ON LEVEL LINE FLUCTUATIONS OF SOS SURFACES ABOVE A WALL

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Abstract. We study the low temperature $(2+1)$D Solid-On-Solid model on $[\![1,L]\!]^2$ with zero boundary conditions and non-negative heights (a floor at height 0). Caputo et al. (2016) established that this random surface typically admits either $\mathfrak{h}$ or $\mathfrak{h}+1$ many nested macroscopic level line loops $\{\mathcal{L}_i\}_{i\geq 0}$ for an explicit $\mathfrak{h} \asymp \log L$, and its top loop $\mathcal{L}_0$ has cube-root fluctuations: e.g., if $\rho(x)$ is the vertical displacement of $\mathcal{L}_0$ from the bottom boundary point $(x,0)$, then $\max \rho(x) = L^{1/3+o(1)}$ over $x \in I_0 := L/2 + [\![-L^{2/3}, L^{2/3}]\!]$. It is believed that rescaling ρ by $L^{1/3}$ and I_0 by $L^{2/3}$ would yield a limit law of a diffusion on $[-1,1]$. However, no nontrivial lower bound was known on $\rho(x)$ for a fixed $x \in I_0$ (e.g., $x = \frac{L}{2}$), let alone on $\min \rho(x)$ in I_0, to complement the bound on $\max \rho(x)$. Here we show a lower bound of the predicted order $L^{1/3}$: for every $\epsilon > 0$ there exists $\delta > 0$ such that $\min_{x\in I_0} \rho(x) \geq \delta L^{1/3}$ with probability at least $1-\epsilon$. The proof relies on the Ornstein–Zernike machinery due to Campanino–Ioffe–Velenik, and a result of Ioffe, Shlosman and Toninelli (2015) that rules out pinning in Ising polymers with modified interactions along the boundary. En route, we refine the latter result into a Brownian excursion limit law, which may be of independent interest. We further show that in a $KL^{2/3} \times KL^{2/3}$ box with boundary conditions $\mathfrak{h}-1, \mathfrak{h}, \mathfrak{h}, \mathfrak{h}$ (i.e., $\mathfrak{h}-1$ on the bottom side and $\mathfrak{h}$ elsewhere), the limit of $\rho(x)$ as $K, L \to \infty$ is a Ferrari–Spohn diffusion.

1. Introduction

We consider the Solid-On-Solid (SOS) model on $\Lambda_L = [\![1,L]\!]^2$, an $L \times L$ square in $\mathbb{Z}^2$, at large inverse-temperature $\beta > 0$, with zero boundary conditions and a floor at height 0: denoting by $\mathsf{x} \sim \mathsf{y}$ a pair of adjacent sites $\mathsf{x}, \mathsf{y} \in \mathbb{Z}^2$, and setting $\varphi_\mathsf{x} = 0$ for all $\mathsf{x} \notin \Lambda_L$, the model assigns a height function $\varphi : \Lambda_L \to \mathbb{Z}_{\geq 0}$ (taking nonnegative integer heights) the probability

$$\pi^0_{\Lambda_L}(\varphi) \propto \exp\Big(-\beta \sum_{\mathsf{x}\sim\mathsf{y}} |\varphi_\mathsf{x} - \varphi_\mathsf{y}| \Big) . \tag{1.1}$$

The model was introduced in the early 1950's (see [5, 39]) to approximate the formation of crystals and the interface separating the plus and minus phases in the low temperature 3D Ising model.

While of interest in any dimension d, the study of the model on $\mathbb{Z}^2$ has special importance, as it is the only dimension associated with the roughening phase transition. For the low temperature 3D Ising model, which the $(2+1)$D SOS model approximates for large β, rigorously establishing the roughening phase transition is a tantalizing open problem which has seen very little progress since being observed some 50 years ago (numerical experiments suggest it takes place at $\beta_\mathrm{R} \approx 0.408$, compared to the critical 3D Ising temperature $\beta_c \approx 0.221$). The corresponding phase transition for the $(2+1)$D SOS *with no floor* $\widehat{\pi}$ (where φ can be negative) was rigorously confirmed as follows: (i) (*localization*) for β large enough, the surface is rigid, in that $\mathrm{Var}(\varphi_\mathsf{x}) = O(1)$ at x in the bulk, and furthermore $|\varphi_\mathsf{x}|$ has an exponential tail [3]; (ii) (*delocalization*) for β small enough, Fröhlich and Spencer [26, 27] famously showed that $\mathrm{Var}(\varphi_\mathsf{x}) \asymp \log L$, just as in the case where φ takes values in $\mathbb{R}$. (iii) Very recently, Lammers [37] showed the phase transition in $\mathrm{Var}(\varphi_\mathsf{x})$ is sharp: there exists $\beta_\mathrm{R} > 0$ such that $\mathrm{Var}(\varphi_\mathsf{x}) \to \infty$ for all $\beta \leq \beta_\mathrm{R}$ whereas it is $O(1)$ for all $\beta > \beta_\mathrm{R}$; numerical experiments suggest that $\beta_\mathrm{R} \approx 0.806$. (See [37] for additional details on the recent developments in the SOS model with no floor and related models of integer-valued height functions.)

Our setting is the low-temperature regime (β large), yet with the restriction that the surface must lie above above a hard wall (the assumption $\varphi \geq 0$). Bricmont, El-Mellouki and Fröhlich [4] showed that this induces *entropic repulsion*, regarded as a key feature of the physics of random surfaces: the restriction $\varphi \geq 0$ propels the surface (despite the energy cost) so as to gain entropy. Namely, it was shown in [4] that $\frac{c}{\beta}\log L \leq \mathbb{E}[\varphi_\mathsf{x} \mid \varphi \geq 0] \leq \frac{C}{\beta}\log L$ for absolute constants $c, C > 0$.

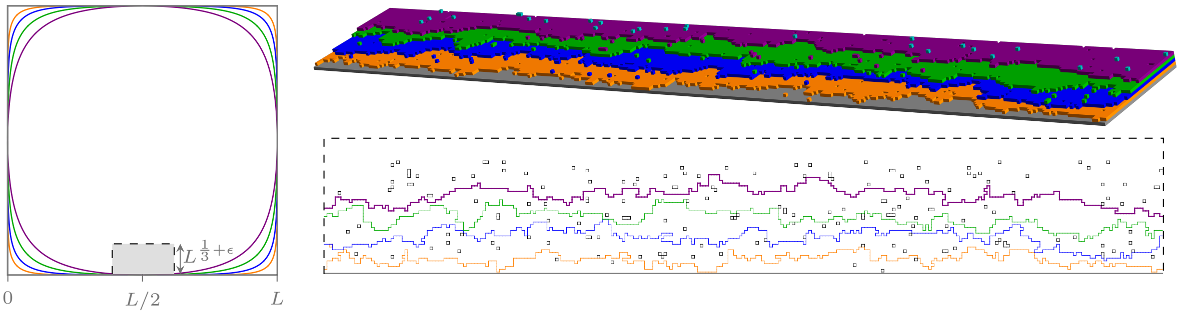

FIGURE 1. Fluctuations of the SOS level lines about the flat portions of their scaling limits. Maximal fluctuation is known to be at most $L^{1/3+\epsilon}$ w.h.p., and it is believed that the distance of the top level line from a given boundary point (e.g., the center-side) is of order $L^{1/3}$.

The gap between these bounds was closed in [13], where it was established that $\mathbb{E}[\varphi_{\mathsf{x}} \mid \varphi \geq 0]$ is $\frac{1}{4\beta}\log L + O(1)$, and moreover, $(1-\epsilon_\beta)L^2$ sites are at such a height with high probability (w.h.p.). The following intuition explains the height asymptotics: if the surface lies rigid about height h, then the cost of raising every site by 1 is $4\beta L$ (incurred at the sites along the boundary); the benefit in doing so would be to gain the ability to feature spikes of depth $h+1$ (forbidden at level h due to the restriction $\varphi \geq 0$), and as such a spike has an energetic cost of about $e^{-4\beta h}$, the entropy gain is about $\mathsf{H}(e^{-4\beta h})L^2$ where $\mathsf{H}(\cdot)$ is the Shannon entropy; the two terms are equated at $h \sim \frac{1}{4\beta}\log L$.

Significant progress in the understanding of the shape of the SOS surface above a hard wall was obtained in the sequel by the same authors [14]. The height-h *level lines* of the surface are the loops formed by placing dual-bonds between every pair $\mathsf{x} \sim \mathsf{y}$ such that $\varphi_{\mathsf{x}} < h$ and $\varphi_{\mathsf{y}} \geq h$. To account for local thermal fluctuations, call a loop *macroscopic* if its length is at least $(\log L)^2$. With this notation, (a more detailed version of) the following theorem was given in [14] (see also [12]):

Theorem ([14, Thms. 1,2,3 and Rem. 1.3])**.** *For β large enough, the $(2+1)D$ SOS model with zero boundary conditions on a square $\Lambda_L = [\![1, L]\!]^2$ above the wall $\varphi \geq 0$, satisfies the following w.h.p.:*

(i) Shape: At least $(1-\epsilon_\beta)L^2$ of the sites $\mathsf{x} \in \Lambda_L$ have height $\varphi_{\mathsf{x}} = \mathfrak{h}^\star$, where the random $\mathfrak{h}^\star$ is either $\lfloor \frac{1}{4\beta}\log L\rfloor$ or $\lfloor \frac{1}{4\beta}\log L\rfloor - 1$. Moreover, there is a unique macroscopic loop at each height $0, 1, \ldots, \mathfrak{h}^\star$, and none above height $\mathfrak{h}^\star$. Further, for a diverging sequence[1] of L's, the sequence of nested loops $\mathcal{L}_0 \subset \mathcal{L}_1 \subset \ldots$, when rescaled to $[0,1]^2$, converges in probability in Hausdorff distance to a deterministic limit defined by a Wulff shape $\mathcal{W}$, which is the convex body of area 1 minimizing the line integral of a surface tension $\tau_\beta(\cdot)$ along its perimeter $\partial\mathcal{W}$; the scaling limit of $\mathcal{L}_k$ (where $\mathcal{L}_0$ is the top level line at height $\mathfrak{h}^\star$) is given by the union of all possible translates of $\mathcal{W}$, rescaled by an explicit radius r_k that is decreasing in k.

(ii) Fluctuations: For a diverging sequence[2] of L's, the maximum displacement of the top level line $\mathcal{L}_0$ from the boundary segment $I \times \{0\}$ for $I = [\![\epsilon_\beta L, (1-\epsilon_\beta)L]\!]$ is $L^{1/3+o(1)}$. That is, if

$$\overline{\rho}(x) = \max\{y \leq L/2 \,:\, (x,y) \in \mathcal{L}_0\}$$

is the maximum y-coordinate of a point (x,y) visited by $\mathcal{L}_0$ in the bottom-half of Λ_L, then

$$\max_{x\in I} \overline{\rho}(x) \leq L^{1/3+\epsilon}\,, \tag{1.2}$$

for any fixed $\epsilon > 0$, whereas for every interval $I' \subset I$ of length $L^{2/3-\epsilon}$,

$$\max_{x\in I'} \overline{\rho}(x) \geq L^{1/3-\epsilon}\,. \tag{1.3}$$

Consider the distance of the top level line loop $\mathcal{L}_0$ from a point $(x_0, 0)$ on the bottom boundary of the box, where the scaling limit is flat—e.g., the center-side $x_0 = L/2$ (see Fig. 1 for a depiction).

[1] Namely, for any sequence of L's where a_L, the fractional part of $\frac{1}{4\beta}\log L$, does not converge to an explicit $\lambda_c(\beta)$.

[2] Namely, for any sequence of L's such that, for the above a_L's and $\lambda_c(\beta)$, one has $\liminf_{L\to\infty} a_L > \lambda_c$.

The above theorem shows that $\overline{\rho}(x_0) \le L^{1/3+\epsilon}$ w.h.p., yet it gives no nontrivial lower bound on it. It is believed that $\overline{\rho}(x_0)$ should have order $L^{1/3}$ (with no poly-log corrections); more precisely, one expects $\overline{\rho}(x_0) \asymp_{\mathtt{P}} L^{1/3}$, where we write $f \lesssim_{\mathtt{P}} g$ if f/g is uniformly tight, and $f \asymp_{\mathtt{P}} g$ if $f \lesssim_{\mathtt{P}} g \lesssim_{\mathtt{P}} f$.

Moreover, one expects that if one were to rescale $\overline{\rho}(x)$ by $L^{1/3}$ along an interval of order $L^{2/3}$ positioned on bottom boundary (within the flat portion of the scaling limit)—take, e.g.,

$$I_0 := [\![\tfrac{L}{2} - L^{2/3}, \tfrac{L}{2} + L^{2/3}]\!]$$

for concreteness—then, after rescaling said interval by $L^{2/3}$ (in the concrete example, to $[-1, 1]$), one would arrive at a limit law of a nontrivial diffusion, a variant of a Ferrari–Spohn diffusion [25]. (This prediction was stated here in terms of $\overline{\rho}(x)$, the maximal vertical displacement of $\mathcal{L}_0$, so as to be well-defined, as $\mathcal{L}_0$ can have many points with a given x-coordinate; the same statement is expected to hold for $\underline{\rho}(x)$ measuring the *minimal* displacement of $\mathcal{L}_0$, as defined in Theorem 1.1.)

