbb{S}^{d-1} \times \mathbb{R}$. More precisely, the so-called *clan-leaders* form a Cox process with intensity proportional to $D_\infty(\theta) e^{-\sqrt{2}r} \mathrm{d}r \mathrm{d}\theta$, where $D_\infty(\theta)$ is the limit of the derivative martingale in direction θ and the decorations are i.i.d. copies of the decoration process of the standard one-dimensional branching Brownian motion. This proves a conjecture of Stasiński, Berestycki and Mallein (Ann. Inst. H. Poincaré 57:1786–1810, 2021), and builds on that paper and on Kim, Lubetzky and Zeitouni (arXiv:2104.07698).

1. Introduction

A (binary) branching Brownian motion (BBM) in dimension $d \geq 1$ is a continuous-time branching particle system in which every particle moves independently as Brownian motions in dimension d, and branches at rate 1 into two daughter particles. For all $t \geq 0$ we write $\mathcal{N}_t$ for the set of particles alive at time t, and for $u \in \mathcal{N}_t$ we set $X_t^{(u)} \in \mathbb{R}^d$ to be the position at time t of particle u. In this article we will describe the structure of the limit extremal point process, i.e. the particles that have travelled the furthest from the origin.

The study of extremal particles in dimension $d = 1$ traces its roots to the work of Fisher [9], Kolmogorov, Petrovskii and Piskunov [12] and McKean [15], and is by now well understood: indeed, using [12], McKean [15] showed that the *rightmost position* $M(t) = \max_{u \in \mathcal{N}_t} X_u(t)$, centred at the median of its law, converges in distribution. The seminal work of Bramson [7] identified the centring $m_t^{(1)} = \sqrt{2}t - \frac{3}{2\sqrt{2}} \log t + O(1)$, and introduced the method of truncated second moment through barriers. Lalley and Sellke [13] were then able to prove that

$$\lim_{t \to \infty} \mathbb{P}(M(t) - m_t^{(1)} \leq x) = \mathbb{E} \exp\{-C D_\infty e^{-\sqrt{2}x}\}$$

where C is a certain constant and D_∞ is the limit of the so-called derivative martingale associated with the branching Brownian motion. Hence, the limiting law of $M(t) - m_t^{(1)}$ is the law of a Gumbel random variable with the random shift $\log(CD_\infty)/\sqrt{2}$, and in fact, it follows from [13] that $M(t)$ has Gumbel fluctuations around $m_t^{(1)} - \log(CD_\infty)/\sqrt{2}$ (see also [1] for extensions to branching random walks).

Finally, it was shown (independently and around the same time) by [2] and [3] that the *extremal point process* converges in distribution

$$\lim_{t \to \infty} \sum_{u \in \mathcal{N}_t} \delta_{X_t^{(u)} - m_t^{(1)}} = \mathcal{L}$$

where the limit point process $\mathcal{L}$ can be described as follows: Let (χ_i) be the atoms of a Poisson process on $\mathbb{R}$ with (random) intensity $CD_\infty e^{-\sqrt{2}x}$. It is shown in [3, 8] that conditionally on $M(t) \geq \sqrt{2}t$ (which is an unusually large displacement), the extremal point process seen from $M(t)$ converges to a limit object $\mathcal{D}$. More precisely:

$$\lim_{t \to \infty} \mathbb{P}\left(\sum_{u \in \mathcal{N}_t} \delta_{X_t^{(u)} - M(t)} \in \cdot \mid M(t) \geq \sqrt{2}t \right) = \mathbb{P}(\mathcal{D} \in \cdot) \tag{1.1}$$

and $\mathcal{D}$ is called the *decoration point process*. Let $\{\mathcal{D}^{(i)}\}_{i \in \mathbb{N}}$ be i.i.d. copies of $\mathcal{D}$, then we have that in distribution

$$\mathcal{L} = \sum_i \sum_{z \in \mathcal{D}^{(i)}} \delta_{\chi_i + z},$$

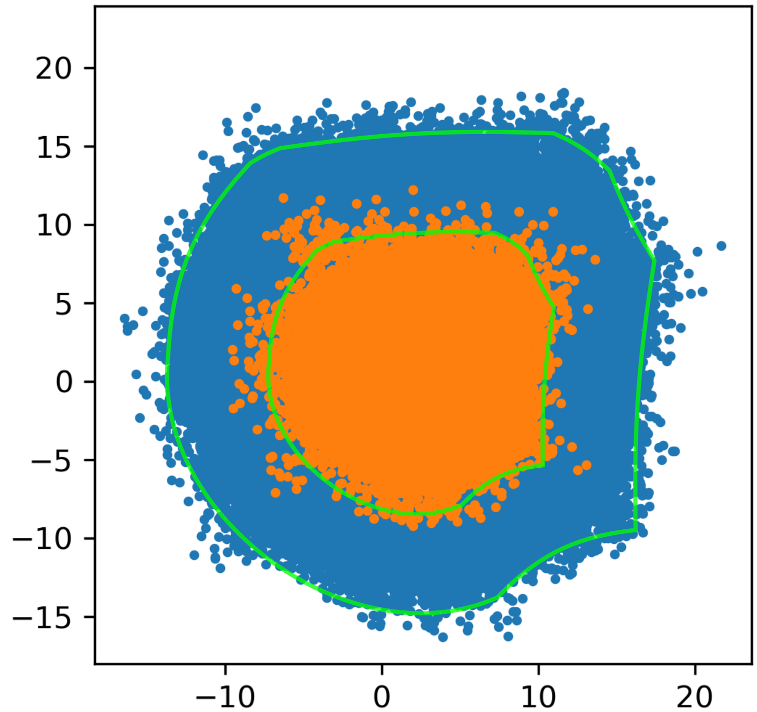

FIGURE 1. A simulation of BBM, $d = 2$, at times $t = 10$ (orange) and $t = 15$ (blue). The green curves depict (approximately) the centring term of the extremal process $m_t^{(d)} + \log D_t(\theta)$.

which we call a *randomly shifted decorated Poisson point process* or $\mathrm{SDPPP}(CD_\infty,\, e^{-\sqrt{2}x}\mathrm{d}x,\, \mathcal{D})$, using the notation from [18].

In words, $\mathcal{L}$ is obtained by decorating each atom χ_i of a randomly shifted Poisson point process by an independent copy of $\mathcal{D}$. Besides their intrinsic interest, the results for one-dimensional branching Brownian motion (and their branching random walks counterparts) have in recent years provided a road map for the analysis of other log-correlated fields, see e.g. [6] for a discussion of the two dimensional discrete GFF and [4] for the (non-Gaussian) sine-Gordon field.

By contrast, the case of the branching Brownian motion in dimension $d > 1$ had until recently received far less attention. To describe what is known, we introduce the polar decomposition for the position of particles, writing

$$R_t^{(u)} = \|X_t^{(u)}\| \in [0, \infty), \qquad \theta_t^{(u)} = \frac{X_t^{(u)}}{R_t^{(u)}} \in \mathbb{S}^{d-1},$$

and $R_t^* := \max_{u \in \mathcal{N}_t} R_t^{(u)}$ for the largest distance travelled by a particle at time t. In [5], Biggins proved that, whatever the dimension d,

$$\lim_{t \to \infty} \frac{R_t^*}{t} = \sqrt{2} \quad \text{a.s.},$$

and Mallein [14] proved that, setting $m_t^{(d)} := \sqrt{2}t + \frac{d-4}{2\sqrt{2}} \log t$, the process $(R_t^* - m_t^{(d)}, t \geq 0)$ is tight. Then Kim, Lubetzky and Zeitouni [11] proved that $R_t^* - m_t^{(d)}$ converges in law to a Gumbel random variable, shifted by an independent random variable $\log Z_\infty$, thus extending the aforementioned results of Bramson [7] and Lalley-Sellke [13] in dimension 1 (however, in contrast with the situation for $d = 1$, the random variable Z_∞ is not constructed as a measurable function of the branching Brownian motion – this matter is resolved here in Corollary 1.6).

The goal of the present paper is to obtain the full description of the limit extremal point process in dimension $d > 1$, that is, to describe the limit of the random point measure on $\mathbb{S}^{d-1} \times \mathbb{R}$ defined by

$$\mathcal{E}_t := \sum_{u \in \mathcal{N}_t} \delta_{(\theta_t^{(u)}, R_t^{(u)} - m_t^{(d)})} .$$

To do that, we first discuss what plays the role of the random shift D_∞. In [17], Stasiński, Berestycki, and Mallein introduced a multidimensional analogue of the derivative martingale: for all $t \geq 0$ and $\theta \in \mathbb{S}^{d-1}$, set

$$D_t(\theta) = \sum_{u \in \mathcal{N}_t} (\sqrt{2}t - X_t^{(u)} \cdot \theta) e^{\sqrt{2} X_t^{(u)} \cdot \theta - 2t} \text{ and } D_\infty(\theta) = \max(0, \liminf_{t \to \infty} D_t(\theta)),$$

where $x \cdot y$ is the usual inner product in $\mathbb{R}^d$. Observe that for each fixed θ, the process $\{X_t^{(u)} \cdot \theta, u \in \mathcal{N}_t\}$ (the projection of the BBM on direction θ) is just a standard one-dimensional BBM, and thus $D_t(\theta)$ is the usual associated derivative martingale. They proved that almost surely, there exists a random set $\Theta \subset \mathbb{S}^{d-1}$ of full Lebesgue measure such that $D_t(\theta)$ converges to $D_\infty(\theta)$ for all $\theta \in \Theta$. Further, letting σ denote the Lebesgue measure on $\mathbb{S}^{d-1}$, they show that that the measure with density $D_t(\theta)\sigma(\mathrm{d}\theta)$ converges weakly to the measure with density $D_\infty(\theta)\sigma(\mathrm{d}\theta)$, almost surely. Explicitly, for any bounded measurable functions $f, g : \mathbb{S}^{d-1} \to \mathbb{R}$, define

$$\langle f, g \rangle := \int_{\mathbb{S}^{d-1}} f(\theta) g(\theta) \sigma(\mathrm{d}\theta) ;$$

then it is shown in [17] that for any bounded measurable $f : \mathbb{S}^{d-1} \to \mathbb{R}$,

$$\lim_{t \to \infty} \langle D_t, f \rangle = \langle D_\infty, f \rangle \text{ a.s.} \tag{1.2}$$

As such, we will often view D_t and D_∞ as measures on $\mathbb{S}^{d-1}$ and write $D_t(A)$ and $D_\infty(A)$ (for $A \subset \mathbb{S}^{d-1}$) to denote $\langle D_t, \mathbb{1}_A \rangle$ and $\langle D_\infty, \mathbb{1}_A \rangle$, respectively.

Our main theorem builds on [11] and [17] and describes the limit extremal point process; it answers in the affirmative Conjecture 1.4 from [17].

Theorem 1.1. *The extremal process converges weakly almost surely in the topology of vague convergence,*

$$\mathcal{E}_t \to \mathcal{E}_\infty,$$

where the limit is a decorated Poisson point process on $\mathbb{S}^{d-1} \times \mathbb{R}_+$ with the following description: let $\gamma > 0$ be the positive constant defined in (2.10) below, and let $\alpha_d := (d-1)/2$. Let $\{(\xi_i, \theta_i)\}_{i \in \mathbb{N}}$ be the atoms of a Poisson point process on $\mathbb{S}^{d-1} \times \mathbb{R}$ with intensity

$$D_\infty(\theta)\sigma(\mathrm{d}\theta) \times \gamma \pi^{-\alpha_d/2} \sqrt{2} e^{-\sqrt{2}x} \mathrm{d}x .$$

Let $\{\mathcal{D}^{(i)}\}_{i \in \mathbb{N}}$ be i.i.d. copies of the decoration point process $\mathcal{D}$ for the one-dimensional BBM as above. Then

$$\mathcal{E}_\infty = \sum_{i=1}^{\infty} \sum_{r \in \mathcal{D}^{(i)}} \delta_{(\theta_i, \xi_i + r)} .$$

Remark 1.2. Theorem 1.1, as well as the other results stated in this article also hold in dimension $d = 1$, where they are usually immediate consequences of known results in [2, 3]. In this case, σ is the measure $\delta_1 + \delta_{-1}$ on the sphere $\mathbb{S}^0 = \{1, -1\}$. In other words, in all dimension σ is the Haar measure on $\mathbb{S}^{d-1}$.