To explain this prediction, note first that it is well-known that a Brownian excursion tilted (penalized) by $\exp(-\lambda A)$, where A is the area under it, has the law of a Ferrari–Spohn diffusion. The entropic repulsion that propels the SOS level line loop $\mathcal{L}$ to height h acts much like an area tilt: if a loop has internal area S, then its probability (roughly) gains a tilt of $\exp(\lambda \mathsf{S})$ for $\lambda = \mathsf{H}(e^{-4\beta h})$ as described earlier, or equivalently, a tilt of $\exp(-\lambda A)$ where $A = L^2 - \mathsf{S}$ is the area exterior to it. Consider $\mathcal{L}_k$, which is at height $\mathfrak{h}^\star - k$: there $\lambda \approx e^{-4\beta(\mathfrak{h}^\star - k)} \approx L^{-1} e^{4\beta k}$ (recall $\mathfrak{h}^\star \approx \frac{1}{4\beta}\log L$), and we see that the rescaling of $\rho(x)$ by $L^{1/3}$ and I_0 by $L^{2/3}$ cancels the L^{-1} factor in λ and translates into a tilt of $\exp(-e^{4\beta k}\hat{A})$ where $\hat{A}$ is the rescaled area, as in the above continuous approximation (see also [33, 35]). Related to this, the famous problem of establishing a Ferrari–Spohn law for the 2D Ising interface under critical prewetting (which may be seen as a version of the SOS problem only with a single contour as opposed to $c \log L$ many) was finally settled in a recent seminal work by Ioffe, Ott, Shlosman and Velenik [30] (prior to that, the $L^{1/3+o(1)}$ fluctuations were established by Velenik [40] and the tightness of the rescaled area was proved by Ganguly and Gheissari [28]). The challenges in handling a diverging number of interacting (non-crossing) contours with distinct area tilts (the k-th one is tilted by $\approx \exp(-e^{4\beta k}\hat{A})$) are such that the simplified problem that has Brownian excursions with area tilts is already nontrivial; see [9, 10, 11, 16] for recent progress on it.

In accordance with this prediction for the scaling limit of $L^{-1/3}\rho(x)$ along I_0, one expects that both $\max_{x\in I_0} \rho(x)$ and $\min_{x \in I_0} \rho(x)$ would be of the same order as our rescaling factor $L^{1/3}$; i.e., to be precise, that $\max \overline{\rho}(x) \lesssim_{\mathtt{P}} L^{1/3}$ and that $\min \underline{\rho}(x) \gtrsim_{\mathtt{P}} L^{1/3}$. Our main result is the latter part (readily implying $\rho(x_0) \gtrsim_{\mathtt{P}} L^{1/3}$ at any given $\epsilon_\beta L \le x_0 \le (1-\epsilon_\beta)L$, e.g., the center-side $x_0 = L/2$).

Theorem 1.1. *Fix β large, and consider the $(2+1)D$ SOS model with zero boundary conditions on $\Lambda_L = [\![1, L]\!]^2$ as per Eq. (1.1) above a wall $\varphi \ge 0$. Let $\mathcal{L}_0$ be the (w.h.p. unique) top macroscopic level line, consider the interval $I_0 = [\![\frac{L}{2} - L^{2/3}, \frac{L}{2} + L^{2/3}]\!]$ centered on the bottom boundary, and let*

$$\underline{\rho}(x) = \min\{y \ge 0 \,:\, (x, y) \in \mathcal{L}_0\}$$

denote the minimum vertical displacement of $\mathcal{L}_0$ from the bottom boundary at the coordinate x. Then for every $\epsilon > 0$ there exists $\delta > 0$ such that for large enough L, with probability at least $1-\epsilon$,

$$\min_{x \in I_0} \underline{\rho}(x) \ge \delta\, L^{1/3}\,. \tag{1.4}$$

As we later explain, we obtain Eq. (1.4) by moving from $\rho(x)$, at a constant probability cost, to a curve whose limit (after the same rescaling) is a Brownian excursion, yielding the $L^{1/3}$ bound. This refines the lower bound in Eq. (1.3) into the estimate $\max_{x\in I_0} \overline{\rho}(x) \ge \min_{x\in I_0} \underline{\rho}(x) \gtrsim_{\mathtt{P}} L^{1/3}$. Note though that one cannot replace the $L^{1/3+o(1)}$ in the upper bound of Eq. (1.2) by $O(L^{1/3})$, as it addresses $\max \overline{\rho}(x)$ over all $I = [\![\epsilon_\beta L, (1-\epsilon_\beta)L]\!]$. Our comparison to a Brownian excursion implies that $\max_{x \in I} \underline{\rho}(x) \ge c L^{1/3}\sqrt{\log L}$ w.h.p.; as we later explain, Theorem 1.3 will imply that $\max_{x\in I} \underline{\rho}(x) \gtrsim_{\mathtt{P}} L^{1/3}(\log L)^{2/3}$, its predicted order (see Remark 1.6 as well as Question 1.7).

To derive the Brownian excursion law, we rely on the powerful Ornstein–Zernike framework as developed by Campanino, Ioffe and Velenik [29, 6, 7, 8], that allows one to couple the interface in hand to a directed random walk. This machinery was the key to several recent advances in the understanding 2D Ising interfaces (e.g., [30] mentioned earlier) and Potts interfaces (e.g., [38, 31]). In fact, the work [31], due to Ioffe, Ott, Velenik and Wachtel, is of particular interest in our setting: there it was shown that the interface of the 2D Potts model in a box with Dobrushin's boundary conditions has the scaling limit of a Brownian excursion for all $\beta > \beta_c$. As in our case, one of the main obstacles is the interaction of the interface with the boundary, and in particular, ruling out the scenario whereby the interface is pinned to the wall. This was achieved for the Potts interface in [31] (and later used as an ingredient in [30]) via a direct analysis of its random cluster counterpart, and then combined with a version of Ornstein–Zernike theory tailored to that model.

Here we instead appeal to the framework of Ioffe, Shlosman and Toninelli [32] to rule out pinning. That approach, while valid only for large enough β (whereas the analysis in [30] holds for all $\beta > \beta_c$), is fairly generic, and applicable to SOS contours as part of the following family of *Ising polymers* (to aid the exposition, we describe it briefly here, deferring its full definition to Sections 2.2 and 2.3). Call a path γ of distinct adjacent edges in $(\mathbb{Z}^2)^* = \mathbb{Z}^2 + (\frac12, \frac12)$ (vertices may repeat according to a splitting rule) a *polymer*, or contour, if it connects the origin $\mathsf{o}^* = (\frac12, \frac12)$ to a marked x_N at distance N from o^* while staying in a half-plane $\mathbb{H}_{\vec{n}}$. The model gives γ a probability proportional to

$$q(\gamma) := \exp\Big(- \beta|\gamma| + \sum_{\mathcal{C}} \Phi(\mathcal{C};\gamma)\Big)\,,$$

where the sum goes over every finite connected subset $\mathcal{C}$ in $\mathbb{Z}^2$ that intersects Δ_γ, the vertex boundary of γ, and the potential function Φ satisfies the following properties: (**P1**) $\Phi(\mathcal{C};\cdot)$ is *local*, in the sense that it only depends on γ through $\mathcal{C}\cap\Delta_\gamma$; (**P2**) $\sup_\gamma |\Phi(\mathcal{C},\gamma)|$ *decays exponentially* in the size of $\mathcal{C}$ (more precisely, in the minimum size of a graph connecting its boundary edges); and (**P3**) Φ is invariant under translations of the form $(\mathcal{C},\gamma)\mapsto(\mathcal{C}+\mathsf{v},\gamma+\mathsf{v})$. The final requirement in [32] is to have that (**P4**) the *surface tension is symmetric*: if one defines the surface tension as

$$\tau_\beta(\vec{n}) := -\lim_{N\to\infty}\frac{1}{N}\log\Big(\sum_\gamma q(\gamma)\Big)\,,$$

then the function $\vec{n}\mapsto\tau_\beta(\vec{n})$ should have all discrete symmetries of $\mathbb{Z}^2$. Under these conditions, the main result of [32] was that modifying the potential function Φ into Φ' along $\partial\mathbb{H}_{\vec{n}}$ does not affect the surface tension. That is, if we let $\Phi'(\mathcal{C},\gamma) = \Phi(\mathcal{C},\gamma)$ whenever $\mathcal{C}$ is fully contained in $\mathbb{H}_{\vec{n}}$, and the modified Φ' still obeys the decay condition in Property (P2), then the modified τ'_β agrees with the original τ_β. Moreover, the corresponding partition functions are comparable (see Theorem 5.1).

The main ingredient in our proof of Theorem 1.1 is the following result, which establishes a Brownian excursion limit law for (a) Ising polymers as defined by [32] in the positive half-plane $\mathbb{H}$, and (b) Ising polymers in a box of side length N. Our proof of Part (a) hinges on the "no pinning" main result of [32] (mentioned above) en route to refining its conclusion and deriving the limit law. Part (b), proved similarly, may be viewed as an analogue of [31] for any Ising polymer at large β.

Theorem 1.2. *Fix β large, and consider the family of Ising polymers γ (see Definitions 2.3 and 2.6) in a domain D, where the potential function Φ' is modified along its boundary ∂D, and D is either*

(a) the positive half-plane $\mathbb{H}$ with the marked end-points $\mathsf{o}^* = (\frac12,\frac12)$ and $\mathsf{x}_N = (\frac12, N-\frac12)$; or

(b) a box of side length N whose bottom corners are the same marked end-points o^* and x_N.

There exists $\sigma > 0$ such that, if $\overline{\gamma}(x) = \max\{y : (x,y)\in\gamma\}$, then $\overline{\gamma}(\lfloor xN\rfloor)/(\sigma\sqrt{N})$ converges weakly to a standard Brownian excursion on $[0,1]$, and the same holds for $\underline{\gamma}(x) = \min\{y : (x,y)\in\gamma\}$.

In particular, Part (b) applies to the SOS model $\widehat{\pi}_\Lambda^{0,1,1,1}$ with no floor *on a box Λ of side length N, for $\beta > \beta_0$ and boundary conditions 0 on the bottom side and 1 elsewhere: namely, the height-1 level line that connects the bottom corners of the box Λ has a scaling limit of a Brownian excursion.*

While Theorem 1.2 addressed level lines in the SOS model (and more generally, Ising polymers) with *no floor*—whereby the scaling limit is a Brownian excursion—its application for Theorem 1.1 (addressing SOS above a floor) used the fact that in that setting the effect of the floor is uniformly bounded. Indeed, in an $L^{2/3} \times L^{2/3}$ box centered on the bottom boundary, the tilting effect of the floor on the top level line (as a Radon–Nikodym derivative) amounts to a factor of $\exp[cA/L]$, where A is the area under the non-tilted curve (note $A \lesssim_{\mathsf{P}} L$ for a Brownian excursion on an interval of length $L^{2/3}$). Since, as mentioned above, a Brownian excursion tilted by an area term is known to converge to a Ferrari–Spohn diffusion, one expects that the top level line of SOS in that box will actually dominate a Ferrari–Spohn diffusion. This is the content of Theorem 1.3 below.

We first define the limiting object formally. Let $\mathsf{Ai}(x)$ denote the Airy function (of the first kind), i.e., the solution to $y''(x) = xy$ with the initial condition $y = 0$ at $x = \infty$. For $\lambda, \sigma > 0$, define $f_{\lambda,\sigma}(x) := \mathsf{Ai}((2\lambda\sigma)^{1/3}x + \omega_1)$, where ω_1 is the "first" zero of Ai ($\omega_1 < 0$ and closest to 0).[3] The stationary Ferrari–Spohn diffusion we consider is the diffusion on $(0, \infty)$ with generator

$$\mathsf{L}\psi = \frac{1}{2}\psi'' + \frac{f'_{\lambda,\sigma}}{f_{\lambda,\sigma}}\psi' \tag{1.5}$$

and Dirichlet boundary condition at 0.

The following result establishes that if we consider the SOS model on a $KL^{2/3} \times KL^{2/3}$ box with boundary conditions $H-1, H, H, H$, where H is the typical height of the top level line (up to 1 integer), then the H-level line will converge weakly to a Ferrari–Spohn diffusion in $(C[-T,T], \|\cdot\|_\infty)$ for any $T > 0$. A direct consequence (via the monotonicity argument in Section 3) is a refinement of Theorem 1.1, showing that $\rho(x)$ from that theorem essentially dominates a Ferrari–Spohn diffusion (thus $\rho(x) \gtrsim_{\mathsf{P}} L^{1/3}$; see Remark 1.5).