Theorem 1.1 is an immediate consequence of the study of the convergence in law of the Laplace transform of $\mathcal{E}_t$ as $t \to \infty$, and the identification of the limit with the Laplace transform of $\mathcal{E}_\infty$.

Proposition 1.3. *Let $\phi : \mathbb{S}^{d-1} \times \mathbb{R} \to \mathbb{R}_{\geq 0}$ be a continuous, compactly-supported, non-negative function. Then*

$$\lim_{t \to \infty} \mathbb{E}\Big[\exp\Big(- \sum_{u \in \mathcal{N}_t} \phi(\theta_t^{(u)}, R_t^{(u)} - m_t^{(d)}) \Big) \Big] = \mathbb{E}\Big[\exp\Big(- \mathfrak{C}_d \int_{\mathbb{S}^{d-1}} C(\phi_\theta) D_\infty(\theta) \sigma(\mathrm{d}\theta) \Big) \Big] ,$$

where

$$\mathfrak{C}_d := \sqrt{\frac{2}{\pi^{1+\alpha_d}}} , \tag{1.3}$$

$\phi_\theta(\cdot) := \phi(\theta, \cdot)$, and $C(\phi_\theta)$ is defined in (2.13).

Remark 1.4. For any $r \in (0,t)$ integer and $u \in \mathcal{N}_t$ we let $[u]_r = \{v \in \mathcal{N}(t) : u \wedge v \geq t - r\}$ where $u \wedge v$ is the time of the most recent common ancestor of u and v. In other words, $[u]_r$ is the set of particles that have branched off u at most r units of time prior to t. We say that $u \in \mathcal{N}_t$ is an *r-clan-leader* if u is the particle which is the furthest away from the origin among $[u]_r$ and we write $\Gamma_r(t) \subset \mathcal{N}(t)$ for the set of all r-clan leaders at time t. Let $r(t)$ be any function such that $r(t) \to \infty$ but $r(t) = o(t)$, and let

$$\mathcal{L}_t := \sum_{u \in \Gamma_{r(t)}(t)} \delta_{(\theta_t^{(u)}, R_t^{(u)})} .$$

Then our proof of Theorem 1.1 will show that $\mathcal{L}_t$ converges in distribution to the Poisson point process on $\mathbb{S}^{d-1} \times \mathbb{R}$ whose atoms are the $\{(\xi_i, \theta_i)\}_{i \in \mathbb{N}}$ from the statement of the theorem. Moreover, it also shows that, writing u_t an element of $\Gamma_{r(t)}(t)$, we have

$$\lim_{t \to \infty} \sum_{v \in [u_t]_{r(t)}} \delta_{\theta_t^{(v)} - \theta_t^{(u_t)}, R_t^{(v)} - R_t^{(u_t)}} = \sum_{d \in \mathcal{D}} \delta_{0,d} .$$

In other words, in the extremal process defined in Theorem 1.1, the Cox process can be identified with the positions of $t/2$-clan-leaders of the branching Brownian motion at time t, and the decoration of each atom with the positions of the clan associated to each clan leader.

A key step in proving Theorem 1.1 will be to be able to use the convergence in distribution of the maximal displacement proved in [11]. However, there the analogue of the derivative martingale is not given by $D_\infty(\mathbb{S}^{d-1})$ but rather by certain random variable Z_∞. We thus need to understand the relation between Z_∞ and the measure $D_\infty(\cdot)$. Let

$$\mathcal{N}_t^{\mathtt{win}} := \{u \in \mathcal{N}_t : R_t^{(u)} \in \sqrt{2}t - [t^{1/6}, t^{2/3}]\} ,$$

and, recalling that $\alpha_d := (d-1)/2$, let

$$\mathfrak{M}_t^{(u)} := (R_t^{(u)})^{-\alpha_d} (\sqrt{2}t - R_t^{(u)}) e^{-(\sqrt{2}t - R_t^{(u)})\sqrt{2}} .$$

The variable Z_∞ is defined in [11] as the limit in distribution of

$$Z_t := \sum_{u \in \mathcal{N}_t^{\mathtt{win}}} \mathfrak{M}_t^{(u)} .$$

We will show that

Theorem 1.5. *Let $f : \mathbb{S}^{d-1} \to \mathbb{R}$ be a continuous function. Then*

$$(2\pi)^{\alpha_d/2} \sum_{u \in \mathcal{N}_L^{\mathtt{win}}} f(\theta_L^{(u)}) \mathfrak{M}_L^{(u)} \xrightarrow[L \to \infty]{p} \langle D_\infty, f \rangle . \tag{1.4}$$

In [11, Remark 1.2], a formal argument was made for the distributional equivalence of $D_\infty(\mathbb{S}^{d-1})$ and a positive constant times Z_∞; the statement above is much stronger. In particular, it implies the following.

Corollary 1.6. *Set $Z_\infty := (2\pi)^{-\alpha_d/2} D_\infty(\mathbb{S}^{d-1})$. Then Z_t converges to Z_∞ in probability.*

Observe that with this definition Z_∞ is a measurable function of the branching Brownian motion.

An immediate consequence of the above convergence in probability and the proof of [11, Theorem 1] is a Lalley-Sellke type description of the limiting law of $R_t^* - m_t$.

Corollary 1.7. *Let $\gamma^* > 0$ be the constant defined in (2.11). Then $\mathbb{P}(R_t^* \leq m_t^{(d)} + y \mid \mathcal{F}_L)$ converges in probability to $\exp(-\gamma^* Z_\infty e^{-y\sqrt{2}})$, as first $t \to \infty$, then $L \to \infty$.*

We note that, as explained in Section 2.5, the constant γ^* above is equal to a dimensional constant times the constant γ appearing in Theorem 1.1.

The structure of the paper is as follows. In Section 2, we describe several technical results. These include: the description by [11] of the trajectories of the norms of extremal particles (those reaching height within constant distance of $m_t^{(d)}$); a simple but key stability result for the process of the angles of extremal particles; and a recollection of convergence results for the F-KPP equation that will utilized throughout the rest of the paper.

In Section 3, we prove Theorem 1.5 by carefully examining the contribution of each particle $v \in \mathcal{N}_L$ to the integral $\langle D_L, f\rangle$, which converges almost surely to the right-hand side of (1.4).

In Section 4, we prove Proposition 1.3 using a key leading-order tail asymptotic on the Laplace functional (Proposition 4.1) in combination with the branching property, as well as Theorem 1.5. We then prove Theorem 1.1 using Proposition 1.3 and the identification of the Laplace transform of $\mathcal{E}_\infty$.

Proposition 4.1 is then proved in Section 5 using information on the trajectories of the extremal BBM particles and a coupling with one-dimensional BBM similar to the one used in [11].

2. Preliminaries

2.1. **Notation for asymptotics.** For functions $f(t)$ and $g(t)$, we write $f \sim g$ to denote the relation $f/g \to 1$ as $t \to \infty$. When needed, we emphasize the dependence on t by writing $f \sim_t g$. We write $f \lesssim g$ to mean there exists some constant $C > 0$ such that for all t sufficiently large, $f(t) \le Cg(t)$. We write $f \asymp g$ to mean $f \lesssim g$ and $g \lesssim f$.

In what follows, we will consider time parameters t and L, where t is sent to infinity before L. We will also consider a parameter $z \in [L^{1/6}, L^{2/3}]$. For functions $f := f(t, L, z)$ and $g := g(t, L, z)$, we write $f \sim_{(\mathrm{u})} g$ to denote the relation

$$\lim_{L\to\infty} \liminf_{t\to\infty} \inf_{z\in[L^{1/6},L^{2/3}]} \frac{f}{g} = \lim_{L\to\infty} \limsup_{t\to\infty} \sup_{z\in[L^{1/6},L^{2/3}]} \frac{f}{g} = 1\,.$$

We write $f = o_u(g)$ if

$$\limsup_{L\to\infty} \limsup_{t\to\infty} \sup_{z\in[L^{1/6},L^{2/3}]} \frac{f}{g} = 0\,.$$

When functions have no dependency in the variable z, we still write $\sim_{(u)}$ and o_u as above, ignoring the sup or inf over z.

2.2. **Trajectories of the norms of the extremal particles.** A key step towards the convergence result of [11] was the following characterization of the trajectories of the norms of particles that reach height $m_t^{(d)} + y$ at time t, where $y \in \mathbb{R}$ is a constant. Let L be a time parameter that is sent to infinity *after* t (so, from the perspective of t, L is just a large constant), and let $\ell := \ell(L)$ be any function such that $\ell \in [1, L^{1/6}]$ and $\ell \to \infty$ as $L \to \infty$. Then, with probability $1 - o_u(1)$, any particle $v \in \mathcal{N}_{t-\ell}$ that produces a descendent $u \in \mathcal{N}_t$ such that $R_t^{(u)} > m_t^{(d)} + y$ did the following:

(1) $R_L^{(v)} \in I_L^{\mathtt{win}} := \sqrt{2}L - [L^{1/6}, L^{2/3}]$;
(2) for $s \in [L, t-\ell]$, $R_s^{(v)}$ was bounded above by $m_t s/t + y$ and below by some explicit function ; and
(3) $R_{t-\ell}^{(v)} \in \frac{m_t}{t}s + y - [\ell^{1/3}, \ell^{2/3}]$.

In words, the norm of v at a constant order time from the beginning and from the end lies in a small window; in between these times, the norm of v stays in a sufficiently tight barrier. See Figure 2 for a depiction of such a trajectory.

This characterization of the extremal trajectories will be key for the proof of Proposition 1.3. More precisely, Proposition 1.3 follows quickly from the tail estimate Proposition 4.1, the proof of which completely relies on the above trajectory characterization. This proof is given in Section 5, where the trajectory characterization is given in full detail along with genealogical information: see Propositions 5.1 and 5.2. Prior to Section 5, we will use (1) multiple times, and so we state it precisely below.

Proposition 2.1 ([11, Theorem 3.1])**.** *For any $y \in \mathbb{R}$, we have*

$$\lim_{L\to\infty} \limsup_{t\to\infty} \mathbb{P}\Big(\exists v \in \mathcal{N}_t :\ R_L^{(v)} \notin I_L^{\mathtt{win}}\,,\, R_t^{(v)} > m_t + y\Big) = 0\,.$$

2.3. **Many-to-one lemma and multidimensional Brownian motions.** Many-to-few lemmas are ubiquitous tools in the study of spatial branching processes. They connect the moments of additive functionals of the branching process with estimates related to a typical trajectory. In this article, we use a simple version of the many-to-one lemma that relates the mean of an additive functional of the branching Brownian motion with a Brownian motion estimate. We refer to [10] for the description of the general settings.

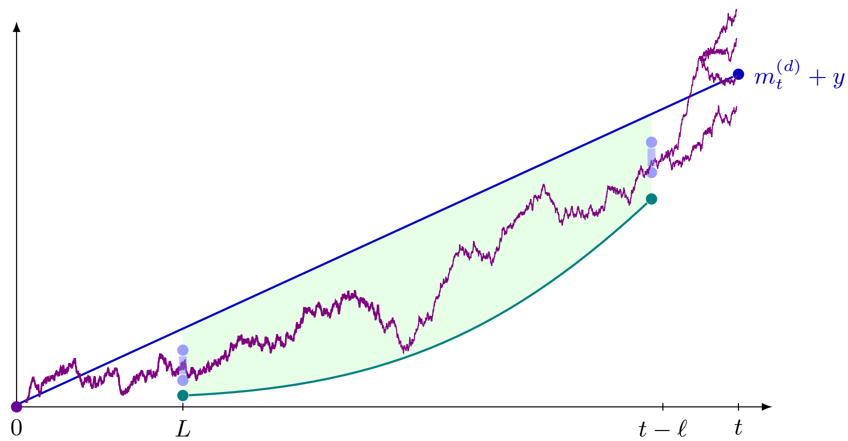

FIGURE 2. Trajectory of the norm of a particle considered in $\mathcal{E}_t$: at time L are at height in $I_L^{\mathtt{win}}$, stay in the shaded (green) region up to time $t-\ell$, at time $t-\ell$ are located in another window, and then produce a descendant that reaches $m_t^{(d)} + y$ at time t.

Lemma 2.2 (Many-to-one lemma)**.** *Fix $d \in \mathbb{N}$, and let $X_\cdot$ denote a d-dimensional Brownian motion. For any $T \geq 0$ and $y \in \mathbb{R}$, and for any measurable function $f : C^d[0,T] \to \mathbb{R}$, we have*

$$\mathbb{E}_x\Bigg[\sum_{v\in\mathcal{N}_T} f\big((X_s)_{s\leq T}\big)\Bigg] = e^T \mathbb{E}_x\Big[f\big((X_s^{(v)})_{s\leq T}\big)\Big]\,. \tag{2.1}$$

Here, $C^d[0,T]$ denotes the set of continuous functions from $[0,T]$ to $\mathbb{R}^d$.

Recall that the norm of a d-dimensional Brownian motion is a d-dimensional Bessel process. In particular, $\{R_s^{(v)}\}_{s>0, v\in cN_s}$ is a branching Bessel process. Throughout, we will write $R_\cdot$ to denote the process given by the norm of standard d-dimensional Brownian motion, and we will write $W_\cdot$ to denote a standard Wiener process. When $R_0 > 0$ and $d \geq 2$, we have the following SDE (see [16, Chapter XI] for a treatment of Bessel processes):

$$\mathrm{d}R_t = \frac{\alpha_d}{R_t}\mathrm{d}t + \mathrm{d}W_t, \tag{2.2}$$

where we recall that $\alpha_d := (d-1)/2$.

We will also use the fact that $\|X_L^{(v)}\| \overset{(d)}{=} L^{1/2}\chi_d$, where χ_d is a chi random variable with d degrees of freedom. Letting $p_L^R(x,y)$ denote the transition density of a d-dimensional Bessel process at time L and p^{χ_d} denote the density of χ_d, we have

$$p_L^R(0,x) = L^{\frac{1}{2}} p^{\chi_d}\big(L^{-\frac{1}{2}}x\big) = c_d L^{-d/2} x^{d-1} e^{-\frac{x^2}{2L}}\,. \tag{2.3}$$

In particular, by integration by parts, there exists $C_d > 0$ such that for all $a, L > 0$ we have

$$\mathbb{P}(R_L \geq a) = \frac{c_d}{L^{1/2}} \int_a^\infty \left(\frac{x}{L^{1/2}}\right)^{d-1} e^{-x^2/2L}\mathrm{d}x \leq C_d \left(\frac{a}{L^{1/2}}\right)^{d-2} e^{-a^2/2L}. \tag{2.4}$$

It is worth noting that considering the polar decomposition of a Brownian motion B in $\mathbb{R}^d$ as the diffusion $((R_t, \theta_t), t \geq 0)$ on $\mathbb{R}_+ \times \mathbb{S}^{d-1}$, then $(R_t, t \geq 0)$ is a d-dimensional Bessel process, and conditionally on the latter, $(\theta_t, t \geq 0)$ is a time-inhomogeneous Brownian motion on the sphere, with diffusion constant $1/R_t^2$ at time t. In particular, θ_t converges in law as $t \to \infty$ to the uniform distribution on the sphere. However, note that conditionally on $\{R_t \geq \epsilon t, t \geq 0\}$, θ_t converges almost surely to a random point of the sphere.

2.4. **Stability of the angular process.** As the radial part of the typical trajectory of a particle at distance $m_t^{(d)}$ at time t has grown linearly over time, we deduce from the above observation that its angular part $\theta_t^{(u)}$ should be converging, and in particular be close to $\theta_s^{(u)}$ for s large enough. The main result of this section confirms this heuristic by stating that the direction of extremal particles at time t are very close to the direction of their ancestor at time L with high probability. This proves that the direction of all individuals in the same clan is identical.

Proposition 2.3. *Fix $y \in \mathbb{R}$. For all L large enough, we have*

$$\limsup_{t\to\infty} \mathbb{P}\big(\exists v \in \mathcal{N}_t : \|\theta_L^v - \theta_t^v\| \geq 2L^{-1/12}\,,\, R_t^v > m_t^{(d)} + y\big) \leq e^{-L^{5/6}}\,, \tag{2.5}$$

where $\|\cdot\|_{\mathbb{S}^{d-1}}$ denotes the metric on $\mathbb{S}^{d-1}$.

This proposition is an immediate consequence of the two following claims and Proposition 2.1.

Claim 2.4. *For all L sufficiently large and t sufficiently large compared to L,*

$$\mathbb{P}\Big(\exists v \in \mathcal{N}_L^{\mathtt{win}}\,,\, u \in \mathcal{N}_{t-L}^v : \|X_t^{(u)} - X_L^{(u)}\| \geq m_{t-L}^{(d)} + L^{5/6}\Big) \leq e^{-L^{5/6}}\,. \tag{2.6}$$

Proof. By the Markov inequality, we have

$$\mathbb{P}\Big(\exists v \in \mathcal{N}_L^{\mathtt{win}},\, u \in \mathcal{N}_{t-L}^v : \|X_t^{(u)} - X_L^{(u)}\| \geq m_{t-L}^{(d)} + L^{5/6}\Big) \leq \mathbb{E}\Bigg[\sum_{v\in\mathcal{N}_L^{\mathtt{win}}} \mathbb{1}_{\{\exists\, u\in\mathcal{N}_{t-L}^v : \|X_t^{(u)} - X_L^{(u)}\| \geq m_{t-L}^{(d)} + L^{5/6}\}}\Bigg].$$

Observe that by the Markov property and the shift-invariance of the d-dimensional Brownian motion, the process $(X_{L+s}^{(u)} - X_L^{(v)}, u \in \mathcal{N}_{L+s}^{(v)} s \geq 0)$ is a branching Brownian motion started from 0, independent of $(X_s^{(v)}, v \in \mathcal{N}_s, s \leq L)$ and therefore, by the many-to-one lemma,

$$\begin{aligned}
&\mathbb{P}\Big(\exists v \in \mathcal{N}_L^{\mathtt{win}}\,,\, u \in \mathcal{N}_{t-L}^v : \|X_t^{(u)} - X_L^{(u)}\| \geq m_{t-L}^{(d)} + L^{5/6}\Big) \\
&\leq \mathbb{E}\left[\#\mathcal{N}_L^{\mathtt{win}}\right] \mathbb{P}(R_{t-L}^* \geq m_{t-L}^{(d)} + L^{5/6}) \leq e^L \mathbb{P}(R_L - \sqrt{2}L \in [-L^{2/3}, -L^{1/6}])\mathbb{P}(R_{t-L}^* \geq m_{t-L}^{(d)} + L^{5/6}),
\end{aligned} \tag{2.7}$$

Using (2.4), we have

$$e^L \mathbb{P}(R_L - \sqrt{2}L \in [-L^{2/3}, -L^{1/6}]) \leq C_d L^{d/2-1} e^{L - \frac{(\sqrt{2}L - L^{2/3})^2}{2L}} \leq C_d L^{d/2-1} e^{\sqrt{2}L^{2/3}}\,.$$

Additionally, applying [14, Equation 1.2], there exists $K_d > 0$ such that for all $t, L > 0$,

$$\mathbb{P}(R_{t-L}^* \geq m_{t-L}^{(d)} + L^{5/6}) \leq K_d e^{-\sqrt{2}L^{5/6}}.$$

As a consequence, (2.7) implies that for all L large enough,

$$\limsup_{t\to\infty} \mathbb{P}\Big(\exists v \in \mathcal{N}_L^{\mathtt{win}}\,,\, u \in \mathcal{N}_{t-L}^v : \|X_t^{(u)} - X_L^{(u)}\| \geq m_{t-L}^{(d)} + L^{5/6}\Big) \leq e^{-L^{5/6}}\,,$$

completing the proof. □

The previous claim states that with high probability, extremal particles at time t stay within distance $m_{t-L} + L^{5/6}$ from their ancestor at time L. We now use simple geometry to conclude that in this case, the direction of extremal particles have to stay close to the direction of their ancestor at time L, as illustrated in Figure 3.

Claim 2.5. *Let $L > 0$ and $x \in \mathbb{R}^d$ such that $L - L^{2/3} \leq \|x\| \leq L - L^{1/6}$. For all L large enough, we have*

$$\limsup_{R\to\infty} \sup_{z\in B(x,R+L^{5/6})\setminus B(0,R+L)} \left\| \frac{x}{\|x\|} - \frac{z}{\|z\|} \right\| \leq L^{-1/12}\,.$$

Proof. For $z \in B(x, R + L^{5/6}) \setminus B(0, R + L)$, straightforward computations yield that

$$\begin{aligned}
\frac{x}{\|x\|} \cdot \frac{z}{\|z\|} &= \frac{\|z\|^2 - \|x-z\|^2}{2\|x\|\|z\|} \geq \frac{(R+L)^2 - (R+L^{5/6})^2}{\|x\|\|z\|} \geq \frac{(2R+L+L^{5/6})(L-L^{5/6})}{(L-L^{2/3})(R+L)} \\
&\geq \frac{1-L^{-1/6}}{1-L^{-1/3}} \frac{1-L/2R}{1-L/R}\,.
\end{aligned}$$

We observe that for all L large enough,

$$\liminf_{R\to\infty} \frac{1-L^{-1/6}}{1-L^{-1/3}} \frac{1-L/2R}{1-L/R} \geq 1 - L^{-1/6}/2\,.$$

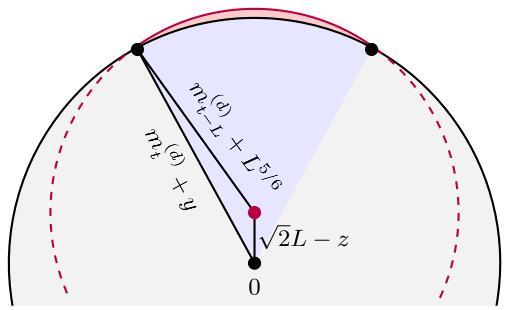

FIGURE 3. The marked angle corresponds to $\|\theta^{(v)}_L - \theta^{(v)}_t\|_{\mathbb{S}^{d-1}}$.

As a result, using that

$$\left\| \frac{x}{\|x\|} - \frac{z}{\|z\|} \right\| = \sqrt{2}\sqrt{1 - \frac{x}{\|x\|} \cdot \frac{z}{\|z\|}} \,,$$

we conclude that for all L large enough,

$$\limsup_{R\to\infty} \sup_{z\in B(x,R+L^{5/6})\setminus B(0,R+L)} \left\| \frac{x}{\|x\|} - \frac{z}{\|z\|} \right\| \le L^{-1/12} \,. \qquad \square$$

We can now complete the proof of Proposition 2.3.

Proof of Proposition 2.3. From Claim 2.5, we have that

$$\begin{aligned} &\mathbb{P}\big(\exists v \in \mathcal{N}_t : \|\theta^v_L - \theta^v_t\| \ge 2L^{-1/12}\,,\, R^v_t > m^{(d)}_t + y\big) \\ &\le \mathbb{P}\left(\exists v \in \mathcal{N}_t : R^{(v)}_L \notin I^{\mathtt{win}}_L, R^{(v)}_t > m^{(d)}_t + y\right) + \mathbb{P}\left(\exists v \in \mathcal{N}_t : R^{(v)}_L \in I^{\mathtt{win}}_L, \|X^{(v)}_t - X^{(v)}_L\| \ge m^{(d)}_{t-L} + L^{5/6}\right) . \end{aligned}$$

From Proposition 2.1 and Claim 2.4, we obtain that both terms are $o_u(1)$, which concludes the proof. $\square$

2.5. **Convergence results for the F-KPP Equation.** Branching Brownian motion is connected to the F-KPP reaction-diffusion equation. More precisely, McKean's representation connects multiplicative functionals of the one-dimensional BBM to solutions of the F-KPP equation:

Proposition 2.6 ([15])**.** *Let $f : \mathbb{R} \to [0,1]$, and let $\{W^{(v)}_s\}_{s\ge 0, v\in N_s}$ denote a one-dimensional BBM. Then, for any $w \in \mathbb{R}$, the F-KPP equation*

$$\partial_t u = \frac{1}{2}\partial^2_x u - u + u^2$$

with initial conditions $u(0,x) = 1 - f(x)$ is solved by

$$u(t,x) := \mathbb{E}\Big[1 - \prod_{v\in N_t} f(x - W^{(v)}_t)\Big] \,. \tag{2.8}$$

We will appeal to the following F-KPP convergence result several times.