Theorem 1.3. *Fix β large and consider the SOS model on a $KL^{2/3} \times KL^{2/3}$ box with a floor at 0 and boundary conditions $H = \lfloor \frac{1}{4\beta} \log L \rfloor$ everywhere except the bottom side, where they are $H - 1$. Suppose that a_L, the fractional part of $\frac{1}{4\beta} \log L$, converges to a limit, and let $\overline{\rho}(x)$ denote the maximum vertical distance of the H-level line (connecting the bottom corners of the box) from the bottom side at horizontal location $x \in \mathbb{R}$. Let $\sigma > 0$ be the constant from Theorem 1.2, Part (b). Then there exists $\lambda > 0$ such that $\overline{\rho}(\lfloor xL^{2/3} \rfloor)/(\sigma L^{1/3})$ converges weakly as $L \to \infty$ followed by $K \to \infty$ to the stationary Ferrari–Spohn diffusion on $(0, \infty)$ with generator L and Dirichlet boundary condition at 0. The same holds for $\underline{\rho}(x)$, the minimum height fluctuation of the level line at $x \in \mathbb{R}$.*

Remark 1.4. Using the same methods, a Ferrari–Spohn diffusion limit may also be derived for Ising polymers with the appropriate area tilt. Furthermore, it is possible to take any diverging sequence of $K := K(L) \in (0, L^{1/20})$. See Theorem 7.1 for a more detailed version of Theorem 1.3.

Remark 1.5. As mentioned above, Theorem 1.2 is proved via coupling the Ising polymer to a 2D directed random walk excursion. The proof of Theorem 1.3, taking into account the floor in the SOS model, proceeds by using the same machinery to couple the polymer to a random walk excursion, yet this time with an area-tilt. We then appeal to the approach of [30, Section 6] for handling the convergence of such 2D random walks to the Ferrari–Spohn diffusion. As a byproduct of this argument, one can read off quantitative results on the model before taking $K, L \to \infty$; namely, the top level line of the SOS model with zero boundary conditions on Λ_L above a wall dominates a random walk excursion with endpoints 0 and an area tilt on any interval of length $L^{2/3}$ at distance at least $\epsilon_\beta L$ from the box corners (a stronger result than Theorem 1.1).

Remark 1.6. Consider the aforementioned stronger version of Theorem 1.1, whereby the vertical displacement $\underline{\rho}(x)$ of the top level line $\mathcal{L}_0$ of the SOS model, along any interval of length $L^{2/3}$ bounded away from the corners, stochastically dominates a random walk with area tilt $e^{-cA/L}$.

[3]The function $f_{\lambda,\sigma}$ is the first eigenfunction of the operator L, see [33, Equation 1.18]

Standard tools will then imply that $\max_{x\in I} \underline{\rho}(x)$ is $\Omega(L^{1/3}(\log L)^{2/3})$ for $I = [\![\epsilon_\beta L, (1-\epsilon_\beta)L]\!]$. Namely, one could show, for some absolute constant $c > 0$, a lower bound on the upper tail of $\underline{\rho}(x_0)$, valid for all $x_0 \in I$, à la Tracy–Widom distribution:

$$\pi^0_{\Lambda_L}(\underline{\rho}(x_0) > aL^{1/3}) \geq e^{-ca^{3/2}} \,.$$

Considering about $L^{1/3}$ such x_0 taken in disjoint boxes of length $L^{2/3}$ each, along with monotonicity arguments similar to those employed in Section 3 will then imply $\max_{x\in I} \underline{\rho}(x) \gtrsim_{\mathsf{P}} L^{1/3}(\log L)^{2/3}$.

It is plausible that this gives the correct order of the upper tail large deviation rate function, and that consequently, $\max_x \underline{\rho}(x) \asymp_{\mathsf{P}} L^{1/3}(\log L)^{2/3}$ (see, e.g., [28], where an estimate of this type was obtained for the 2D Ising interface in critical prewetting, as well the work of Alexander [1] on local roughness of droplet boundaries in the random cluster model).

Question 1.7. Let $x_0 \in [\![\epsilon_\beta L, (1-\epsilon_\beta)L]\!]$. What is the rate function $a \mapsto -\log \pi^0_{\Lambda_L}(\underline{\rho}(x_0) > aL^{1/3})$?

As for lower tails, to our knowledge these are still open for the 2D Ising under critical prewetting, where one expects the Ising interface to reach the bottom in $O_{\mathsf{P}}(1)$ locations along $[\![\epsilon_\beta L, (1-\epsilon_\beta)L]\!]$. One should stress though that it is unclear that the SOS large deviations would take after the behavior of the 2D Ising interface under prewetting—particularly for the lower tails, where in SOS there are $\Theta(\log L)$ level lines below $\mathcal{L}_0$, all of which must cooperate with a downward deviation.

Question 1.8. Let $x_0 \in [\![\epsilon_\beta L, (1-\epsilon_\beta)L]\!]$. What is the order of $\pi^0_{\Lambda_L}(\underline{\rho}(x_0) = 0)$?

The paper is organized as follows. In Section 2, we formalize the setting of Ising polymers, as well as the inputs we need from Ornstein–Zernike theory. We also establish that the SOS model satisfies the required hypotheses of Ising polymers, and thus Theorem 1.2 is applicable to it. Section 3 proves Theorem 1.1, addressing the SOS measure π, using a monotonicity argument and the conclusion of Theorem 1.2 on $\widehat{\pi}$, the SOS measure without a floor. In Section 4, we introduce a random walk in $\mathbb{H}$ that is closely related to Ising polymers, and state the key limit theorems for this random walk. In turn, Section 5 provides the proof of Theorem 1.2 modulo these random walks results that are deferred to Section 6. In Section 7, we prove Theorem 1.3.

2. Cluster expansion, Ising polymers, and Ornstein–Zernike theory

In this section, we review the tools needed for our proofs—notably, cluster expansion, prior work on Ising polymers, and Ornstein–Zernike theory. In several cases, we will need variants of existing results, which are not covered by the results proved in the literature. In those cases, we provide proofs of these analogues (either in the main text or in the appendix).

Throughout the paper, we say that an event holds with high probability (w.h.p.) if its probability tends to 1 as the system size (typically, L or N) tends to ∞. For two functions $f : \mathbb{N} \to (0,\infty)$ and $g : \mathbb{N} \to (0,\infty)$, write $f \sim g$ to denote that $\lim_{N\to\infty} f(N)/g(N) = 1$; write $f \lesssim g$ when there exists a constant $K > 0$ such that $f(N) \leq Kg(N)$ for all $N \in \mathbb{N}$; and write $f \asymp g$ when $f \lesssim g$ and $g \lesssim f$.

2.1. **Contours and cluster expansion.** A contour γ is a collection of bonds $(e_i)_{i=1}^m$ in the dual lattice $(\mathbb{Z}^2)^*$, where all bonds are distinct except possibly e_1 and e_m may coincide, every two consecutive edges share a vertex, and the path formed is simple except in accordance with a splitting rule: if the pair e_i, e_{i+1} and e_j, e_{j+1} all intersect at a vertex x, then the two other end-points of e_i, e_{i+1} are on the same side of the line through x with slope 1 (from southwest to northeast), and similarly for e_j, e_{j+1} (this is the *northeast* splitting rule). We call γ an *open contour* if $e_1 \neq e_m$.

In the context of the SOS model in a finite, connected $\Lambda \Subset \mathbb{Z}^2$ under 0 boundary conditions, for any h, the h-level lines (recall that for any configuration φ and integer h, these are the bonds dual to $\mathsf{x} \sim \mathsf{y}$ with $\varphi_{\mathsf{x}} < h$ and $\varphi_{\mathsf{y}} \geq h$) give rise to a collection of disjoint loops after applying the global splitting rule. In the presence of a boundary condition $\xi \in \{0,1\}^{\partial\Lambda}$ consisting of a connected stretch of 0's (and 1's elsewhere), this gives rise to a unique open contour among the height-1 level lines (accompanied by a collection of closed contours). We refer to this path as the open 1-contour. Let Δ_γ^+ and Δ_γ^- denote the set of sites (of $\mathbb{Z}^2$) immediately above and below γ, respectively, and define

$$\Delta_\gamma := \Delta_\gamma^+ \cup \Delta_\gamma^- .$$

Note that the sites in Δ_γ^+ have height ≥ 1, while the sites in Δ_γ^- have height ≤ 0. Each γ divides Λ into two regions, Λ_γ^+ and Λ_γ^-, where we write Λ_γ^+ to denote the region that contains Δ_γ^+ as part of its inner boundary. The next proposition addresses the law of this unique SOS open contour γ.

Proposition 2.1 ([13, Lem. A.2])**.** *Consider the SOS model $\widehat{\pi}_\Lambda^\xi$ on any finite, connected $\Lambda \subset \mathbb{Z}^2$ and under any boundary condition $\xi \in \{0,1\}$ that induces a unique open 1-contour γ. Then there exists a constant $\beta_0 > 0$ such that for all $\beta \geq \beta_0$,*

$$\widehat{\pi}_\Lambda^\xi(\gamma) \propto \exp\Big(-\beta|\gamma| + \sum_{\mathcal{C}\cap\Delta_\gamma\neq\emptyset} \phi(\mathcal{C};\gamma)\mathbb{1}_{\{\mathcal{C}\subset\Lambda\}}\Big)\,, \tag{2.1}$$

for some "decoration functions" $\{\phi(\mathcal{C},\gamma)\}_{\mathcal{C}\subset\mathbb{Z}^2}$ satisfying the following properties:

(i) If $\mathcal{C}$ is not connected, then $\phi(\mathcal{C};\gamma)=0$.
(ii) The decoration function $\phi(\mathcal{C};\cdot)$ depends on γ only through $\mathcal{C}\cap\Delta_\gamma$.
(iii) For all $\mathsf{v}\in\mathbb{Z}^2$, $\phi(\mathcal{C};\gamma)=\phi(\mathcal{C}+\mathsf{v};\gamma+\mathsf{v})$.
(iv) Letting $d(\mathcal{C})$ denote the cardinality of the smallest connected set of bonds of $\mathbb{Z}^2$ containing all boundary bonds of $\mathcal{C}$ (i.e., bonds connecting $\mathcal{C}$ to $\mathcal{C}^c$), we have the decay bound

$$\sup_\gamma |\phi(\mathcal{C};\gamma)| \leq \exp\Big(-(\beta-\beta_0)d(\mathcal{C})\Big)\,. \tag{2.2}$$

Furthermore, defining the interface partition function

$$\mathcal{G}_\Lambda^\xi := \sum_\gamma e^{-\beta|\gamma|+\sum_{\mathcal{C}\subset\Lambda}\phi(\mathcal{C};\gamma)}\,,$$

we have

$$\mathcal{G}_\Lambda^\xi = \widehat{Z}_\Lambda^\xi/\widehat{Z}_\Lambda^0\,. \tag{2.3}$$

The decoration functions come from *cluster expansion* applied to the partition functions in Λ_γ^+ and Λ_γ^-. In Appendix A, we recall cluster expansion for the SOS model and provide the proof of Proposition 2.1, as the expression for the decoration function ϕ is needed to verify that it meets the criteria of modified Ising polymers (Definition 2.6). In light of Property (i), from now on we will write $\mathcal{C}$ to denote a connected subset (or *cluster*) of $\mathbb{Z}^2$.

Next, we recall the notion of surface tension for the SOS model.

Definition 2.2 (Dobrushin boundary conditions, surface tension)**.** Fix $\vec{u}\in\mathbb{S}^1$ with $\theta_{\vec{u}}\in[0,\pi/2)$, where $\theta_{\vec{u}}$ is the angle $\vec{u}$ makes with the positive horizontal axis. Set $\Lambda_{N,M} := [\![1,N]\!]\times[\![-M,M]\!]$, and let $\xi(\vec{u})$ denote the boundary condition defined by $\xi(\vec{u})_v=0$ for all $v\in\mathbb{Z}^2$ lying on or below $\mathrm{span}(\vec{u})$, and $\xi(\vec{u})_v=1$ otherwise. The set of boundary conditions relevant to us is $\xi(\mathsf{e}_1)$ and we will denote it by $0,1,1,1$ (after the values induced by $\xi(\mathsf{e}_1)$ on the four sides of the box $\Lambda_{N,M}$). Define $d_{N,\vec{u}} := N/\cos(\theta_{\vec{u}})$. Then the *surface tension* $\tau_\beta^{\mathrm{SOS}}(\vec{u})$ is defined by

$$\tau_\beta^{\mathrm{SOS}}(\vec{u}) := \lim_{N\to\infty}\lim_{M\to\infty} -\frac{1}{d_{N,\vec{u}}}\log\Big(\mathcal{G}_{\Lambda_{N,M}}^{\xi(\vec{u})}\Big)\,. \tag{2.4}$$

Using the symmetry of the SOS model, $\tau_\beta^{\mathrm{SOS}}$ trivially extends to an even function on $\mathbb{S}^1$ possessing all lattice symmetries. Finally, $\tau_\beta^{\mathrm{SOS}}$ extends to a function on all of $\mathbb{R}^2$ via homogeneity:

$$\tau_\beta^{\mathrm{SOS}}(\mathsf{x}) := \|\mathsf{x}\|\,\tau_\beta^{\mathrm{SOS}}\big(\tfrac{\mathsf{x}}{\|\mathsf{x}\|}\big)\,,\ \text{for all } \mathsf{x}\in\mathbb{R}^2\,.$$

The proof of the existence of $\tau_\beta^{\text{SOS}}(\mathsf{x})$ as well as many of its properties can be found in [20, §1–2].