Proposition 2.7 ([3, Proposition 3.2, Lemma 4.6, Lemma 4.8])**.** *Let $u(t,x)$ solve the F-KPP equation with initial condition $g(x) \in [0,1]$ satisfying*

$$\sup\{y : g(y) > 0\} < \infty \,.$$

Then there exists a positive, finite constant C_g depending only on g such that for any constant $c \in \mathbb{R}$, we have

$$C_g e^{\sqrt{2}c} = \lim_{\ell\to\infty} \int_0^\infty w e^{\sqrt{2}w} u(\ell, \sqrt{2}\ell + w + c)\mathrm{d}w = \lim_{\ell\to\infty} \int_{\ell^{1/3}}^{\ell^{2/3}} w e^{\sqrt{2}w} u(\ell, \sqrt{2}\ell + w + c)\mathrm{d}w \,.$$

We make explicit how the Proposition 2.7 follows from [3]. Equation (3.3) of Proposition 3.2 of [3] tells us that the following limit exists:

$$C_g := \lim_{\ell\to\infty} \int_0^\infty w e^{\sqrt{2}w} u(\ell, \sqrt{2}\ell + w)\mathrm{d}w \,. \tag{2.9}$$

Lemma 4.6 of [3] tells us that C_g is equal to the limit as ℓ tends to infinity of the above integral taken only over $[\ell^{1/2-\delta}, \ell^{1/2+\delta}]$, for any $\delta \in (0, 1/2)$. The shift by c resulting in the $e^{\sqrt{2}c}$ pre-factor is now a direct consequence of Lemma 4.8 of [3].

An important application of McKean's theorem is to the case $u(0, x) = \mathbb{1}_{\{x<0\}}$. Proposition 2.6 states that $\mathbb{P}(W_\ell^* > x)$ solves the F-KPP equation, whence Proposition 2.7 yields the following positive constant:

$$\gamma := C_{\mathbb{1}_{\{x<0\}}} = \lim_{\ell\to\infty} \int_0^\infty w e^{\sqrt{2}w} \mathbb{P}(W_\ell^* > \sqrt{2}\ell + w) \mathrm{d}w \,. \tag{2.10}$$

This constant appears in the limiting law of the re-centered maximum of BBM in every dimension. In [13], Lalley and Sellke showed that there exists some positive constant $C > 0$

$$\lim_{t\to\infty} \mathbb{P}(W_t^* - m_t^{(d)} \le y) = \mathbb{E}\big[\exp\big(- CZ^{(1)} e^{-y\sqrt{2}}\big)\big] \,,$$

where $Z^{(1)}$ denotes the derivative martingale from one-dimensional BBM. (The constant C was identified as $C = \gamma\sqrt{2/\pi}$, see [3].) In [11], the main result (Theorem 1) states that

$$\mathbb{P}(R_t^* - m_t^{(d)} \le y) = \mathbb{E}\Big[\exp\Big(- \sqrt{\tfrac{2^{1+\alpha_d}}{\pi}}\gamma Z_\infty e^{-y\sqrt{2}}\Big)\Big] \,.$$

In contrast to the above, in [11, Theorem 1], the constant in front of Z_∞ is not written out explicitly. Instead it is just called "γ^*". The following expression for γ^* is given by [11, Lemma 5.1 and Proposition 5.4]:

$$\gamma^* = \sqrt{\tfrac{2^{1+\alpha_d}}{\pi}} \lim_{\ell\to\infty} \int_{\ell^{1/3}}^{\ell^{2/3}} w e^{\sqrt{2}w} \mathbb{P}(W_\ell^* > \sqrt{2}\ell + w) \mathrm{d}w \; = \sqrt{\tfrac{2^{1+\alpha_d}}{\pi}}\gamma. \tag{2.11}$$

This is shown to be a positive constant in [11, Section 5.4] using purely probabilistic methods, without reference to the F-KPP equation. Proposition 2.7 tells us that the above limit actually equals γ.

We will often consider the solution u_ϕ to the F-KPP equation with initial conditions $u(0, x) = 1 - e^{-\phi(-x)}$, for a non-negative, compactly supported function $\phi : \mathbb{R} \to \mathbb{R}_{\ge 0}$. Proposition 2.6 states that, for any $w \in \mathbb{R}$,

$$u_\phi(\ell, \sqrt{2}\ell + w) = \mathbb{E}\Big[1 - \exp\Big(- \sum_{u\in\mathcal{N}_\ell} \phi(W_\ell^{(u)} - \sqrt{2}\ell - w)\Big)\Big] \,. \tag{2.12}$$

Furthermore, Proposition 2.7 states that the following limit exists, and is positive and finite:

$$C(\phi) := \lim_{\ell\to\infty} \int_0^\infty w e^{\sqrt{w}} \mathbb{E}\Big[1 - \exp\Big(- \sum_{u\in\mathcal{N}_\ell} \phi(W_\ell^{(u)} - \sqrt{2}\ell - w)\Big)\Big] \mathrm{d}w \,. \tag{2.13}$$

The constant $C(\phi)$ appears numerous times in the sequel.

3. Proof of Theorem 1.5 (equivalence of Z_∞ and D_∞)

We prove Theorem 1.5 by identifying the main contribution to D_∞, and showing that it coincides with the main contribution of Z_∞. Towards this end, in the following lemma, for any $L, x > 0$ and $\psi \in \mathbb{S}^{d-1}$, we estimate the contribution of a particle located at $x\psi \in \mathbb{R}^d$ to $\langle D_L, f(\cdot)\rangle$.

Lemma 3.1. *Let $f : \mathbb{S}^{d-1} \to (0, \infty)$ be a continuous function and fix constants $K > 0$ and $\epsilon > 0$. Then, uniformly in $\psi \in \mathbb{S}^{d-1}$ and $x \in [\epsilon L, \sqrt{2}L - K \log L]$, we have that*

$$\int_{\mathbb{S}^{d-1}} f(\theta)(\sqrt{2}L - x\psi.\theta) e^{\sqrt{2}(x\psi.\theta - \sqrt{2}L)} \sigma(\mathrm{d}\theta) \sim_{L\to\infty} (2\pi)^{(d-1)/4} f(\psi) x^{-\alpha_d} (\sqrt{2}L - x) e^{\sqrt{2}(x-\sqrt{2}L)} \,. \tag{3.1}$$

Furthermore, there exists $C_d > 0$ such that for all L large enough and $x \in [\sqrt{2}L - K \log L, \sqrt{2}L + K \log L]$, we have that

$$\Big| \int_{\mathbb{S}^{d-1}} (\sqrt{2}L - x\psi.\theta) e^{\sqrt{2}(x\psi.\theta - \sqrt{2}L)} \sigma(\mathrm{d}\theta) \Big| \le C_d (\log L) L^{-\alpha_d} e^{\sqrt{2}(x-\sqrt{2}L)} \,. \tag{3.2}$$

Proof. For $\psi \in \mathbb{S}^{d-1}$ and $\delta > 0$, define the subset $B(\psi,\delta) = \{\theta \in \mathbb{S}^{d-1} : \theta.\psi > \cos(\delta)\} \subset \mathbb{S}^{d-1}$. We first bound

$$\left|\int_{\mathbb{S}^{d-1}\setminus B(\psi,\delta)} f(\theta)(\sqrt{2}L - x\psi.\theta)e^{\sqrt{2}x\psi.\theta}\sigma(\mathrm{d}\theta)\right| \leq \|f\|_\infty \int_{\mathbb{S}^{d-1}\setminus B(\psi,\delta)} |\sqrt{2}L - x\psi.\theta|e^{\sqrt{2}x\psi.\theta}\sigma(\mathrm{d}\theta)$$
$$= \mathrm{vol}(\mathbb{S}^{d-2})\|f\|_\infty \int_\delta^\pi |\sqrt{2}L - x\cos(\phi)|\sin(\phi)^{d-2}e^{\sqrt{2}x\cos(\phi)}\mathrm{d}\phi\,,$$

where we used the change of variables $\theta \mapsto (\phi, \vec{y}) \in [0,\pi] \times \mathbb{S}^{d-2}$, so that the sphere $\mathbb{S}^{d-1}$ is parametrized by $R_\psi(\cos\theta, \vec{y}\sin\theta)$, where R_ψ is a fixed rotation sending e_1 to ψ. Therefore, for all δ small enough that $\cos(\delta) \leq 1 - \delta^2/4$, there exists a constant $K_d > 0$ such that for all L large enough and $x \geq \epsilon L$, we have

$$\left|\int_{\mathbb{S}^{d-1}\setminus B(\psi,\delta)} f(\theta)(\sqrt{2}L - x\psi.\theta)e^{\sqrt{2}x\psi.\theta-\sqrt{2}L}\sigma(\mathrm{d}\theta)\right| \leq K_d\|f\|_\infty L e^{\sqrt{2}(x-\sqrt{2}L)-\sqrt{2}\epsilon\delta^2L/4}\,. \tag{3.3}$$

Note that the right-hand side of (3.3) is dominated by the right-hand of (3.1). In particular, as L becomes large, the mass of $\int_{\mathbb{S}^{d-1}} f(\theta)(\sqrt{2}L - x\psi.\theta)e^{\sqrt{2}x\psi.\theta}\sigma(\mathrm{d}\theta)$ concentrates on $B(\psi,\delta)$— we show this now.

The continuous function f on the compact space $\mathbb{S}^{d-1}$ is uniformly continuous; hence, for all $\eta > 0$, there is $\delta = \delta(\eta)$ small enough so that

$$(f(\psi) - \eta)\int_{B(\psi,\delta)} (\sqrt{2}L - x\psi.\theta)e^{\sqrt{2}x\psi.\theta}\sigma(\mathrm{d}\theta) \leq \int_{B(\psi,\delta)} f(\theta)(\sqrt{2}L - x\psi.\theta)e^{\sqrt{2}x\psi.\theta}\sigma(\mathrm{d}\theta) \tag{3.4}$$
$$\leq (f(\psi) + \eta)\int_{B(\psi,\delta)} (\sqrt{2}L - x\psi.\theta)e^{\sqrt{2}x\psi.\theta}\sigma(\mathrm{d}\theta)\,.$$

Therefore, to complete the proof, it is enough to compute the asymptotic behaviour for large x and L of

$$I_{d,\delta}(L,x) = \int_{B(\psi,\delta)} (\sqrt{2}L - x\psi.\theta)e^{\sqrt{2}x\psi.\theta}\sigma(\mathrm{d}\theta) = \mathrm{vol}(\mathbb{S}^{d-2})\int_0^\delta (\sqrt{2}L - x\cos(\phi))\sin(\phi)^{d-2}e^{\sqrt{2}x\cos(\phi)}\mathrm{d}\phi\,,$$

which is done using Laplace's method, as follows. Let $\eta > 0$, and fix $C_d > 0$ large enough and $\delta = \delta(\eta) > 0$ small enough such that for all $0 \leq \phi \leq \delta$, we have

$$1 - \phi^2/2 \leq \cos(\phi) \leq 1 - (1-\eta)\phi^2/2 \text{ and } \phi^{d-2} - C_d\phi^d \leq \sin(\phi)^{d-2} \leq \phi^{d-2}\,.$$

With this notation, we observe that

$$\left|I_{d,\delta}(L,x) - \mathrm{vol}(\mathbb{S}^{d-2})(\sqrt{2}L - x)\int_0^\delta \phi^{d-2}e^{\sqrt{2}x\cos(\phi)}\mathrm{d}\phi\right| \leq (\sqrt{2}L + x)(2C_d + 1/2)\int_0^\delta \phi^d e^{\sqrt{2}x\cos(\phi)}\mathrm{d}\phi\,.$$

From Laplace's method, we have that for all $k \geq 0$,

$$\int_0^\delta \phi^k e^{\sqrt{2}x\cos(\phi)}\mathrm{d}\phi \sim_{x\to\infty} e^{\sqrt{2}x}\frac{1}{2}\left(\frac{\sqrt{2}}{x}\right)^{(k+1)/2}\Gamma((k+1)/2)\,.$$

As a result, uniformly over $\epsilon L \leq x \leq \sqrt{2}L - K\log L$, we have

$$(\sqrt{2}L - x)\int_0^\delta \phi^{d-2}e^{\sqrt{2}x\cos(\phi)}\mathrm{d}\phi \gg (\sqrt{2}L + x)\int_0^\delta \phi^d e^{\sqrt{2}x\cos(\phi)}\mathrm{d}\phi\,.$$

Thus, for any fixed $K > 0$ and uniformly in $x \in [\epsilon L, \sqrt{2}L - K\log L]$, we have

$$I_{d,\delta}(L,x) \sim_{L\to\infty} \mathrm{vol}(\mathbb{S}^{d-2})(\sqrt{2}L - x)e^{\sqrt{2}x}\frac{\Gamma((d-1)/2)}{2}\left(\frac{\sqrt{2}}{x}\right)^{(d-1)/2}$$
$$= (2\pi)^{(d-1)/4}x^{-(d+1)/2}(\sqrt{2}L - x)e^{\sqrt{2}x}\,.$$

Similarly, if $x \in [\sqrt{2}L - K\log L, \sqrt{2}L + K\log L]$, we have that for some constant $C_d > 0$,

$$I_{d,\delta}(L,x) \leq C_d(\log L)L^{-(d-1)/2}e^{\sqrt{2}x}\,. \tag{3.5}$$

Finally, using (3.3) and (3.4), we obtain that

$$\int_{\mathbb{S}^{d-1}} f(\theta)(\sqrt{2}L - x\psi.\theta)e^{\sqrt{2}(x\psi.\theta-\sqrt{2}L)}\sigma(\mathrm{d}\theta) \sim_{L\to\infty} f(\psi)I_{d,\delta}(L,x)e^{-2L}$$
$$\sim_{L\to\infty} f(\psi)(2\pi)^{(d-1)/4}x^{-(d+1)/2}(\sqrt{2}L - x)e^{\sqrt{2}(x-\sqrt{2}L)},$$

uniformly in $\psi \in \mathbb{S}^{d-1}$ and $x \in [\epsilon L, \sqrt{2}L - K\log L]$. The upper bound for $x \in [\sqrt{2}L - K\log L, \sqrt{2}L + K\log L]$ is obtained using (3.5) and (3.4). □

Next, we show that particles that are not in the window $I_L^{\mathtt{win}}$ (defined in Section 2.2) at time L do not contribute to the left-hand side of (1.4).

Lemma 3.2.

$$\sum_{u\in\mathcal{N}_L} \mathbb{1}_{\{R_L^{(u)}\notin I_L^{\mathtt{win}}\}}(R_L^{(u)})^{-\alpha_d}(1+|\sqrt{2}L - R_L^{(u)}|)e^{-(\sqrt{2}L-R_L^{(u)})\sqrt{2}} \xrightarrow[L\to\infty]{p} 0.$$

Proof. Let Z'_L denote the expression on the left-hand side of the above display, and fix $\epsilon > 0$. From a union bound, the many-to-one lemma (Lemma 2.2), the Bessel density at time L (2.3), and standard estimates of Gaussian integrals, it follows that there exists a constant $K_d > 0$ depending only on the dimension d such that

$$\mathbb{P}(\exists u \in \mathcal{N}_L : R_L^{(u)} > \sqrt{2}L + K_d\log L) \le \mathbb{E}\Big[\sum_{u\in\mathcal{N}_L} \mathbb{1}_{\{R_L^{(u)}>\sqrt{2}L+K_d\log L\}}\Big]$$
$$= e^L\mathbb{P}(R_L > \sqrt{2}L + K_d\log L) \lesssim L^{-1/2}. \quad (3.6)$$

For brevity, let us write $B(L) := \sqrt{2}L + K_d\log L$. Then, using the Markov inequality and the many-to-one lemma (Lemma 2.2), we have

$$\mathbb{P}(|Z'_L| > \epsilon) \le \epsilon^{-1}e^L\mathbb{E}\Big[\mathbb{1}_{\{R_L\notin I_L^{\mathtt{win}},R_L\in[0,B(L)]\}}(R_L)^{-\alpha_d}(1+|\sqrt{2}L-R_L|)e^{-(\sqrt{2}L-R_L)\sqrt{2}}\Big] + o_L(1),$$

where the $o_L(1)$ term comes from (3.6). We now integrate over the density $p_L^R(0,\cdot)$ of R_L, so that the last display is given by

$$\epsilon^{-1}e^L\int_{[0,B(L)]\setminus[L^{1/6},L^{2/3}]} \frac{p_L^R(0,B(L)-w)}{(B(L)-w)^{\alpha_d}}(1+|w-K_d\log L|)e^{-\sqrt{2}(w-K_d\log L)}\mathrm{d}w + o_L(1)$$
$$= c_dL^{-\frac{d}{2}}\epsilon^{-1}\int_{[0,B(L)]\setminus[L^{1/6},L^{2/3}]}(B(L)-w)^{\alpha_d}(1+|w-K_d\log L|)e^{-\frac{(K_d\log L-w)^2}{2L}}\mathrm{d}w + o_L(1). \quad (3.7)$$

The Gaussian term in the right-hand side of (3.7) ensures that the latter is dominated by the integral over the interval $[0, L^{1/6}]$. The integral over this interval (including the $c_dL^{-d/2}\epsilon^{-1}$ pre-factor) is bounded by $\epsilon^{-1}L^{-1/2}L^{1/6}L^{1/6} = o_L(1)$. Thus, we have shown that for any $\epsilon > 0$, $\mathbb{P}(|Z'_L| > \epsilon)$ tends to 0 as L tends to ∞, which concludes the proof. □

Proof of Theorem 1.5. In this proof, the term "with high probability" means "with probability approaching 1 as $L \to \infty$". Recall from (1.2) that

$$\sum_{u\in\mathcal{N}_L}\int_{\mathbb{S}^{d-1}} f(\theta)(\sqrt{2}L - \theta.X_L^{(u)})e^{\sqrt{2}\theta.X_L^{(u)}-2L}\sigma(\mathrm{d}\theta) \xrightarrow{a.s.} \langle f, D_\infty\rangle. \quad (3.8)$$

We proceed by restricting the locations of the contributing particles at time L. To start, by (3.6) with high probability, all norms of particles at time L are bounded by $\sqrt{2}L + K_d\log L$, for some large, positive constant K_d. Furthermore, we can show that

$$\left|\mathbb{E}\Big[\sum_{u\in\mathcal{N}_L}\mathbb{1}_{\{R_L^{(u)}\le\sqrt{2}L-L^{2/3}\}}\int_A f(\theta)(\sqrt{2}L-\theta.X_L^{(u)})e^{\sqrt{2}\theta.X_L^{(u)}-2L}\sigma(\mathrm{d}\theta)\Big]\right| \lesssim L^{1/6}e^{-\frac{L^{1/3}}{2}}. \quad (3.9)$$

Indeed, (3.9) follows by applying the triangle inequality and the many-to-one lemma to its left-hand side, yielding an upper-bound of

$$e^L|\max f|\mathbb{E}\Big[\big|\sqrt{2}L - W_L\big|e^{-\sqrt{2}(\sqrt{2}L-W_L)}\mathbb{1}_{\{W_L\le\sqrt{2}L-L^{2/3}\}}\Big]\,, \tag{3.10}$$

where W_L denotes a standard Brownian motion at time L, and we have used the fact that $\theta.X_L^{(u)}$ is equal to W_L in distribution for any θ and u. Taking $\tilde{W}_L := \sqrt{2}L - W_L$ and applying the Girsanov transform gives that (3.10) equals $\mathbb{E}[|\tilde{W}_L|\mathbb{1}_{\{\tilde{W}_L\ge L^{2/3}\}}]$, from which (3.9) follows.

We can further restrict the locations of the $R_L^{(u)}$ by observing that

$$\sum_{u\in\mathcal{N}_L}\mathbb{1}_{\{R_L^{(u)}\in[\sqrt{2}L-K_d\log L,\sqrt{2}L+K_d\log L]\}}\int_{\mathbb{S}^{d-1}} f(\theta)(\sqrt{2}L-\theta.X_L^{(u)})e^{\sqrt{2}\theta.X_L^{(u)}-2L}\sigma(\mathrm{d}\theta)$$

converges to 0 in probability due to the triangle inequality, the upper bound (3.2), and Lemma 3.2. Together with (3.9) and (3.6), we have thus far shown that

$$\langle D_L', f\rangle := \sum_{u\in\mathcal{N}_L}\mathbb{1}_{\{R_L^{(u)}\in[\sqrt{2}L-L^{2/3},\sqrt{2}L-K_d\log L]\}}\int_{\mathbb{S}^{d-1}} f(\theta)(\sqrt{2}L-\theta.X_L^{(u)})e^{\sqrt{2}\theta.X_L^{(u)}-2L}\sigma(\mathrm{d}\theta)$$

converges to $\langle D_\infty, f\rangle$ in probability as L tends to infinity. The uniform asymptotic in equation (3.1) of Lemma 3.1 shows that the above expression is equal to

$$(1+o(1))(2\pi)^{(d-1)/4}\sum_{u\in\mathcal{N}_L}\mathfrak{M}_{L,R_L^{(u)}}\mathbb{1}_{\{R_L^{(u)}\in[\sqrt{2}L-L^{2/3},\sqrt{2}L-K_d\log L]\}}\,.$$

Now, Lemma 3.2 shows that

$$\sum_{u\in\mathcal{N}_L^{\mathtt{win}}}\mathfrak{M}_{L,R_L^{(u)}}\mathbb{1}_{\{R_L^{(u)}\notin I_L^{\mathtt{win}}\}}\xrightarrow[L\to\infty]{(p)} 0\,,$$

which concludes the proof. ∎

4. Proof of Theorem 1.1 and Proposition 1.3

Proposition 1.3 will follow from Theorem 1.5 and the following estimate, which can be seen as a two-fold extension of [11, Theorem 3.2].

Proposition 4.1. *Let $\phi : \mathbb{S}^{d-1}\times\mathbb{R}\to\mathbb{R}_{\ge 0}$ be a compactly supported, continuous function. For any $\theta\in\mathbb{S}^{d-1}$, define the (positive) constant*

$$C_d(\phi_\theta) := \sqrt{\frac{2^{1+\alpha_d}}{\pi}}C(\phi_\theta)\,, \tag{4.1}$$

where $\phi_\theta(x) := \phi(\theta, x)$, and $C(\phi_\theta)$ is defined in (2.13). Then

$$\lim_{L\to\infty}\limsup_{t\to\infty}\sup_{\substack{z\in[L^{1/6},L^{2/3}],\\ \theta\in\mathbb{S}^{d-1}}}\left|\frac{\mathbb{E}_{\theta(\sqrt{2}L-z)}\left[1-\exp\left(-\langle\overline{\mathcal{E}}_{L,t},\phi\rangle\right)\right]}{\mathfrak{M}_{L,z}} - C_d(\phi_\theta)\right| = 0, \tag{4.2}$$

where

- $I_L^{\mathtt{win}} = [\sqrt{2}L - L^{2/3}, \sqrt{2}L - L^{1/6}]$,
- $\overline{\mathcal{E}}_{L,t} = \sum_{u\in\mathcal{N}_{t-L}}\delta_{\left(R_{t-L}^{(u)}-m_t^{(d)},\ \theta_{t-L}^{(u)}\right)}$ *, and*
- $\mathfrak{M}_{L,z} = (\sqrt{2}L-z)^{-\alpha_d}ze^{-z\sqrt{2}}$.

Remark that when compared to Theorem 3.2 in [11], Proposition 4.1 adds information on the direction of large particles. It states that, if the initial particle is close to θz for large $z\in\mathbb{R}_+$, then with high probability, the direction of the farthest particle from the origin will be in a small neighbourhood of θ. Additionally, while the former result only deals with the tail of R_{t-L}^*, which can be seen as the Laplace functional computed with $\phi(x) = \infty\mathbb{1}_{\{x>0\}}$, equation (4.2) extends the Laplace functional to a broader class of functions.

In what follows, we show how Theorem 1.5 and Proposition 4.1 imply Proposition 1.3. We then use Proposition 1.3 to prove Theorem 1.1. The proof of Proposition 4.1 is postponed to Section 5.