2.2. **The free Ising polymer model.** In this section, we define the class of Ising polymer models, as given by [32].[4] The reader is also referred to [20] for many useful results on such polymer models. It will be shown that the SOS open contour from Eq. (2.1) falls in this class.

Recall that we always write $\mathcal{C}$ to denote a finite, connected subset of $\mathbb{Z}^2$. For every contour γ, consider any *decoration function* $\Phi(\mathcal{C};\gamma)$ satisfying the following four properties:

(P1) *Locality*: $\Phi(\mathcal{C};\cdot)$ depends on γ only through $\mathcal{C}\cap\Delta_\gamma$.

(P2) *Decay*: There exists some $\chi > 1/2$ such that, for all $\beta > 0$ sufficiently large,

$$\sup_\gamma |\Phi(\mathcal{C},\gamma)| \le \exp(-\chi\beta(d(\mathcal{C})+1))\,, \tag{2.5}$$

where $d(\mathcal{C})$ is defined as in Property (iv) of Proposition 2.1.

(P3) *Translational symmetry*: for all $\mathsf{v}\in\mathbb{Z}^2$, $\Phi(\mathcal{C};\gamma) = \Phi(\mathcal{C}+\mathsf{v};\gamma+\mathsf{v})$.

(P4) *Symmetry of the surface tension*: the surface tension $\tau_\beta(\mathsf{x})$ defined below in Eq. (2.8) possesses all discrete symmetries of $\mathbb{Z}^2$ (rotations by $\pi/4$ and reflections w.r.t. axes and the diagonals $y=\pm x$).

Towards specifying the probability that the model assigns to each polymer—as well as the surface tension τ_β mentioned in Property (P4)—define the *free (polymer) weight* via

$$q(\gamma) = \exp\Big(-\beta|\gamma| + \sum_{\mathcal{C}\cap\Delta_\gamma\neq\emptyset}\Phi(\mathcal{C};\gamma)\Big)\,, \tag{2.6}$$

where, here and throughout the article, sums over $\mathcal{C}$ are assumed to only go over *connected* subsets $\mathcal{C}\subset\mathbb{Z}^2$. Next, for any $\mathsf{x}\in(\mathbb{Z}^2)^*$, consider the partition function going over all contours γ with start-point given by the dual origin $\mathsf{o}^* := (1/2,1/2)$ and end-point x:

$$\mathcal{G}(\mathsf{x}) := \sum_{\gamma:\mathsf{o}^*\to\mathsf{x}} q(\gamma)\,.$$

For any set of contours E, consider also the partition function going over all contours in E with end-points 0 and x:

$$\mathcal{G}(\mathsf{x}\mid E) := \sum_{\gamma:\mathsf{o}^*\to\mathsf{x}} q(\gamma)\mathbb{1}_{\{\gamma\in E\}}\,. \tag{2.7}$$

Lastly, define the *Ising polymer surface tension* $\tau_\beta(\cdot)$ via[5]

$$\tau_\beta(\vec{n}) := -\lim_{N\to\infty}\frac{1}{N}\log\mathcal{G}(N\vec{n}) \qquad \text{for } \vec{n}\in\mathbb{S}^1, \tag{2.8}$$

where the limit is taken over N such that $N\vec{n}$ is in $(\mathbb{Z}^2)^*$. By homogeneity, τ_β extends to all of $\mathbb{R}^2$.

Definition 2.3 (Free Ising polymer)**.** The *free Ising polymer model* in a subset $D\subset\mathbb{R}^2$ is given by the probability measure over contours $\gamma:\mathsf{o}^*\to\mathsf{x}$ contained in D:

$$\mathbf{P}^{\mathsf{x}}(\cdot\mid\gamma\subset D) := \frac{\mathcal{G}(\mathsf{x}\mid\gamma\subset D,\gamma\in\cdot)}{\mathcal{G}(\mathsf{x}\mid\gamma\subset D)}\,, \tag{2.9}$$

for a partition function $\mathcal{G}$ as above with a decoration function Φ satisfying Properties (P1) to (P4).

[4]The Ising polymer model in [32] had a weaker decay condition, taking $d(\mathcal{C})$ in Property (P2) to be the L^∞-diameter of $\mathcal{C}$, whereas our arguments require $d(\mathcal{C})$ to be the minimum size of a connected set containing its boundary.

[5]It is common to define τ_β with the $1/\beta$ pre-factor (such was the case in [20] as well as [14]). We do not include the pre-factor here since related Ornstein–Zernike works (e.g., [34, 30]) do not include this pre-factor, and this keeps various definitions (e.g., $\mathcal{W}$, W^{h}, $\mathbb{P}^{\mathsf{h}_{\mathsf{x}}}$ defined below) consistent with those works. In [32], τ_β does have a $1/\beta$ pre-factor, though it seems to be a typo, as their inputs from the Ornstein–Zernike theory come from the aforementioned [34] (and as such their calculations are consistent with the above definition of τ_β). We will prove the Ornstein–Zernike facts we require here, to make the proof more self-contained and avoid potential consistency issues.

Below, we list some needed properties of τ_β, proven in [20].

Proposition 2.4 (Surface tension properties, [20])**.** *There exists $\beta_0 > 0$ such that for all $\beta \geq \beta_0$:*

(i) the formula in Eq. (2.8) converges uniformly,
(ii) the surface tension $\tau_\beta : \mathbb{R}^2 \to \mathbb{R}^2$ is analytic,
(iii) (Strong triangle inequality) For any two non-collinear vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{Z}^2$, we have

$$\tau_\beta(\mathsf{u}) + \tau_\beta(\mathsf{v}) > \tau_\beta(\mathsf{u}+\mathsf{v})\,.$$

Proof. Properties (i) and (ii) follow from the combination of Theorem 4.8 and Proposition 4.12 of [20].[6] Lastly, Property (iii) appeared in [20, Proposition 1, Section 4.21]. ☐

For our main application of Theorem 1.2, the SOS model, the following result relates the surface tension $\tau_\beta^{\mathrm{SOS}}$ defined in Eq. (2.4) to the Ising polymer surface tension τ_β defined above. We suspect it is known, though could not find an exact reference for it, and as it follows from our other arguments in Section 5, we include its proof in Section 5.3 for completeness.

Proposition 2.5. *Fix $\beta \geq \beta_0$. Let $q(\gamma)$ denote the free polymer weight as per Eq. (2.6) with $\Phi(\mathcal{C};\gamma)$ taken to be $\phi(\mathcal{C};\gamma)$ from Proposition 2.1. Consider the Ising polymer surface tension τ_β corresponding to the free polymer weights $q(\gamma)$ as per Eq. (2.8). For all $\mathsf{u} \in \mathbb{R}^2$,*

$$\tau_\beta^{\mathrm{SOS}}(\mathsf{u}) = \tau_\beta(\mathsf{u})\,.$$

Observe that the decoration function $\phi(\mathcal{C};\gamma)$ appearing in the law of the unique open 1-contour under $\widehat{\pi}_\Lambda^\xi$ satisfies Properties (P1) to (P4). Indeed, Proposition 2.1 states that the decoration function $\phi(\mathcal{C};\gamma)$ satisfies Properties (P1) to (P3). Property (P4) holds for $\tau_\beta^{\mathrm{SOS}}$ because of the symmetry of the SOS model, and thus for τ_β thanks to Proposition 2.5.

However, the open 1-contour in the SOS model is not a free Ising polymer, due to the $\mathbb{1}_{\{\mathcal{C}\subset\Lambda\}}$ term appearing in (2.1), which introduces an interaction of γ with the boundary via the decoration function ϕ. This served as one of the motivations of [32] to study modified Ising polymers—the generalization of the free Ising polymers described above to allow domain-induced modifications on the "free" weights (the SOS open contour does belong to that family of models: see Observation 2.7).

2.3. **The modified Ising polymer model.** For any $D \subset \mathbb{R}^2$ and for any decoration function $\Phi(\mathcal{C};\gamma)$ satisfying Properties (P1) to (P4), consider any function $\Phi_D(\mathcal{C};\gamma)$ satisfying

(M1) $\Phi_D(\mathcal{C};\gamma) = \Phi(\mathcal{C};\gamma)$ for any $\mathcal{C} \subset D$, and
(M2) $\Phi_D(\mathcal{C};\gamma)$ satisfies the same decay bound in Eq. (2.5) for all $\mathcal{C}$.

Call $\Phi_D(\mathcal{C};\gamma)$ a *modified decoration function* (with modifications outside of D), and define the *modified polymer weight*

$$q_D(\gamma) = \exp\Big(-\beta|\gamma| + \sum_{\mathcal{C}\cap\Delta_\gamma\neq\emptyset} \Phi_D(\mathcal{C};\gamma)\Big)$$

as well as the partition function

$$\mathcal{G}_D(\mathsf{x}) := \sum_{\gamma:\mathsf{o}^*\to\mathsf{x},\gamma\subset D} q_D(\gamma) \quad \text{and} \quad \mathcal{G}_D(\mathsf{x}\mid E) := \sum_{\gamma:\mathsf{o}^*\to\mathsf{x},\gamma\subset D} q_D(\gamma)\mathbb{1}_{\{\gamma\in E\}} \text{ for a set of contours } E.$$

Definition 2.6 (Modified Ising polymer)**.** The *modified Ising polymer* in a subset $D \subset \mathbb{R}^2$ is given by the probability measure on contours $\gamma : \mathsf{o}^* \to \mathsf{x}$ contained in D:

$$\mathbf{P}_D^{\mathsf{x}}(\cdot) := \frac{\mathcal{G}_D(\mathsf{x}\mid\gamma\in\cdot)}{\mathcal{G}_D(\mathsf{x})}\,, \tag{2.10}$$

[6]These results are stated in [20] for the Ising polymer model $\mathcal{G}(\mathsf{x})$. The analogue of $\mathcal{G}$ is defined in [20, Eq. 4.3.11] (where for us, $h = 0$). Though that formula involves decoration functions Φ coming from the cluster expansion of the Ising model, the results in [20, Section 4] that follow this formula only use Properties (iii) and (iv) of our Proposition 2.1. This is stated in the final paragraph of [20, Section 4.3].

where the partition function $\mathcal{G}_D$ is defined for a decoration function Φ satisfying Properties (P1) to (P4), and a modified decoration function Φ_D satisfying Properties (M1) and (M2).

Note that the SOS open contour with law $\widehat{\pi}_\Lambda^\xi$ given by Eq. (2.1) is of the form above, with $\Phi(\mathcal{C};\gamma)=\phi(\mathcal{C};\gamma)$, $D=\Lambda$, and $\Phi_\Lambda(\mathcal{C};\gamma)=\phi(\mathcal{C};\gamma)\mathbb{1}_{\{\mathcal{C}\subset\Lambda\}}$ for $\phi(\mathcal{C};\gamma)$ from Proposition 2.1 (in which this choice of $\Phi_D(\mathcal{C};\gamma)$ clearly satisfies Properties (M1) and (M2)). Namely, the following holds:

Observation 2.7 (SOS is a modified Ising polymer)**.** *Fix $\beta\geq\beta_0$, and consider the SOS model $\widehat{\pi}_\Lambda^\xi$ in a finite, connected subset $\Lambda\subset\mathbb{Z}^2$ and boundary condition $\xi\in\{0,1\}$ that induces a unique open 1-contour γ. Let $\bar\Lambda$ denote the region in $\mathbb{R}^2$ enclosed by $\partial\Lambda$ (i.e., the region enclosed by the $\mathbb{Z}^2$-edges connecting the boundary vertices of Λ). Assume for convenience that the start-point of γ is o^*, and denote its end-point by x. Then γ has the Ising polymer law $\mathbf{P}^{\mathsf{x}}_{\bar\Lambda}$ defined with decoration weights $\Phi(\mathcal{C};\gamma):=\phi(\mathcal{C};\gamma)$ and $\Phi_{\bar\Lambda}(\mathcal{C};\gamma):=\phi(\mathcal{C};\gamma)\mathbb{1}_{\{\mathcal{C}\subset\Lambda\}}$.*

It will be very convenient to view Ising polymers as connected paths in the lattice $\mathbb{Z}^2$ rather than the dual lattice $(\mathbb{Z}^2)^*$ via the translation map $\iota:\mathsf{o}^*\mapsto 0$, and this is the convention we will follow for the remainder of the article, excluding Section 3.

Let us now re-state Theorem 1.2 in the above language, which is the form in which we will prove it (Section 5.7). For $N\in\mathbb{N}$, define $Q:=[0,N]^2$.