4.1. **Proof of Proposition 1.3 using Proposition 4.1.**

Proof of Proposition 1.3. Fix some $\phi : \mathbb{S}^{d-1} \times \mathbb{R} \to \mathbb{R}_{\geq 0}$ satisfying the conditions of Proposition 1.3. Let $-y$ be the infimum of the support of ϕ on $\mathbb{R}$. We are interested in the quantity

$$g_t(\phi) := \mathbb{E}\Big[\exp\Big(-\sum_{u\in\mathcal{N}_t}\phi(\theta_t^{(u)}, R_t^{(u)} - m_t^{(d)})\Big)\Big].$$

Since $\phi(\theta_t^{(u)}, R_t^{(u)} - m_t^{(d)} - y)$ is nonzero only if $R_t^{(u)} > m_t^{(d)} + y$, we see from Proposition 2.1 that only particles $u \in \mathcal{N}_t$ that are descendants of particles in $\mathcal{N}_L^{\mathtt{win}}$ contribute. Thus, applying the branching property at time L as well as the Markov property shows that

$$g_t(\phi) = \mathbb{E}\Big[\prod_{u\in\mathcal{N}_L^{\mathtt{win}}}\mathbb{E}_{X_L^{(u)}}\left(\exp\left(-\langle\overline{\mathcal{E}}_{L,t},\phi\rangle\right)\right)\Big] + o_u(1).$$

Set $z_L(u) := \sqrt{2}L - R_L^{(u)}$. Then, using Proposition 4.1, we can follow the steps leading up to [11, Equation 3.7] to obtain a non-negative sequence ϵ_L tending to 0 as L tends to infinity such that for all L large and t large enough compared to L, we have

$$\left|g_t(\phi) - \mathbb{E}\Big[\exp\Big(-\sum_{u\in\mathcal{N}_L^{\mathtt{win}}} C_d(\phi_{\theta_L^{(u)}})\mathfrak{M}_{L,z_L(u)}\Big)\Big]\right| \leq \epsilon_L.$$

From Theorem 1.5, we have convergence of the above Laplace transform to

$$\mathbb{E}\Big[\exp\Big(-(2\pi)^{-\alpha_d/2}\int_{\mathbb{S}^{d-1}} C_d(\phi_\theta)D_\infty(\theta)\sigma(\mathrm{d}\theta)\Big)\Big],$$

as L tends to infinity. Recalling now the definition of $C_d(\phi_\theta)$ from (4.1) and of $C(\phi)$ from (2.13), the previous two displays yield a non-negative sequence ϵ'_L tending to 0 as L tends to infinity such that for all L large and t large enough compared to L, we have

$$\left|g_t(\phi) - \mathbb{E}\Big[\exp\Big(-\sqrt{\frac{2}{\pi^{1+\alpha_d}}}\int_{\mathbb{S}^{d-1}} C(\phi_\theta)D_\infty(\theta)\sigma(\mathrm{d}\theta)\Big)\Big]\right| \leq \epsilon'_L.$$

Since $g_t(\phi)$ has no L-dependence, it follows from the previous two displays that

$$g_t(\phi) \xrightarrow[t\to\infty]{} \mathbb{E}\Big[\exp\Big(-\sqrt{\frac{2}{\pi^{1+\alpha_d}}}\int_{\mathbb{S}^{d-1}} C(\phi_\theta)D_\infty(\theta)\sigma(\mathrm{d}\theta)\Big)\Big], \tag{4.3}$$

as desired. □

4.2. **Proof of Theorem 1.1.** We recall that as observed in Remark 1.2, the statement of Proposition 1.3 is valid in all dimensions, including $d = 1$, where for $d = 1$, we have $\mathbb{S}^{d-1} = \{-1, 1\}$, and $\sigma = \delta_1 + \delta_{-1}$ (in other words, in all dimensions σ is the Haar measure on $\mathbb{S}^{d-1}$).

Proof of Theorem 1.1. We note that Proposition 1.3 already gives the claimed convergence, so that we only need to prove the claimed identification of the limit.

By [2] or [3], in dimension 1, for all continuous compactly supported function φ on $\mathbb{R}$, we have that

$$\lim_{t\to\infty}\mathbb{E}\left(e^{-\sum_{u\in\mathcal{N}_t}\varphi(X_t^{(u)} - m_t^{(1)})}\right) = \mathbb{E}\left(e^{-\langle\mathcal{E}_\infty^{(1)},\varphi\rangle}\right),$$

with $\mathcal{E}_\infty^{(1)}$ an SDPPP($\sqrt{2}\gamma D_\infty$,$e^{-\sqrt{2}x}$, $\mathcal{D}$), where $m_t^{(1)}$ denotes m_t corresponding to dimension $d = 1$ and D_∞ is the same as $D_\infty(\theta)$ for $d = 1$ with $\theta = 1$, i.e. it is the standard one dimensional derivative martingale. Using Campbell's formula on the right hand side of the last display, we obtain that

$$\mathbb{E}\left(e^{-\langle\mathcal{E}_\infty^{(1)},\varphi\rangle}\right) = \mathbb{E}\left(\exp\left(-\sqrt{2}\gamma D_\infty\int_{\mathbb{R}}\left(1 - \mathbb{E}\left(e^{-\langle\mathcal{D},\varphi(x+\cdot)\rangle}\right)\right)e^{-\sqrt{2}x}\mathrm{d}x\right)\right).$$

Then using Proposition 1.3 in dimension $d = 1$ with $\varphi(x)\mathbb{1}_{\{\theta=1\}}$, we obtain that

$$\lim_{t\to\infty}\mathbb{E}\left(e^{-\langle\mathcal{E}_t^1,\varphi\rangle}\right) = \mathbb{E}\Big[\exp\Big(-\mathfrak{C}_1 C(\varphi)D_\infty\Big)\Big], \tag{4.4}$$

where $\mathfrak{C}_1 = \sqrt{2/\pi}$. (Alternatively, (4.4) is obtained in [3, Proposition 3.2].) As a result, we have

$$\mathbb{E}\Big[\exp\Big(-\mathfrak{C}_1 C(\varphi) D_\infty\Big)\Big] = \mathbb{E}\left(\exp\left(-\sqrt{2}\gamma D_\infty \int_{\mathbb{R}} \left(1 - \mathbb{E}\left(e^{-\langle \mathcal{D}, \varphi(x+\cdot)\rangle}\right)\right) e^{-\sqrt{2}x} \mathrm{d}x\right)\right).$$

and therefore

$$\mathfrak{C}_1 C(\varphi) = \sqrt{2}\gamma \int_{\mathbb{R}} \left(1 - \mathbb{E}\left(e^{-\langle \mathcal{D}, \varphi(x+\cdot)\rangle}\right)\right) e^{-\sqrt{2}x} \mathrm{d}x, \tag{4.5}$$

using that the Laplace transform of D_∞ is strictly decreasing as D_∞ is non-negative and non-degenerate.

Returning to $d > 1$, thanks to this identification, we can then compute for ϕ continuous compactly supported on $\mathbb{S}^{d-1} \times \mathbb{R}$, again by Campbell's formula,

$$\begin{aligned}\mathbb{E}\left[e^{-\langle \mathcal{E}_\infty, \phi\rangle}\right] &= \mathbb{E}\Big[\exp\Big(-\sqrt{2}\gamma\pi^{-\alpha_d/2} \int_{\mathbb{S}^{d-1}} D_\infty(\theta) \int_{\mathbb{R}} \left(1 - \mathbb{E}\left[e^{-\langle \mathcal{D}, \phi_\theta(x+\cdot)\rangle}\right]\right) e^{-\sqrt{2}x} \mathrm{d}x \sigma(\mathrm{d}\theta)\Big)\Big] \\ &= \mathbb{E}\Big[\exp\Big(-\mathfrak{C}_d \int_{\mathbb{S}^{d-1}} C(\phi_\theta) D_\infty(\theta) \sigma(\mathrm{d}\theta)\Big)\Big],\end{aligned}$$

by (4.5), using that $\mathfrak{C}_d = \pi^{-\alpha_d/2}\mathfrak{C}_1$. □

5. Proof of Proposition 4.1

We finish this article with a proof of Proposition 4.1, which uses the geometrical result Proposition 2.3 to take care of the directional constraint, the results of [11] to characterize the typical trajectories of particles contributing to the Laplace functional, and a coupling with one-dimensional branching Brownian motion on the last time interval of length ℓ. Throughout this section, let $\ell := \ell(L)$ satisfy the properties

$$1 \le \ell \le L^{1/6} \quad \text{and} \quad \lim_{L\to\infty} \ell = \infty. \tag{5.1}$$

The precise dependence of ℓ on L does not play a role. For convenience, we will write $\tilde{t} := t - L$. Since we will send t to infinity before L, the parameter L should be thought of as an order one quantity (compared to t). Also, we fix a constant $y \in \mathbb{R}$ in what follows until further specified.

5.1. **Description of the extremal particles by [11].** The (modified) second moment method used to prove [11, Theorem 3.2] implies the following: for a branching Bessel process started from the window $I_L^{\mathtt{win}}$ (at time 0), particles $u \in \mathcal{N}_{\tilde{t}}$ that reach height $m_t^{(d)} + y$ or higher at time $\tilde{t}$ follow a well-controlled trajectory until time $\tilde{t} - \ell$; furthermore, among particles $v \in \mathcal{N}_{\tilde{t}-\ell}$ that follow such a trajectory, at most one produces a descendent that reaches height $m_t^{(d)} + y$. These two results are reproduced below as Propositions 5.1 and 5.2, respectively. We will use these two ideas in the proof of Proposition 4.1 to great effect. Before we begin, let us lay out some notation that will be familiar from [11].

Let $\mathbf{y}(b) := \frac{m_t}{t}(t-\ell) + y - b$. For any $v \in \mathcal{N}_{\tilde{t}-\ell}$, we define the event

$$\mathfrak{T}(v) := \mathfrak{T}_{t,L,y}(v) = \Big\{\max_{u \in \mathcal{N}_\ell^v} R_{\tilde{t}}^{(u)} > m_t^{(d)} + y\Big\}. \tag{5.2}$$

We will also make use of certain "barrier events" to restrict the paths of our BBM particles. For functions $f, g : [0,\infty) \to \mathbb{R}$, a set $I \subset [0,\infty)$, and a real-valued process $X.$, we call events of the following form *barrier events*:

$$\overline{\mathcal{B}}_I^f(X.) := \{X_u \le f(u),\ \forall u \in I\} \quad \text{and} \quad \underline{\mathcal{B}}_I^f(X.) := \{X_u \ge f(u),\ \forall u \in I\}.$$

Recall the barriers $B_0(s) := B_0(s; t, L)$ ([11, Equation 4.38]) and $Q_z(s) := Q_z(s; t, L, y)$ ([11, Equation 5.5]), whose precise definitions we will not use here. A crucial barrier event used throughout this subsection will be the event that a process $X.$ is bounded above by the linear barrier $\frac{m_t^{(d)}}{t}(\cdot + L) + y$ (where we write $f(\cdot + r)$ to denote the function $u \mapsto f(u + r)$) and bounded below by $Q_z(\cdot)$ on a certain time interval $I \subset \mathbb{R}_{\ge 0}$. We will denote this event by $\mathcal{B}_I^{\hat{\hat{}}}(X.) := \mathcal{B}_{I,y,z,L,t}^{\hat{\hat{}}}(X.)$; note that

$$\mathcal{B}_I^{\hat{\hat{}}}(X.) = \overline{\mathcal{B}}_I^{\frac{m_t^{(d)}}{t}(\cdot + L) + y}(X.) \cap \underline{\mathcal{B}}_I^{Q_z}(X.). \tag{5.3}$$

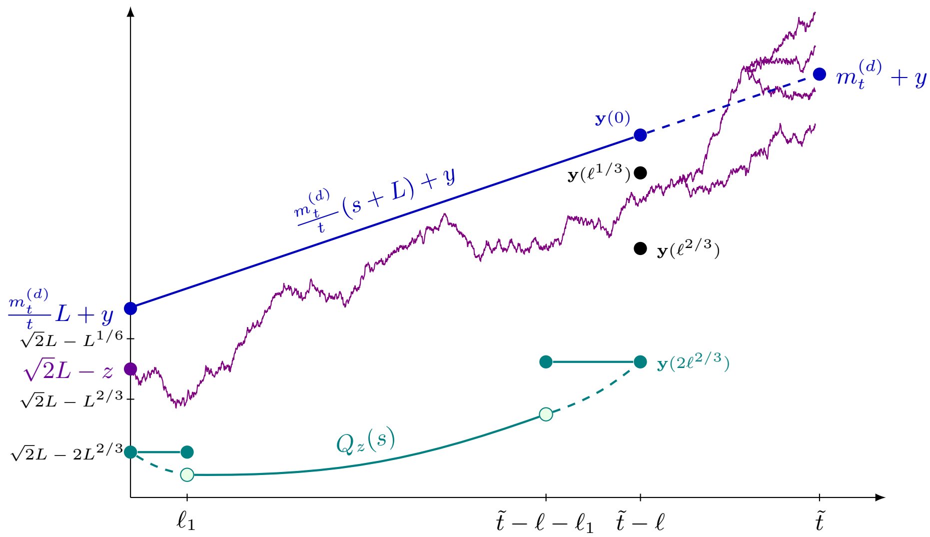

FIGURE 4. The event $G_{L,t}(v) \cap \mathfrak{T}(v)$ from (5.5): the trajectory $R_\cdot^{(v)}$ of a given particle $v \in \mathcal{N}_{\tilde{t}-\ell}$ is bounded above by the blue line and below by the solid green curves that comprise Q_z on $[0, \tilde{t}-\ell]$, $R_{\tilde{t}-\ell}^{(v)}$ lies in the window $[\mathbf{y}(\ell^{2/3}), \mathbf{y}(\ell^{1/3})]$, and v produces a descendent in $\mathcal{N}_{\tilde{t}}$ that exceeds $m_t^{(d)} + y$.

Now, for any $v \in \mathcal{N}_{\tilde{t}-\ell}$, define the events

$$F_{L,t}(v) := \overline{\mathcal{B}}^{B_0}_{[0,\tilde{t}-\ell]}(R_\cdot^{(v)}) \cap \left\{ R_{\tilde{t}-\ell}^{(v)} > \frac{t}{\sqrt{d}} \right\}, \text{ and} \tag{5.4}$$

$$G_{L,t}(v) := \mathcal{B}^{\frown}_{[0,\tilde{t}-\ell]}(R_\cdot^{(v)}) \cap \left\{ \mathbf{y}\big(R_{\tilde{t}-\ell}^{(v)}\big) \in [\ell^{1/3}, \ell^{2/3}] \right\} . \tag{5.5}$$

The event $G_{L,t}(v) \cap \mathfrak{T}(v)$ is depicted in Figure 4. In Proposition 4.1, we consider a d-dimensional BBM started from a point on the sphere of radius $\sqrt{2}L - z$, where $z \in [L^{1/6}, L^{2/3}]$. In particular, we seek to demonstrate asymptotic statements uniformly over z in this interval. For such statements, we will use heavily the notation set forth in Section 2.1.

The following results are from [11].

Proposition 5.1 ([11])**.**

$$\mathbb{E}_{\sqrt{2}L-z}\Bigg[\sum_{u \in \mathcal{N}_{\tilde{t}-\ell}} \mathbb{1}_{F_{L,t}(v)^c} \Bigg] = o_u(\mathfrak{M}_{L,z}) \,, \ \textit{and}$$

$$\mathbb{E}_{\sqrt{2}L-z}\Bigg[\sum_{u \in \mathcal{N}_{\tilde{t}-\ell}} \mathbb{1}_{F_{L,t}(v) \setminus G_{L,t}(v)} \mathbb{1}_{\mathfrak{T}(v)} \Bigg] = o_u(\mathfrak{M}_{L,z}) \,.$$

The above proposition follows from [11, Lemma 4.3 and Claim 5.5] respectively and their proofs.[1] The result below is from [11, Lemma 5.2] (see also there equation (7.2)).

Proposition 5.2 ([11])**.**

$$\mathbb{E}_{\theta(\sqrt{2}L-z)}\Bigg[\sum_{v \neq w \in \mathcal{N}_{\tilde{t}-\ell}} \mathbb{1}_{\{G_{L,t}(v) \cap G_{L,t}(w)\}} \Bigg] = o_u(\mathfrak{M}_{L,z})$$

[1]The upper-bounds in these results of [11] are stated for quantities of the form $\mathbb{P}(\cup_{u \in \mathcal{N}_{\tilde{t}-\ell}} A_v)$, while Proposition 5.1 bounds (larger) first-moment quantities of the form $\mathbb{E}[\sum_{u \in \mathcal{N}_{\tilde{t}-\ell}} \mathbb{1}_{A_v}]$. However, the first step in the proof of each result of [11] is to bound $\mathbb{P}(\cup_{u \in \mathcal{N}_{\tilde{t}-\ell}} A_v)$ by the corresponding first-moment quantity, so that the results there are actually shown via upper-bounds on the $\mathbb{E}[\sum_{u \in \mathcal{N}_{\tilde{t}-\ell}} \mathbb{1}_{A_v}]$. Thus, we indeed have Proposition 5.1.

In the proof of Proposition 4.1, we will use Proposition 5.1 to show that the only particles in $\mathcal{N}_{\tilde{t}}$ contributing to $\mathbb{E}_{\theta(\sqrt{2}L-z)}[1-\exp(-\langle\overline{\mathcal{E}}_{L,t},\phi\rangle)]$ are those that descended from particles $v\in\mathcal{N}_{\tilde{t}-\ell}$ that performed the event $G_{L,t}(v)$. Proposition 5.2 shows that only one such particle in $v\in\mathcal{N}_{\tilde{t}-\ell}$ will perform $G_{L,t}(v)$.

5.2. **Coupling with a one-dimensional BBM.** Eventually, we will encounter an expression of the form

$$\mathbb{E}_{\mathbf{y}(w)}\Big[\exp(-\langle\mathcal{E}_\ell,\phi(\cdot-c)\rangle)\Big]=\mathbb{E}_{\mathbf{y}(w)}\Big[1-\sum_{u\in\mathcal{N}_\ell}\phi(R_\ell^{(u)}-m_t^{(d)}-y-c)\Big]\,,$$

where $\phi:\mathbb{R}\to\mathbb{R}$ is a continuous function and $w\in[\ell^{1/3},\ell^{2/3}]$ (c.f. the definition of $G_{L,t}(v)$ in (5.5)). We will approximate this expression via a coupling with the one-dimensional branching Brownian motion, which will then via a famous formula of McKean [15] give an expression in terms of the solution to the F-KPP equation, for which asymptotics are by now well-understood (see Proposition 2.7). Let us discuss these matters now.

In light of the SDE for the d-dimensional Bessel process,

$$\mathrm{d}R_s=\frac{\alpha_d}{R_s}\mathrm{d}s+\mathrm{d}W_s\,,$$

one might expect that since $\mathbf{y}(w)$ is of order t, a branching Bessel process on $[0,\ell]$ started from $\mathbf{y}(w)$ may be coupled to be "very close" with probability going to 1 as t goes to infinity with a one-dimensional branching Brownian motion (for which many more results are known). Indeed, this was shown in [11]. Before stating this result, let us state the coupling. Consider the natural coupling of a one-dimensional BBM and a d-dimensional branching Bessel process obtained by using the same branching tree for both processes (hence the same set of particles in both processes at all times), and the same driving Brownian motion for each edge in the tree (to be used in each of the SDEs by the two processes for evaluating the location of the corresponding particle). Thus, for all $s>0$, each $v\in N_s$, is associated to a Bessel process $R_\cdot^{(v)}$ and a 1-d Brownian motion $W_\cdot^{(v)}$ satisfying the SDE

$$dR_r^{(v)}=\frac{\alpha_d}{R_r^{(v)}}\mathrm{d}r+\mathrm{d}W_r^{(v)}\,.$$

Note that $R_r^{(v)}\geq W_r^{(v)}$.

Proposition 5.3 ([11])**.** *Consider the above-defined coupling of 1-dimensional BBM $\{W_s^{(v)}\}_{s\geq0,v\in N_s}$ and a branching d-dimensional Bessel process $\{R_s^{(v)}\}_{s\geq0,v\in N_s}$ started at x, for some $x>0$. Fix $\ell>0$, and let*

$$\mathcal{G}_x=\Big\{\min_{v\in\mathcal{N}_\ell}\inf_{0\leq s\leq\ell}R_s^{(v)}\geq x/4\Big\}\,.$$

Then there exists some constant $C_d>0$ such that for large enough x (in terms of ℓ),

$$\sup_{0\leq s\leq\ell}\sup_{v\in N_s}\Big|R_s^{(v)}-W_s^{(v)}\Big|\mathbb{1}_{\mathcal{G}_x}\leq C_d\ell/x\,.$$

Furthermore, $\mathbb{P}_x(\mathcal{G}_x^c)\leq(2+e^\ell)e^{-x^2/8\ell}$.

The first bound is given in [11, Claim 6.2], while the bound on $\mathbb{P}_x(\mathcal{G}_x^c)$ is given in the proof of [11, Corollary 6.3]. As a consequence of the coupling result in Proposition 5.3, we have the following corollary, which shows that functionals of a branching Bessel process may be replaced by functionals of one-dimensional BBM, up to a negligible error, if the time interval on which the process is considered is much shorter than the initial position of the process.

Corollary 5.4. *Fix $\phi:\mathbb{R}\to\mathbb{R}$ a uniformly continuous function, $y\in\mathbb{R}$, and $c>0$. Let $\ell:=\ell(L)$ be as in (5.1). Couple $\{W_s^{(v)}\}_{s\geq0,v\in N_s}$ and $\{R_s^{(v)}\}_{s\geq0,v\in N_s}$ as above. Then uniformly over $x:=x(t)$ such that $x\to\infty$ as $t\to\infty$, we have*

$$\mathbb{E}_x\Big[\exp(-\langle\mathcal{E}_\ell,\phi(\cdot-c)\rangle)-\exp\Big(-\sum_{u\in\mathcal{N}_\ell}\phi(W_\ell^{(u)}-m_t^{(d)}-c)\Big)\Big]=o_u(L^{-5/2})\,.$$

Remark 5.5. The uniform continuity assumption on ϕ in Corollary 5.4 may be removed if the range of x is taken to be any ball around $m_t^{(d)}$ with radius given by a function of L.

Remark 5.6. Corollary 5.4 complements [11, Corollary 6.3], which may be thought of as Corollary 5.4 with $\phi(a):=\infty\mathbb{1}_{\{a>0\}}$.

Proof of Corollary 5.4. Since ϕ is uniformly continuous, there exists some ϵ_t such that $\epsilon_t \to 0$ as $t \to \infty$ and

$$\epsilon_t := \sup_{a\in\mathbb{R}} |\phi(a + C_d\ell/t) - \phi(a)| \,.$$

Further, by Markov, $\mathbb{P}(|\mathcal{N}_\ell| > \exp(L^3)) \le \exp(\ell - L^3) = o(-L^{5/2})$. Let

$$f(\ell,t) := \exp\Big(-\sum_{u\in\mathcal{N}_\ell} \phi(W^{(u)}_\ell - m^{(d)}_t - c)\Big)\,.$$

On the event $\mathcal{G}_x \cap \{|\mathcal{N}_\ell \le \exp(L^3)\}$, we have

$$\exp\big(-\langle \mathcal{E}_\ell, \phi(\cdot - c)\rangle\big) \in f(\ell,t)\cdot[\exp(-\epsilon_t e^{L^3}), \exp(\epsilon_t e^{L^3})]\,.$$

The above implies

$$\Big|\mathbb{E}_x\Big[\exp\big(-\langle \mathcal{E}_\ell, \phi(\cdot - c)\rangle\big) - f(\ell,t)\Big]\Big| \le \mathbb{E}_x\Big[(1 - e^{-\epsilon_t e^{L^3}})f(\ell,t)\Big] + 2\Big(\mathbb{P}(\mathcal{G}_x^c) + \mathbb{P}(|\mathcal{N}_\ell| > e^{L^3})\Big)\,.$$

Here, we have used the fact that the quantity inside the expectation on the left hand side is bounded in absolute value by 2. Since $f(\ell,t) \le 1$, the first term in the second line vanishes as $t\to\infty$; thus, it is $o_u(L^{5/2})$ uniformly over x such that $x \to 0$ as $t \to \infty$. The same is true for $\mathbb{P}(\mathcal{G}_x^c)$ by Proposition 5.3. This concludes the proof. ☐

5.3. **Proof of Proposition 4.1.** We are now ready to prove Proposition 4.1.