Theorem 2.8. *Fix $\beta>0$ large, take D to be either $\mathbb{H}$ or Q, and set $\mathsf{x}:=(N,0)$. Consider an Ising polymer $\gamma\sim\mathbf{P}^{\mathsf{x}}_D(\cdot)$. There exists $\sigma>0$ such that, if $\overline{\gamma}(x)=\max\{y:(x,y)\in\gamma\}$, then $\overline{\gamma}(\lfloor xN\rfloor)/(\sigma\sqrt{N})$ converges weakly to a standard Brownian excursion in the Skorokhod space $(D[0,1],\|\cdot\|_\infty)$ and the same holds for $\underline{\gamma}(x)=\min\{y:(x,y)\in\gamma\}$.*

Remark 2.9. The variance σ^2 with σ as in Theorem 2.8 (and by extension, the one in Theorem 1.2) is given above Theorem 4.3. It is related to the curvature of the Wulff shape (defined in Section 2.6) associated to the Ising polymer. See [30, Appendix B] for details.

2.4. **Non-negative decoration functions and a product structure.** Following [32, Section 3.1], we employ a construction going back to [21] that allows us to consider decoration functions which are non-negative. In this discussion, we allow for the case $D=\mathbb{R}^2$, in which case $\Phi_D(\mathcal{C};\gamma)=\Phi(\mathcal{C};\gamma)$ and $q_D(\gamma)=q(\gamma)$.

For a contour γ, we consider the set of (not necessarily distinct) bonds in $\mathbb{Z}^2$ (recall we have applied the translation map $\iota:0\to\mathsf{o}^*$):

$$\nabla_\gamma:=\bigcup_{b=(\mathsf{y},\mathsf{y}+\mathsf{e}_i)\in\gamma}\{b,b+\mathsf{e}_i,b-\mathsf{e}_i\}\,.$$

To be clear, b is a bond in γ, e_i denotes a standard basis vector, and $b\pm\mathsf{e}_i$ denotes the bond obtained by translating b by $\pm\mathsf{e}_i$. Thus, ∇_γ contains three bonds for each bond of γ. Define

$$\Phi'_D(\mathcal{C};\gamma):=|\mathcal{C}\cap\nabla_\gamma|\,e^{-\chi\beta(d(\mathcal{C})+1)}+\Phi_D(\mathcal{C};\gamma)\,,$$

where $|\mathcal{C}\cap\nabla_\gamma|$ is equal to the number of bonds, counted with multiplicity, in ∇_γ that $\mathcal{C}$ intersects:

$$|\mathcal{C}\cap\nabla_\gamma|=\sum_{b=(\mathsf{y},\mathsf{y}+\mathsf{e}_i)\in\gamma}\left(\mathbb{1}_{\{b\cap\mathcal{C}\neq\emptyset\}}+\mathbb{1}_{\{b+\mathsf{e}_i\cap\mathcal{C}\neq\emptyset\}}+\mathbb{1}_{\{b-\mathsf{e}_i\cap\mathcal{C}\neq\emptyset\}}\right).$$

Observe that $\Phi'_D\geq 0$ by Eq. (2.5) and Property (M2). For any fixed bond $b\in\mathbb{Z}^2$, let $c(\beta)$ denote the value of

$$c(\beta):=\sum_{\mathcal{C}\subset\mathbb{Z}^2\,,\,\mathcal{C}\cap b\neq\emptyset}e^{-\chi\beta(d(\mathcal{C})+1)}\,.$$

Note that

$$c(\beta)=\sum_{m\geq1}e^{-\chi\beta(m+1)}\left|\{\mathcal{C}\subset\mathbb{Z}^2:\mathcal{C}\ni\mathsf{b},d(\mathcal{C})=m\}\right|\leq e^{-\chi\beta}\,,\tag{2.11}$$

where we used that $\left|\{\mathcal{C} \subset (\mathbb{Z}^2)^* : \mathcal{C} \ni \mathsf{b}, d(\mathcal{C}) = m\}\right| \le e^{cm}$ for some $c > 0$. (To see this, replace the marked $\mathsf{b} \in \mathcal{C}$ by a marked boundary edge $e_0 \in \partial\mathcal{C}$ with the same y-coordinate (say) at the cost of a factor of m; then, when enumerating the smallest connected set of edges of $\mathbb{Z}^2$ containing the marked e_0 and specified edges $\partial\mathcal{C}$, regard $\partial\mathcal{C}$ as the vertices of a 6-regular graph (whose vertices are the $\mathbb{Z}^2$ bonds and two are adjacent if they share an end-point; that is, the line graph of $\mathbb{Z}^2$), and recall that the number of m-vertex connected subgraphs of a graph with maximum degree Δ is at most $(e\Delta)^m$.) Using the fact that $\mathcal{C} \cap \Delta_\gamma \neq \emptyset$ implies $\mathcal{C} \cap \nabla_\gamma \neq \emptyset$, we have

$$\sum_{\mathcal{C}\cap\Delta_\gamma\neq\emptyset} \Phi_D(\mathcal{C};\gamma) = -3c(\beta)|\gamma| + \sum_{\mathcal{C}\cap\nabla_\gamma\neq\emptyset} \Phi'_D(\mathcal{C};\gamma)\,,$$

Thus, we have

$$q_D(\gamma) = \exp\Big(-(\beta + 3c(\beta))|\gamma| + \sum_{\mathcal{C}\cap\nabla_\gamma\neq\emptyset} \Phi'_D(\mathcal{C};\gamma)\Big)\,.$$

Since $f(\beta) := \beta + 3c(\beta)$ is strictly increasing for all β large enough, we will henceforth redefine $\beta = f(\beta)$ so that we may drop the laborious $3c(\beta)$ from our weights:

$$q_D(\gamma) = \exp\Big(-\beta|\gamma| + \sum_{\mathcal{C}\cap\nabla_\gamma\neq\emptyset} \Phi'_D(\mathcal{C};\gamma)\Big)\,. \tag{2.12}$$

A useful comparison to record at this stage is that, for any $D \subset \mathbb{R}^2$,

$$\left|\log \frac{q_D(\gamma)}{q(\gamma)}\right| \le 6e^{-\chi\beta}|\gamma|\,. \tag{2.13}$$

Remark 2.10. For any domain $D \subset \mathbb{R}^2$, the non-negative decoration functions $\Phi'(\mathcal{C};\gamma)$ still satisfy Properties (P1) to (P4) above, and the modified non-negative decoration functions $\Phi'_D(\mathcal{C};\gamma)$ still satisfy Properties (M1) and (M2).

We next uncover the product structure of q_D. Defining $\Psi_D(\mathcal{C},\gamma) := \big(\exp(\Phi'_D(\mathcal{C};\gamma)) - 1\big)\mathbb{1}_{\{\mathcal{C}\cap\nabla_\gamma\neq\emptyset\}}$, we may write

$$\exp\Big(\sum_{\mathcal{C}\cap\nabla_\gamma\neq\emptyset} \Phi'_D(\mathcal{C};\gamma)\Big) = \prod_{\mathcal{C}\cap\nabla_\gamma\neq\emptyset} \Big((e^{\Phi'_D(\mathcal{C};\gamma)} - 1) + 1\Big) = \sum_{\underline{\mathcal{C}}=\{\mathcal{C}_i\}} \prod_i \Psi_D(\mathcal{C}_i;\gamma)\,,$$

where the sum goes over all possible finite collections $\underline{\mathcal{C}}$ of clusters. Given this, for any contour γ and any collection $\underline{\mathcal{C}}$ of clusters, we define the *animal weight* $q_D(\Gamma)$ of the *animal* $\Gamma = [\gamma, \underline{\mathcal{C}}]$ by

$$q_D(\Gamma) = q_D([\gamma,\underline{\mathcal{C}}]) := e^{-\beta|\gamma|} \prod_{\mathcal{C}\in\underline{\mathcal{C}}} \Psi_D(\mathcal{C};\gamma)\,, \tag{2.14}$$

and observe

$$q_D(\gamma) = \sum_{\Gamma=[\gamma,\underline{\mathcal{C}}]} q_D(\Gamma)\,.$$

The above allows us to consider the free and modified Ising polymer measures as probability measures on animals:

$$\mathbf{P}^{\mathsf{x}}(\cdot \mid \gamma \subset D) = \frac{\mathcal{G}(\mathsf{x} \mid \gamma \subset D, \Gamma \in \cdot)}{\mathcal{G}(\mathsf{x} \mid \gamma \subset D)} \qquad \text{and} \qquad \mathbf{P}^{\mathsf{x}}_D(\cdot) := \frac{\mathcal{G}_D(\mathsf{x} \mid \Gamma \in \cdot)}{\mathcal{G}_D(\mathsf{x})}\,. \tag{2.15}$$

Due to the product structure of $q_D(\Gamma)$, it is often convenient to consider animals rather than contours; indeed, we will see in Section 4.1 that the product structure begets a connection with a random walk, which is crucial to our analysis.

When D is taken to be $\mathbb{R}^2$, we will omit D from the notation laid out above.

2.5. **Notation for Ising polymers and animals.** Below, we set notation that will be used throughout the article in the context of Ising polymers and animals. Recall that we Ising polymers as connected paths in the lattice $\mathbb{Z}^2$ rather than the dual lattice $(\mathbb{Z}^2)^*$ via the translation map $\iota : \mathsf{o}^* \mapsto 0$, and this is the convention we will follow for the remainder of the article, excluding Section 3.

- We write $Q := [0, N]^2$.
- For a point $\mathsf{u} \in \mathbb{Z}^2$, we will write u_1 and u_2 to denote the x-coordinate and y-coordinate of u respectively.
- For a subset $E \subset \mathbb{Z}^2$, we'll write $\mathsf{u} + E$ to denote the translation of E by the vector defined by u.
- For a contour γ, we write $|\gamma|$ to denote the number of bonds in γ. We write $(\gamma(0), \dots, \gamma(|\gamma|))$ to denote the ordered vertices of γ.
- For an animal $\Gamma := [\gamma, \underline{\mathcal{C}}]$, we write $|\Gamma| := |\gamma|$, and $\mathsf{X}(\Gamma) := \mathsf{X}(\gamma)$ to denote the displacement of γ; that is, the vector in $\mathbb{Z}^2$ given by the end-point of γ minus the start-point of γ.
- For a pair of contours $\gamma := (\gamma(0), \dots, \gamma(|\gamma|))$ and $\gamma' := (\gamma'(0), \dots, \gamma'(|\gamma'|))$, we define their concatenation to be
$$\gamma \circ \gamma' := \big(\gamma(0), \dots, \gamma(|\gamma|), \mathsf{X}(\Gamma) + \gamma'(0), \dots, \mathsf{X}(\Gamma) + \gamma'(|\gamma'|)\big)\,.$$
Similarly, for a pair of animals $\Gamma := [\gamma, \underline{\mathcal{C}}]$ and $\Gamma' := [\gamma', \underline{\mathcal{C}}']$, we define their concatenation to be
$$\Gamma \circ \Gamma' := [\gamma \circ \gamma',\ \underline{\mathcal{C}} \cup \underline{\mathcal{C}}']\,.$$
- For $\mathsf{u}, \mathsf{v} \in \mathbb{Z}^2$, we write $\gamma : \mathsf{u} \to \mathsf{v}$ to indicate that $\gamma(0) = \mathsf{u}$ and $\gamma(|\gamma|) = \mathsf{v}$. We write $\Gamma : \mathsf{u} \to \mathsf{v}$ to indicate that $\Gamma = [\gamma, \underline{\mathcal{C}}]$ for some contour $\gamma : \mathsf{u} \to \mathsf{v}$.
- For a set $D \subset \mathbb{R}$, we say $\Gamma = [\gamma, \underline{\mathcal{C}}] \subset D$ if γ as well as all clusters $\mathcal{C} \in \underline{\mathcal{C}}$ are contained in D.
- For any $D \subset \mathbb{R}^2$, we define the set of contours
$$\mathcal{P}_D(\mathsf{u}, \mathsf{v}) := \{\gamma : \gamma \subset D\,,\, \gamma : \mathsf{u} \to \mathsf{v}\}\,.$$
When $\mathsf{u} = 0$, we will simply write $\mathcal{P}_D(\mathsf{v})$. We'll write $\Gamma \in \mathcal{P}_D(\mathsf{u}, \mathsf{v})$ for animals Γ to mean $\Gamma = [\gamma, \underline{\mathcal{C}}]$ for some contour $\gamma \in \mathcal{P}_D(\mathsf{u}, \mathsf{v})$.

2.6. **The Wulff Shape.** Now, define the *Wulff shape*
$$\mathcal{W} := \bigcap_{\mathsf{y} \in \mathbb{R}^2} \{\mathsf{h} \in \mathbb{R}^2 : \mathsf{h} \cdot \mathsf{y} \le \tau_\beta(\mathsf{y})\}\,,$$
which is clearly closed and convex (as it is the intersection of half-spaces). Observe that
$$\mathcal{W} = \overline{\left\{\mathsf{h} \in \mathbb{R}^2 : \sum_{\mathsf{y} \in \mathbb{Z}^2} e^{\mathsf{h} \cdot \mathsf{y}} \mathcal{G}(\mathsf{y}) < \infty\right\}}\,.$$
Indeed, from Eq. (2.8), we have
$$\log \mathcal{G}(\mathsf{y}) = -\tau_\beta(\mathsf{y})\big(1 + o_{\|\mathsf{y}\|_1}(1)\big)\,. \tag{2.16}$$
It follows that the sum in the second expression for $\mathcal{W}$ converges if and only if $\mathsf{h} \cdot \mathsf{y} < \tau_\beta(\mathsf{y})$ for all $\mathsf{y} \in \mathbb{Z}^2$ large, which by homogeneity and continuity of τ_β is equivalent to the same holding for all $\mathsf{y} \in \mathbb{R}^2$. Including the equality case $\mathsf{h} \cdot \mathsf{y} = \tau_\beta(\mathsf{y})$ in the first expression for $\mathcal{W}$ is equivalent to taking the closure of the set in the second expression.

2.7. **Cone-points, the irreducible decomposition of animals, and weight factorization.** Let us define the *forward cone* $\mathcal{Y}^{\blacktriangleleft} := \{(x,y) \in \mathbb{Z}^2 : |y| \le x\}$ and the *backward cone* $\mathcal{Y}^{\blacktriangleright} := -\mathcal{Y}^{\blacktriangleleft}$. We will also need $\mathcal{Y}_\delta^{\blacktriangleleft} := \{(x,y) \in \mathbb{Z}^2 : |y| \le \delta x\}$ for $\delta > 0$. Given a contour γ and an animal $\Gamma = [\gamma, \underline{\mathcal{C}}]$, we say that $\mathsf{u} \in \gamma$ is a *cone-point* for γ if

$$\gamma \subset \mathsf{u} + \mathcal{Y}^{\blacktriangleright} \cup \mathsf{u} + \mathcal{Y}^{\blacktriangleleft},$$

and we say $\mathsf{u} \in \gamma$ is a *cone-point* for Γ if

$$\Gamma \subset \mathsf{u} + \mathcal{Y}^{\blacktriangleright} \cup \mathsf{u} + \mathcal{Y}^{\blacktriangleleft}.$$

Recall from Section 2.5 that the previous display means that the forward and backward cones emanating from u fully contain γ as well as all clusters $\mathcal{C}$ in $\underline{\mathcal{C}}$. Of course if $\mathsf{u} \in \gamma$ is a cone-point for Γ, then u is a cone-point for γ as well. Note that for any $\mathsf{u}, \mathsf{v} \in \mathbb{Z}^2$, $\mathsf{u} \in \mathsf{v} + \mathcal{Y}^{\blacktriangleleft}$ if and only if $\mathsf{v} \in \mathsf{u} + \mathcal{Y}^{\blacktriangleright}$.

If Γ has two cone-points u and v, the contour part of Γ between u and v as well as all associated clusters are entirely contained within the "diamond" $\mathsf{u} + \mathcal{Y}^{\blacktriangleleft} \cap \mathsf{v} + \mathcal{Y}^{\blacktriangleright}$. An animal is called *left-irreducible* if it contains no cone-points and if it is entirely contained in the backwards cone emanating from its end-point. Similarly, an animal is called *right-irreducible* if it contains no cone-points and it is contained in the forward cone emanating from its start-point. An animal is called *irreducible* if it is both left- and right-irreducible. We let $\mathsf{A_L}, \mathsf{A_R}$ and A be the sets of left-irreducible, right-irreducible and irreducible animals, respectively, with start-point at the origin.

Consider now an animal Γ with at least two cone-points, and decompose it into (left-/right-) irreducible animals as follows:

$$\Gamma = \Gamma^{(L)} \circ \Gamma^{(1)} \circ \cdots \circ \Gamma^{(n)} \circ \Gamma^{(R)}, \tag{2.17}$$

for some $n \ge 1$, $\Gamma^{(L)} \in \mathsf{A_L}$, $\Gamma^{(R)} \in \mathsf{A_R}$, and $\Gamma^{(i)} \in \mathsf{A}$ for $i = 1, \dots, n$. For $i \in \{L, 1, \dots, n, R\}$, we will write

$$\Gamma^{(i)} =: [\gamma^{(i)}, \underline{\mathcal{C}}_i].$$

From the definition of a cone-point, any cluster $\mathcal{C} \in \underline{\mathcal{C}}_i$ will be such that $\mathcal{C} \cap \nabla_{\gamma^{(j)}} = \emptyset$ for $j \ne i$. Informally, this means that any cluster of $\Gamma^{(i)}$ will *not* be a cluster of $\Gamma^{(j)}$. Thus, the weights $q_D(\Gamma)$ "factorize" into a product of the weights of the irreducible pieces:

$$q_D(\Gamma) = \prod_{i \in \{L,1,\dots,n,R\}} e^{-\beta|\gamma^{(i)}|} \prod_{\mathcal{C} \in \underline{\mathcal{C}}_i} \Psi_D(\mathcal{C}; \gamma) = q_D(\Gamma^{(L)}) q_D(\Gamma^{(R)}) \prod_{i=1}^n q_D(\Gamma^{(i)}) \tag{2.18}$$

When $\gamma \subset Q$, note that the shape of the forward and backward cones necessitates

$$\Gamma^{(1)} \circ \cdots \Gamma^{(n)} \subset [1, N-1] \times (-\infty, N],$$

so that Eq. (2.18) becomes

$$q_Q(\Gamma) = q_Q(\Gamma^{(L)}) q_Q(\Gamma^{(R)}) \prod_{i=1}^n q_{\mathbb{H}}(\Gamma^{(i)}). \tag{2.19}$$

We write $\mathsf{Cpts}(\gamma)$ and $\mathsf{Cpts}(\Gamma)$ to denote the set of cone-points of γ and the set of cone-points of Γ, respectively. Let $n := n(\Gamma) = |\mathsf{Cpts}(\Gamma)| - 1$ as above, so that n is equal to the number of irreducible pieces of Γ. We label the cone-points of Γ by $\zeta^{(1)}, \dots, \zeta^{(n+1)}$, where the ordering is strictly increasing in the x-coordinates, i.e.,

$$\zeta_1^{(i)} < \zeta_1^{(i+1)}.$$

The set $\mathsf{Cpts}(\Gamma)$ therefore takes on a natural meaning as an ordered $n(\Gamma)-$tuple, and so we will abuse notation and write

$$\mathsf{Cpts}(\Gamma) = (\zeta^{(i)})_{i=1}^{n+1} = \left(X(\Gamma^{(L)}), X(\Gamma^{(L)}) + X(\Gamma^{(1)}), \dots, X(\Gamma^{(L)}) + \cdots + X(\Gamma^{(R)})\right). \tag{2.20}$$

The following result of [32] states that the free weight of contours in $\mathbb{Z}^2$ that have large length and contain a sub-linear number of cone-points compared to the distance of the end-point is exponentially small compared to $\mathcal{G}$.

Lemma 2.11 ([32, Eq. (4.5)][7])**.** *Fix any $\epsilon, \delta \in (0,1)$. There exist $\beta_0 \in (0,\infty)$, $\nu_0 > 0$, $\delta_0 > 0$, and $c > 0$ such that the following bounds hold uniformly over $\beta \geq \beta_0$, $\mathsf{y} \in \mathcal{Y}_\delta^{◀} \setminus \{0\}$, and $r \geq 1+\epsilon$:*

$$\mathcal{G}(\mathsf{y} \mid |\gamma| \geq r \left\|\mathsf{y}\right\|_1) \leq ce^{-\nu_0\beta r\|\mathsf{y}\|_1}\mathcal{G}(\mathsf{y}) \tag{2.21}$$

$$\mathcal{G}(\mathsf{y} \mid \mathsf{Cpts}(\gamma) < 2\delta_0 \left\|\mathsf{y}\right\|_1) \leq ce^{-\nu_0\beta\|\mathsf{y}\|_1}\mathcal{G}(\mathsf{y}) \tag{2.22}$$

In [32][8], it is claimed without proof that the an analogue of Lemma 2.11 also holds for animals. For completeness, we supply the result as well as a proof below.

Proposition 2.12. *For any $\delta \in (0,1)$, there exist $\nu > 0$ and $c > 0$ such that the following bounds hold uniformly over $\beta \geq \beta_0$, and $\mathsf{y} \in \mathcal{Y}_\delta^{◀} \setminus \{0\}$:*

$$\mathcal{G}(\mathsf{y} \mid \mathsf{Cpts}(\Gamma) < \delta_0\|\mathsf{y}\|_1) \leq ce^{-\nu\beta\|\mathsf{y}\|_1}\mathcal{G}(\mathsf{y}),$$

where δ_0 is as in Lemma 2.11.

Proof. The idea is to show that, for a typical $\Gamma = [\gamma, \underline{\mathcal{C}}]$, many of the cone-points for γ are also cone-points for Γ. We do this by showing that typically, $\mathcal{C}$ does not contain many clusters (more precisely, the total $d(\cdot)$-size is not too big). The result then follows from Eq. (2.22).

Our starting point is the following, which comes from Property (P2) and Remark 2.10 and states that for some constant $c_1 > 0$, for all $\beta > 0$ sufficiently large, and for any animal $[\gamma, \underline{\mathcal{C}}]$,

$$q([\gamma, \underline{\mathcal{C}}]) \leq e^{-\beta|\gamma|} \exp\left(-c_1\beta \sum_{\mathcal{C}\in\underline{\mathcal{C}}} d(\mathcal{C})\right). \tag{2.23}$$

Now, define the events

$$A = \{\Gamma = [\gamma, \underline{\mathcal{C}}] \mid \mathsf{Cpts}(\Gamma) < \delta_0\|\mathsf{y}\|_1\}$$
$$B = \{\Gamma = [\gamma, \underline{\mathcal{C}}] \mid \mathsf{Cpts}(\gamma) \geq 2\delta_0\|\mathsf{y}\|_1\,,\ |\gamma| \leq 1.1\|\mathsf{y}\|_1\}$$

Then, from Eqs. (2.21) and (2.22), we have

$$\frac{\mathcal{G}(\mathsf{y} \mid A)}{\mathcal{G}(\mathsf{y})} = \frac{\mathcal{G}(\mathsf{y} \mid A, B)}{\mathcal{G}(\mathsf{y})} + O(e^{-\nu_0\beta\|\mathsf{y}\|_1}).$$

Now, consider a contour γ with at least $2\delta_0\|\mathsf{y}\|_1$ cone-points and an animal Γ such that $\Gamma = [\gamma, \underline{\mathcal{C}}]$ for some set of clusters $\underline{\mathcal{C}}$. Note that if Γ has less than $\delta_0\|\mathsf{y}\|_1$ cone-points, then the clusters of $\underline{\mathcal{C}}$ must intersect γ in such a way that at least $\delta_0\|\mathsf{y}\|_1$ cone-points of γ are not cone-points of Γ. This necessitates that the sum of $d(\mathcal{C})$ over $\mathcal{C} \in \underline{\mathcal{C}}$ exceeds $\delta_0\|\mathsf{y}\|_1$. Hence, we can write

$$\mathcal{G}(\mathsf{y} \mid A, B) \leq \sum_{\substack{\gamma:0\to\mathsf{y} \\ |\gamma|\leq 1.1\|\mathsf{y}\|_1}} \sum_{m\geq\delta_0\|\mathsf{y}\|_1} \sum_{n=1}^{m}\sum_{k=1}^{n} \binom{3|\gamma|}{k} \sum_{\substack{\underline{n}=(n_1,\dots,n_k) \\ n_1+\dots+n_k=n}} \sum_{\substack{\underline{m}=(m_1,\dots,m_n) \\ m_1+\dots+m_n=m}} \left(\prod_{i=1}^{k} |G_{i,\underline{n},\underline{m}}|\right) e^{-\beta|\gamma|-c_1\beta m},$$

where: the second sum accounts for the possible values of $m = \sum_{\mathcal{C}\in\underline{\mathcal{C}}} d(\mathcal{C})$; the third sum accounts for the possible values of $n = |\underline{\mathcal{C}}|$ given m; the fourth sum and the binominal coefficient account for the number of possible bonds of ∇_γ that the clusters of $\underline{\mathcal{C}}$ intersect; the fifth sum accounts for the number of clusters that intersect a particular bond of ∇_γ (given k and an arbitrary labeling of the bonds $b_1, \dots, b_k$); the sixth sum accounts for the possible $d(\cdot)$-value of each of the n clusters given

[7]In [32], $1+\epsilon$ is replaced by an unspecified constant r_0; however, it is trivial to see that r_0 may be taken to be arbitrarily close to 1 in [32, Lemma 4], which is the same r_0 as in [32, Eq. 4.5]. Additionally, our condition $\mathsf{y} \in \mathcal{Y}_\delta^{◀} \setminus \{0\}$ appears as $\mathsf{x} \in \mathcal{Q}_+$ in [32].