Proof of Proposition 4.1. Define $-y$ to be the infimum of the (compact) support of ϕ in the real coordinate, that is,

$$\inf\{x \in \mathbb{R} : \sup_{\theta\in\mathbb{S}^{d-1}} \phi(\theta, x) > 0\}\,.$$

Define the function $\overline{\phi} : \mathbb{S}^{d-1}\times\mathbb{R} \to \mathbb{R}_{\ge 0}$ as

$$\overline{\phi}(\sigma, x) := \phi(\sigma, x + y)\,,$$

so that $\overline{\phi}$ is supported on $\mathbb{S}^{d-1}\times\mathbb{R}_{\ge 0}$. We seek the leading-order asymptotics of

$$\Upsilon := \mathbb{E}_{\theta(\sqrt{2}L - z)}\Big[1 - \exp\Big(-\sum_{v\in\mathcal{N}_{\tilde{t}-\ell}}\sum_{u\in\mathcal{N}^v_\ell} \overline{\phi}(\theta^{(u)}_{\tilde{t}}, R^{(u)}_{\tilde{t}} - m^{(d)}_t - y)\Big)\Big]\,.$$

Note that only particles $u \in \mathcal{N}_{\tilde{t}}$ such that $R^{(u)}_{\tilde{t}} \ge m^{(d)}_t + y$ contribute to the exponential, since $\overline{\phi}_\sigma(x) := \overline{\phi}(\sigma, x)$ is only supported on $x \ge 0$; thus, we may add the indicator $\mathfrak{T}(v)$ to the sum in the exponential without change. Furthermore, Proposition 2.3 states that

$$\sup_{\theta\in\mathbb{S}^{d-1}} \mathbb{P}_{\theta(\sqrt{2}L-z)}\Big(\bigcap_{\{u\in\mathcal{N}_{\tilde{t}},\ R^{(u)}_{\tilde{t}} > m^{(d)}_t + y\}} \Big\{d_{\mathbb{S}^{d-1}}(\theta, \theta^{(u)}_{\tilde{t}}) < L^{-1/12}\Big\}\Big) > 1 - e^{-L^{5/6}}$$

Define the function

$$\psi_\theta(v) := \psi_{\theta,y,\ell,L,t}(v) = \sum_{u\in\mathcal{N}^v_\ell} \overline{\phi}_\theta(R^{(u)}_{\tilde{t}} - m^{(d)}_t - y)\,.$$

Then the above gives the following expression

$$\Upsilon = (1 + o_u(1))\mathbb{E}_{\theta(\sqrt{2}L-z)}\Big[1 - \exp\Big(-\sum_{v\in\mathcal{N}_{\tilde{t}-\ell}} \psi_\theta(v)\mathbb{1}_{\mathfrak{T}(v)}\Big)\Big] + o_u(\mathfrak{M}_{L,z})\,,$$

where the $o_u(\cdot)$'s hold uniformly in θ. Now,

$$\begin{aligned}
&\mathbb{E}_{\theta(\sqrt{2}L-z)}\Big|\exp\Big(-\sum_{v\in\mathcal{N}_{\tilde{t}-\ell}} \psi_\theta(v)\mathbb{1}_{\mathfrak{T}(v)}\Big) - \exp\Big(-\sum_{v\in\mathcal{N}_{\tilde{t}-\ell}} \psi_\theta(v)\mathbb{1}_{G_{L,t}(v)\cap\mathfrak{T}(v)}\Big)\Big| \\
&\le 2\mathbb{P}_{\theta(\sqrt{2}L-z)}\Big(\cup_{v\in\mathcal{N}_{\tilde{t}-\ell}} G_{L,t}(v)^c \cap \mathfrak{T}(v)\Big) \\
&\le 2\mathbb{E}_{\theta(\sqrt{2}L-z)}\Big[\sum_{v\in\mathcal{N}_{\tilde{t}-\ell}} \mathbb{1}_{F_{L,t}(v)^c} + \mathbb{1}_{F_{L,t}(v)\setminus G_{L,t}(v)}\mathbb{1}_{\mathfrak{T}(v)}\Big] = o_u(\mathfrak{M}_{L,z})\,,
\end{aligned}$$

where the last step follows from Proposition 5.1. Thus, we have

$$\Upsilon = (1+o_u(1))\hat{\Upsilon} + o_u(\mathfrak{M}_{L,z})\,, \tag{5.6}$$

where

$$\hat{\Upsilon} := \mathbb{E}_{\theta(\sqrt{2}L-z)}\Big[1-\exp\Big(-\sum_{v\in\mathcal{N}_{\tilde{t}-\ell}}\psi_\theta(v)\mathbb{1}_{G_{L,t}(v)\cap\mathfrak{T}(v)}\Big)\Big]\,.$$

Let $\mathfrak{E}_{\tilde{t}-\ell}$ denote the event that $\mathbb{1}_{\{G_{L,t}(v)\cap\mathfrak{T}(v)\}}=1$ for at most one $v\in\mathcal{N}_{\tilde{t}-\ell}$. Using the identity $1-\exp(\sum_{i=1}^n x_i)=\sum_{i=1}^n(1-e^{x_i})$ when at most one of the x_i is nonzero, as well as the trivial bound $e^{-x}\le 1$, we find

$$\begin{aligned}\hat{\Upsilon} &\le \mathbb{E}_{\theta(\sqrt{2}L-z)}\Big[\sum_{v\in\mathcal{N}_{\tilde{t}-\ell}}\big(1-e^{-\psi_\theta(v)\mathbb{1}_{G_{L,t}(v)\cap\mathfrak{T}(v)}}\big);\mathfrak{E}\Big]+\mathbb{P}_{\theta(\sqrt{2}L-z)}(\mathfrak{E}^c_{\tilde{t}-\ell})\\ &\le \mathbb{E}_{\theta(\sqrt{2}L-z)}\Big[\sum_{v\in\mathcal{N}_{\tilde{t}-\ell}}\big(1-e^{-\psi_\theta(v)\mathbb{1}_{G_{L,t}(v)\cap\mathfrak{T}(v)}}\Big]+o_u(\mathfrak{M}_{L,z})\end{aligned}$$

(the last line follows from Proposition 5.2), as well as

$$\begin{aligned}&\mathbb{E}_{\theta(\sqrt{2}L-z)}\Big[\sum_{v\in\mathcal{N}_{\tilde{t}-\ell}}\big(1-e^{-\psi_\theta(v)\mathbb{1}_{G_{L,t}(v)\cap\mathfrak{T}(v)}}\big)\Big]\\ &\le \hat{\Upsilon}+\mathbb{E}_{\theta(\sqrt{2}L-z)}\Big[\sum_{v\in\mathcal{N}_{\tilde{t}-\ell}}\big(1-e^{-\psi_\theta(v)\mathbb{1}_{G_{L,t}(v)\cap\mathfrak{T}(v)}}\big);\mathfrak{E}^c_{\tilde{t}-\ell}\Big]\\ &\le \hat{\Upsilon}+2\mathbb{E}_{\theta(\sqrt{2}L-z)}\Big[\sum_{v\neq w\in\mathcal{N}_{\tilde{t}-\ell}}\mathbb{1}_{\{G_{L,t}(v)\cap\mathfrak{T}(v)\cap G_{L,t}(w)\cap\mathfrak{T}(w)\}}\Big]=\hat{\Upsilon}+o_u(\mathfrak{M}_{L,z})\end{aligned}$$

(again, the last equality follows from Proposition 5.2). Equation (5.6) and the above inequalities then yield

$$\Upsilon=(1+o_u(1))\mathbb{E}_{\theta(\sqrt{2}L-z)}\Big[\sum_{v\in\mathcal{N}_{\tilde{t}-\ell}}\big(1-e^{-\psi_\theta(v)\mathbb{1}_{G_{L,t}(v)\cap\mathfrak{T}(v)}}\big)\Big]+o_u(\mathfrak{M}_{L,z})\,.$$

Note that the quantity inside of the expectation depends only on the *norms* of our BBM particles. Further, since $\overline{\phi}_\theta$ is supported on $\mathbb{R}_{\ge 0}$, it follows that $\psi_\theta(v)$ is nonzero only on the event $\mathfrak{T}(v)$. Thus, the proof of Proposition 4.1 will be complete if we can show the following asymptotic, uniformly over θ:

$$\mathbb{E}_{\sqrt{2}L-z}\Big[\sum_{v\in\mathcal{N}_{\tilde{t}-\ell}}\big(1-e^{-\psi_\theta(v)\mathbb{1}_{G_{L,t}(v)}}\big)\Big]\underset{(\mathrm{u})}{\sim}C_d(\phi_\theta)\mathfrak{M}_{L,z} \tag{5.7}$$

With the identity $1-e^{x\mathbb{1}_A}=(1-e^x)\mathbb{1}_A$ and the many-to-one lemma (Lemma 2.2), the left-hand side of (5.7) simplifies as

$$e^{\tilde{t}-\ell}\mathbb{E}_{\sqrt{2}L-z}\Big[\big(1-e^{-\psi_\theta(v)}\big)\mathbb{1}_{G_{L,t}(v)}\Big]\,.$$

We can expand this expectation by applying the Girsanov transform (given by [11, Equation 2.7]) to convert the Bessel process $(R_s^{(v)})_{[0,\tilde{t}-\ell]}$ to a Brownian motion $(W_s^{(v)})_{[0,\tilde{t}-\ell]}$, and then integrating over the (Brownian) transition density $p^{W^{(v)}_\cdot}_{\tilde{t}-\ell}$. Letting $\mathbf{x}(z):=\sqrt{2}L-z$, this gives the previous display as

$$\begin{aligned}&e^{\tilde{t}-\ell}\mathbb{E}_{\sqrt{2}L-z}\left[\left(\frac{W^{(v)}_{\tilde{t}-\ell}}{\mathbf{x}(z)}\right)^{\alpha_d}\exp\left(\int_0^{\tilde{t}-\ell}\frac{\alpha_d-\alpha_d^2}{{W_s^{(v)}}^2}\mathrm{d}s\right)\big(1-e^{-\psi_\theta(v)}\big)\mathbb{1}_{G_{L,t}(v)}\right]\\ &\underset{(\mathrm{u})}{\sim}e^{\tilde{t}-\ell}\int_{\ell^{1/3}}^{\ell^{2/3}}p^{W^{(v)}}_{\tilde{t}-\ell}\big(\mathbf{x}(z),\mathbf{y}(w)\big)\left(\frac{\mathbf{y}(w)}{\mathbf{x}(z)}\right)^{\alpha_d}\mathbb{P}^{\mathbf{y}(w)}_{\mathbf{x}(z),\tilde{t}-\ell}\Big(\mathcal{B}^{\hat{\ }}_{[0,\tilde{t}-\ell]}(W^{(v)}_\cdot)\Big)\,\mathbb{E}_{\mathbf{y}(w)}[\Psi_{\theta,\ell}]\mathrm{d}w\,,\end{aligned}$$

where

$$\Psi_{\theta,\ell}:=1-\exp(\langle\mathcal{E}_\ell,\overline{\phi}_\theta(\cdot-y)\rangle)\,.$$

In the last line, we have used the fact that on the event $G_{L,t}(v)$ (or, more specifically, on $\mathcal{B}^{\diamond}_{[0,\tilde{t}-\ell]}(W^{(v)}_{\cdot})$), we have

$$\exp\left(\int_0^{\tilde{t}-\ell} \frac{\alpha_d - \alpha_d^2}{{W_s^{(v)}}^2} \mathrm{d}s\right) \underset{(\mathrm{u})}{\sim} 1\,.$$

We have also used the Markov property at time $\tilde{t}-\ell$. Now, from equations (6.53), (6.54) and (6.60) of [11], the last display is asymptotically equivalent (in the sense of $\sim_{(\mathrm{u})}$) to

$$\sqrt{\frac{2^{1+\alpha_d}}{\pi}} e^{-y\sqrt{2}} \mathfrak{M}_{L,z} \int_{\ell^{1/3}}^{\ell^{2/3}} w e^{w\sqrt{2}} \mathbb{E}_{\mathbf{y}(w)}[\Psi_{\theta,\ell}] \mathrm{d}w\,.$$

Corollary 5.4 and Proposition 2.7 tell us that

$$\int_{\ell^{1/3}}^{\ell^{2/3}} w e^{w\sqrt{2}} \mathbb{E}_{\mathbf{y}(w)}[\Psi_{\theta,\ell}] \mathrm{d}w \underset{(\mathrm{u})}{\sim} e^{\sqrt{2}y} C(\phi_\theta)\,,$$

where $C(\phi_\theta)$ is defined in (2.13). This concludes the proof. □

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