[8]See the paragraph containing equation (4.9) in [32]

that their total $d(\cdot)$-sum is m, and where $\underline{m} = (m_1, \dots, m_n)$ is ordered such that $m_1, \dots, m_{n_1}$ represent the $d(\cdot)$-values of the n_1 clusters intersecting b_1, $m_{n_1+1}, \dots, m_{n_1+n_2}$ represent the $d(\cdot)$-values of the n_2 clusters intersecting b_2 and so on; each $G_{i,\underline{n},\underline{m}}$ denotes the set of all possible collections of the n_i clusters intersecting b_i given the aforementioned $\underline{n}$ and $\underline{m}$; and lastly, the exponential is the cumulative upper bound on $q([\gamma, \underline{\mathcal{C}}])$ given by Eq. (2.23) (given that the cumulative $d(\cdot)$ is m).

Now, similar to Eq. (2.11), the number of clusters $\mathcal{C}$ with $d(\mathcal{C}) = r$ and intersecting some fixed bond b is bounded from above by $e^{c_2 r}$, and therefore

$$\prod_{i=1}^{k} |G_{i,\underline{n},\underline{m}}| \le e^{c_2(m_1+\cdots+m_n)} = e^{c_2 m}\,.$$

Hence, $\mathcal{G}(\mathsf{y} \mid A, B)$ is upper-bounded by

$$\begin{aligned}
&\sum_{\substack{\gamma:0\to\mathsf{y}\\ |\gamma|\le 1.1\|\mathsf{y}\|_1}} e^{-\beta|\gamma|} \sum_{m\ge\delta_0\|\mathsf{y}\|_1} \sum_{n=1}^{m}\sum_{k=1}^{n} \binom{3|\gamma|}{k} \sum_{\substack{\underline{n}=(n_1,\dots,n_k)\\ n_1+\cdots+n_k=n}} \sum_{\substack{\underline{m}=(m_1,\dots,m_n)\\ m_1+\cdots+m_n=m}} e^{-c_1\beta m} e^{c_2 m}\\
\le &\sum_{\substack{\gamma:0\to\mathsf{y}\\ |\gamma|\le 1.1\|\mathsf{y}\|_1}} e^{-\beta|\gamma|} \sum_{m\ge\delta_0\|\mathsf{y}\|_1} e^{-c_3\beta m} \sum_{n=1}^{m}\sum_{k=1}^{n} \binom{3|\gamma|}{k}\binom{n-1}{k-1}\binom{m-1}{n-1}\\
\le &\sum_{\substack{\gamma:0\to\mathsf{y}\\ |\gamma|\le 1.1\|\mathsf{y}\|_1}} e^{-\beta|\gamma|} \sum_{m\ge\delta_0\|\mathsf{y}\|_1} e^{-c_3\beta m} 2^m(2^m-1)2^{3|\gamma|} \le\; e^{-c_4\beta\|\mathsf{y}\|_1} \sum_{\gamma:0\to\mathsf{y}} e^{-\beta|\gamma|}\,.
\end{aligned}$$

Finally, the non-negativity of the decorations Φ' implies

$$\mathcal{G}(\mathsf{y}) = \sum_{\substack{\Gamma=[\gamma,\underline{\mathcal{C}}]\\ \gamma:0\to\mathsf{y}}} q([\gamma,\underline{\mathcal{C}}]) = \sum_{\gamma:0\to\mathsf{y}} \exp\Big(-\beta|\gamma| + \sum_{\substack{\mathcal{C}\subset\Lambda\\ \mathcal{C}\cap\Delta_\gamma\neq\emptyset}} \Phi'(\mathcal{C};\gamma)\Big) \ge \sum_{\gamma:0\to\mathsf{y}} e^{-\beta|\gamma|}\,,$$

and thus

$$\frac{\mathcal{G}(\mathsf{y} \mid A, B)}{\mathcal{G}(\mathsf{y})} \le e^{-c_5\beta\|\mathsf{y}\|_1}\,,$$

concluding the proof. ☐

Remark 2.13. Note that the proof of Proposition 2.12 is such that, for any Ising polymer model $\mathcal{G}_D$ or $\mathcal{G}(\cdot \mid \gamma \subset D)$ satisfying the analogous bounds in Lemma 2.11, Proposition 2.12 follows with $\mathcal{G}$ replaced by $\mathcal{G}_D$ or $\mathcal{G}(\cdot \mid \gamma \subset D)$. This is used when we prove the existence of cone-points for Γ in these models in Lemma 5.7.

2.8. **Ornstein–Zernike theory and its applications.** For $\mathsf{h} \in \mathbb{R}^2$, we define $W^{\mathsf{h}}(\cdot)$ by

$$W^{\mathsf{h}}(\Gamma) := e^{\mathsf{h}\cdot\mathsf{X}(\Gamma)} q(\Gamma)$$

for any animal Γ (not necessarily irreducible). For $\mathsf{y} \neq 0$, define the *dual* parameter h_{y} by

$$\mathsf{h}_{\mathsf{y}} = \nabla\tau_\beta(\mathsf{y})\,.$$

Convexity and homogeneity of $\tau_\beta(\mathsf{y})$ imply that $\mathsf{h}_{\mathsf{y}} \in \partial\mathcal{W}$ (see the first definition of $\mathcal{W}$).

Proposition 2.14 below, which forms the main result of the Ornstein–Zernike theory for the Ising polymer models, was stated in [32, Thm. 5] in a slightly different setting (namely, the cones are defined differently here), pointing to [34] as its relevant input. Indeed, one may infer this result via an Ornstein–Zernike analysis as in [34, Sec 3.3 and 3.4], but since (a) the models considered in that work do not include Ising polymers, and (b) our setting differs somewhat from that of [32], we include a full proof of this proposition in Appendix B.

Proposition 2.14. *For any $\delta \in (0,1)$, there exists $\beta_0 > 0$ such that for all $\beta > \beta_0$ and for any $\mathsf{y} \in \mathcal{Y}_\delta^{\blacktriangleleft} \setminus \{0\}$, the collection of weights $W^{\mathsf{h}_\mathsf{y}}$ defines a probability distribution on the set A of irreducible animals. To emphasize that $W^{\mathsf{h}_\mathsf{y}}$ defines a probability distribution (on irreducible animals), and for consistency with [32], we use the notation*

$$\mathbb{P}^{\mathsf{h}_\mathsf{y}}(\Gamma) := e^{\mathsf{h}_\mathsf{y}\cdot \mathsf{X}(\Gamma)} q(\Gamma) = W^{\mathsf{h}_\mathsf{y}}(\Gamma)\,. \tag{2.24}$$

Let $\mathbb{E}^{\mathsf{h}_\mathsf{y}}$ denote expectation under $\mathbb{P}^{\mathsf{h}_\mathsf{y}}$. Then

$$\mathbb{E}^{\mathsf{h}_\mathsf{y}}[\mathsf{X}(\Gamma)] = \alpha \mathsf{y}\,, \tag{2.25}$$

for some constant $\alpha := \alpha(\beta, \mathsf{y}) > 0$— in particular, the expectation of $\mathsf{X}(\Gamma)$ under $\mathbb{P}^{\mathsf{h}_\mathsf{y}}$ is collinear to y. Finally, there exists a "mass-gap" constant $\nu_g > 0$ such that for all $\beta > 0$ large, $\mathsf{y} \in \mathcal{Y}_\delta^{\blacktriangleleft} \setminus \{0\}$, and $k \geq 1$,

$$\sum_{\Gamma \in \mathsf{A}_\mathsf{L} \cup \mathsf{A}_\mathsf{R}} \mathbb{P}^{\mathsf{h}_\mathsf{y}}(\Gamma) \mathbb{1}_{\{|\Gamma| \geq k\}} \leq C e^{-\nu_g \beta k}\,, \tag{2.26}$$

where $C := C(\beta) > 0$ is a positive constant.

Note that, since $\mathsf{h}_\mathsf{y} \cdot \mathsf{y} = \tau_\beta(\mathsf{y})$, we have the following from Eq. (2.18)

$$q(\Gamma) = e^{-\tau_\beta(\mathsf{y})} \mathbb{P}^{\mathsf{h}_\mathsf{y}}(\Gamma^{(L)}) \mathbb{P}^{\mathsf{h}_\mathsf{y}}(\Gamma^{(R)}) \prod_{i=1}^{n} \mathbb{P}^{\mathsf{h}_\mathsf{y}}(\Gamma^{(i)})\,, \tag{2.27}$$

for any Γ with at least two cone-points. For $\mathsf{u}, \mathsf{v} \in \mathbb{Z}^2$ and a set of animals E, introduce

$$\mathcal{A}(\mathsf{u}, \mathsf{v}; E) = \sum_{n \geq 1} \sum_{\Gamma^{(1)}, \dots, \Gamma^{(n)} \in \mathsf{A}} \prod_{i=1}^{n} \mathbb{P}^{\mathsf{h}_\mathsf{y}}(\Gamma^{(i)}) \mathbb{1}_{\{\mathsf{u} + \gamma^{(1)} \circ \cdots \circ \gamma^{(n)} \in \mathcal{P}_{\mathbb{H}}(\mathsf{u},\mathsf{v})\}} \mathbb{1}_{\{\mathsf{u} + \Gamma^{(1)} \circ \cdots \circ \Gamma^{(n)} \in E\}}\,. \tag{2.28}$$

When E is taken to be the set of all possible animals, we'll simply write $\mathcal{A}(\mathsf{u}, \mathsf{v}; E) = \mathcal{A}(\mathsf{u}, \mathsf{v})$. Then, for $D = Q$ or $\mathbb{H}$, Eq. (2.27) allows us to write

$$\begin{aligned} &\mathcal{G}(\mathsf{y} \mid \gamma \subset D, |\mathsf{Cpts}(\Gamma)| \geq 2) \\ &\quad = e^{-\tau_\beta(\mathsf{y})} \sum_{\substack{\Gamma^{(L)} \in \mathsf{A}_\mathsf{L} \\ \gamma^{(L)} \subset D}} \mathbb{P}^{\mathsf{h}_\mathsf{y}}(\Gamma^{(L)}) \sum_{\substack{\Gamma^{(R)} \in \mathsf{A}_\mathsf{R} \\ \gamma^{(R)} \subset D}} \mathbb{P}^{\mathsf{h}_\mathsf{y}}(\Gamma^{(R)})\ \mathcal{A}(\mathsf{X}(\Gamma^{(L)}), \mathsf{y} - \mathsf{X}(\Gamma^{(R)}))\,. \end{aligned} \tag{2.29}$$

Note that the sum over n in (2.28) above closely resembles the probability of a random walk event. This product structure of probability weights $\mathbb{P}^{\mathsf{h}_\mathsf{y}}$ naturally leads to considering the random walk model related to the law of $\mathsf{Cpts}(\Gamma)$ described in Section 4.1.

3. Proof of Theorem 1.1 modulo Theorem 1.2

In this section, we infer Theorem 1.1 from Theorem 1.2(a). Throughout the section, let $H_L = \lfloor \frac{1}{4\beta} \log L \rfloor$. Recall that, as per Eq. (1.4), we wish to give a lower bound on the event

$$E := \Big\{ \min_{x \in I_0} \underline{\rho}(x) \geq \delta\, L^{1/3} \Big\}\,.$$

The fact that the definition of $\underline{\rho}(x)$ pertains to the height-$\mathfrak{h}^*$ (top) level line, where $\mathfrak{h}^*$ is random in $\{H_{L-1}, H_L\}$ (as mentioned in the Introduction, the results of [14] determine it w.h.p. for side lengths L outside of a critical subsequence) make this event delicate to work with. We will derive the required lower bound on E via more tractable events, tailored to the analysis of a local rectangle of side-length $\asymp L^{2/3}$ located at the center of the bottom side of Λ_L.

Let

$$R_0 := [\![\tfrac{L}{2} - 3L^{2/3}, \tfrac{L}{2} + 3L^{2/3}]\!] \times [\![0, 2L^{2/3}]\!]\,,$$

and let $R_1 \subset \Lambda_L$ be the set of $\mathsf{x} \in \Lambda_L$ at distance at most $(\log L)^2 + 1$ from R_0. For fixed $\delta > 0$, define the horizontal line $\mathfrak{L} := I_0 \times \{\lfloor \delta L^{1/3} \rfloor\})$. Theorem 1.1 will follow from the next proposition:

Proposition 3.1. *Fix $\beta \geq \beta_0$. Let B_n be the event that there exists a simple path in R_0 of length $\geq (\log L)^2$ that intersects the line $\mathfrak{L}$, and whose sites have height $\geq H_L - n$. Let C_n be the event that there is no simple path in R_1 of length $\geq (\log L)^2$ whose sites have height $\geq H_L - n + 1$ in R_1. Then, for every fixed $n \geq 0$ and $\epsilon > 0$ there exists $\delta > 0$ such that $\pi^0_{\Lambda_L}(B_n \cap C_n) < \epsilon$ for all L large.*

Proof of Theorem 1.1. Observe that

$$\pi^0_{\Lambda_L}(E^c,\, \mathfrak{h}^* = H_L) \leq \pi^0_{\Lambda_L}(B_0 \cap C_0) + o(1)\,.$$

Indeed, E^c and [14, Theorem 2a] imply that the height-$\mathfrak{h}^*$ level line loop $\mathcal{L}^*$ must encompass a path crossing $\mathcal{L}$ as in B_0, while C_0 holds w.h.p. (also by [14, Theorem 2a]). Similarly,

$$\pi^0_{\Lambda_L}(E^c,\, \mathfrak{h}^* = H_L - 1) \leq \pi^0_{\Lambda_L}(B_1 \cap C_1) + o(1)\,,$$

again using that the height-$\mathfrak{h}^*$ level line loop $\mathcal{L}^*$ must encompass a path crossing $\mathfrak{L}$ as in B_1 and that C_1 holds w.h.p. when $\mathfrak{h}^* = H_L - 1$, by [14, Theorem 2a]. The conclusion now follows from Proposition 3.1. ☐

Proof of Proposition 3.1. Fix $n \geq 0$ and $\epsilon > 0$. The following procedure will reveal the "outermost" chain $\mathfrak{C}$ of $\mathbb{Z}^2$-connected sites of height at most $H_L - n$ that encloses, together with the southern boundary of Λ_L, the box R_0. For each site x in the north, east, and west boundaries of R_1, reveal its $\mathbb{Z}^2$-connected component of sites $\mathsf{y} \in R_1$ for which $\varphi_\mathsf{y} > H_L - n$. Let U denote the collection of all revealed vertices, and let $\mathfrak{C}$ be the exterior boundary of U contained in R_1. By definition, $\varphi_\mathsf{y} \leq H_L - n$ for all $\mathsf{y} \in \mathfrak{C}$. Now, let W denote the collection of sites whose exterior boundary is formed by $\mathfrak{C}$ and the southern boundary of Λ_L. On the event C_n, each connected component comprising U has diameter strictly less than $(\log L)^2$; in particular, $R_0 \subset W$.

Now, condition on $\mathfrak{C}$, and let $\xi(\mathfrak{C})$ denote the boundary condition of W given by the heights on $\mathfrak{C}$ and the sites of height 0 on $\partial\Lambda$. From the domain Markov property, we may deduce our desired bound $\pi^0_{\Lambda_L}(B_n \cap C_n) < \epsilon$ if we can show

$$\pi^{\xi(\mathfrak{C})}_W(B_n) < \epsilon\,,$$

uniformly over $\mathfrak{C}$. Define $\mathsf{x}_\ell := \lfloor \frac{L}{2} - 2L^{2/3} \rfloor$ and $\mathsf{x}_r := \mathsf{x}_\ell + \lfloor 4L^{2/3} \rfloor$. Using that all heights on $\mathfrak{C}$ are bounded by $H_L - n$, that all heights on $\partial\Lambda_L$ are 0 (in particular, bounded by $H_L - n - 1$), and that B_n is an increasing event, we have

$$\pi^{\xi(\mathfrak{C})}_W(B_n) \leq \pi^{\texttt{legs}}_W(B_n)\,, \tag{3.1}$$

where the boundary conditions `legs` are $H_L - n - 1$ on the horizontal line of sites $[\![\mathsf{x}_\ell, \mathsf{x}_r]\!] \times \{-1\}$ and $H_L - n$ elsewhere (see Fig. 2, noting the "legs" of sites of height $H_L - n$ protruding inwards from the bottom boundary of W). Thus, it suffices to show the right-hand is bounded by ϵ.

Moving to $\pi^{\texttt{legs}}_W$ has two main advantages. The first is that there exists a *unique* open $H_L - n$-contour γ under this measure, and in particular, the sites of height $\geq H_L - n$ along γ contain a simple path of length larger than $(\log L)^2$ (typically of order $L^{2/3}$). Further, a classical Peierls argument shows that, w.h.p. under $\pi^{\texttt{legs}}_W$, there are no closed contours of length larger than $\log L$. In particular, defining

$$\underline{\rho}^\gamma(x) := \min\{y \geq 0 : (x,y) \in \gamma\} \text{ for } x \in \mathbb{R},$$

we have

$$\pi^{\texttt{legs}}_W(B_n) = \pi^{\texttt{legs}}_W\big(\min_{x \in I_0} \underline{\rho}^\gamma(x) < \delta L^{1/3}\big) + o(1)\,. \tag{3.2}$$

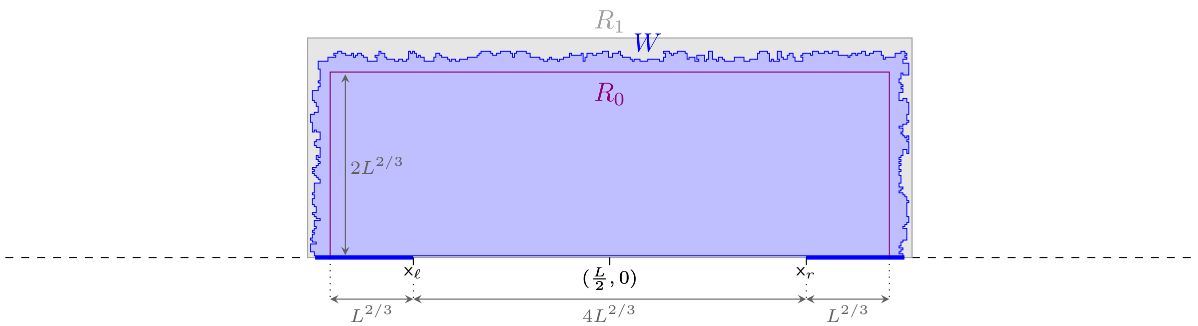

FIGURE 2. The rectangles R_0 (purple) and R_1 (gray), and in between the region W (blue) whose boundary conditions are raised via monotonicity to $H_L - n$ except on the segment between x_ℓ and x_r, where they are set to $H_L - n - 1$.

The second advantage of moving to π_W^{legs} is that we may couple γ under the no-floor measure $\widehat{\pi}_W^{\mathsf{legs}}$ with a modified Ising polymer law in the half-space $\mathbf{P}_{\mathbb{H}}^{\mathsf{x}}$ (see Definition 2.6), for which we have Theorem 1.2. After exhibiting this in Claim 3.2, we compare $\widehat{\pi}_W^{\mathsf{legs}}$ and π_W^{legs} via Eq. (A.8).

We begin with the argument relating $\widehat{\pi}_W^{\mathsf{legs}}$ to $\mathbf{P}_{\mathbb{H}}^{\mathsf{x}}$. From now on, we shift W to the left so that $\mathsf{x}_\ell = 0$ and $\mathsf{x}_r = \lfloor 4L^{2/3} \rfloor$. Note that γ is a contour in W with start-point $\mathsf{o}^* \in (\mathbb{Z}^2)^*$ and end-point $\mathsf{x} := \mathsf{x}_r - (1/2, 1/2)$. We will couple $\widehat{\pi}_W^{\mathsf{legs}}$ with the Ising polymer law $\mathbf{P}_{\mathbb{H}}^{\mathsf{x}}$, defined using the SOS decoration weights $\Phi(\mathcal{C};\gamma) := \phi(\mathcal{C};\gamma)$ and $\Phi_{\mathbb{H}}(\mathcal{C};\gamma) := \phi(\mathcal{C};\gamma)\mathbb{1}_{\{\mathcal{C}\subset\mathbb{H}\}}$ from Proposition 2.1. This coupling is natural in light of the very far distance of $\mathfrak{C}$ from both o^* and x. Define $N := \lfloor 4L^{2/3} \rfloor$. Recall the set $\mathcal{P}_D(\mathsf{u},\mathsf{v})$ from Section 2.5. Let $\bar{W}$ denote the region in $\mathbb{R}^2$ enclosed by W, i.e., the region enclosed by the $\mathbb{Z}^2$-edges connecting the boundary vertices of W. Then $\gamma \in \mathcal{P}_{\bar{W}}(\mathsf{o}^*,\mathsf{x})$. Lower the boundary conditions of $\widehat{\pi}_W^{\mathsf{legs}}$ to $0,1,1,1$ using the shift-invariance of the no-floor model. Then Observation 2.7 states that $\gamma \sim \widehat{\pi}_W^{\mathsf{legs}}$ has the Ising polymer law $\mathbf{P}_{\bar{W}}^{\mathsf{x}}$, and the decoration weights are defined via $\Phi(\mathcal{C};\gamma) := \phi(\mathcal{C};\gamma)$ and $\Phi_{\bar{W}}(\mathcal{C};\gamma) := \phi(\mathcal{C};\gamma)\mathbb{1}_{\{\mathcal{C}\subset\bar{W}\}}$.

Claim 3.2. *Extend $\widehat{\pi}_W^{\mathsf{legs}}$ to a probability measure on all contours in $\mathcal{P}_{\mathbb{H}}(\mathsf{o}^*,\mathsf{x})$ via $\widehat{\pi}_W^{\mathsf{legs}}(\gamma) = 0$ for all $\gamma \notin \mathcal{P}_{\bar{W}}(\mathsf{o}^*,\mathsf{x})$. For some constant $C > 0$ and for all $\beta > 0$ large enough, we have*

$$\big\|\mathbf{P}_{\mathbb{H}}^{\mathsf{x}} - \widehat{\pi}_W^{\mathsf{legs}}\big\|_{\mathrm{TV}} \le Ce^{-\beta C^{-1}N}\,.$$

Proof of Claim 3.2. We use the identification of the law of the contour induced by $\widehat{\pi}_W^{\mathsf{legs}}$ with $\mathbf{P}_{\bar{W}}^{\mathsf{x}}$ throughout this proof.

The main input for the proof is the following: for any $\delta \in (0,1)$ and for all $\beta, N > 0$ large,

$$\max\Big(\widehat{\pi}_W^{\mathsf{legs}}(|\gamma| \ge (1+\delta)N)\,,\, \mathbf{P}_{\mathbb{H}}^{\mathsf{x}}(|\gamma| \ge (1+\delta)N)\Big) \le e^{-\beta\delta N/2}\,. \tag{3.3}$$

We prove Eq. (3.3) now. Using the decay of $\phi(\mathcal{C};\gamma)$ (Proposition 2.1(iv)) as in Eq. (2.11), we have the following for any $\gamma : \mathsf{o}^* \to \mathsf{x}$:

$$\sum_{\mathcal{C}\cap\Delta_\gamma\neq\emptyset} |\phi(\mathcal{C};\gamma)| \le \sum_{\mathsf{y}\in\Delta_\gamma}\sum_{\mathcal{C}\ni\mathsf{y}} \sup_\gamma |\phi(\mathcal{C};\gamma)| \le 3|\gamma|e^{-(\beta-\beta_0)}\,. \tag{3.4}$$

Since $|\gamma|$ is always larger than N, we have

$$\min\Big(\mathcal{G}_{\mathbb{H}}(\mathsf{x})\,,\, \mathcal{G}_{\bar{W}}(\mathsf{x})\Big) \ge \exp\Big(-(\beta + 3e^{-(\beta-\beta_0)})N\Big)\,. \tag{3.5}$$

On the other hand, Eq. (3.4) implies

$$\max\Big(\mathcal{G}_{\mathbb{H}}(\mathsf{x}\mid|\gamma|\geq(1+\delta)N)\,,\,\mathcal{G}_{\bar{W}}(\mathsf{x}\mid|\gamma|\geq(1+\delta)N)\Big)$$
$$\leq\sum_{\substack{\gamma\subset\mathbb{H}\\|\gamma|\geq(1+\delta)N}}\exp\Big(-(\beta-3e^{-(\beta-\beta_0)})|\gamma|\Big)\leq e^{-(\beta-c)(1+\delta)N}\,,$$

where $c>0$ is a constant independent of β and N. The above two bounds yield Eq. (3.3). Note that a consequence of (3.3) is

$$\widehat{\pi}_W^{\texttt{legs}}(d_{L^\infty}(\gamma,\mathfrak{C})\leq(1-2\delta)N)\leq e^{-\beta\delta N/2}\,. \tag{3.6}$$

Now, fix any $\delta\in(0,1/2)$. Observe that

$$q_{\mathbb{H}}(\gamma)/q_{\bar{W}}(\gamma)=\exp\Big(\sum_{\mathcal{C}\cap\Delta_\gamma\neq\emptyset}\phi(\mathcal{C};\gamma)\mathbb{1}_{\{\mathcal{C}\subset\mathbb{H}\setminus W\}}\Big)$$

For any contour γ such that $d_{L^\infty}(\gamma,\mathfrak{C})>(1-2\delta)N$, we have the following for all $\beta>0$ large:

$$\sum_{\mathcal{C}\cap\Delta_\gamma\neq\emptyset}|\phi(\mathcal{C};\gamma)|\mathbb{1}_{\{\mathcal{C}\subset\mathbb{H}\setminus W\}}\leq\sum_{\mathsf{y}\in\Delta_\gamma}\sum_{\mathcal{C}\ni\mathsf{y}}\sup_\gamma|\phi(\mathcal{C};\gamma)|\leq|\gamma|e^{-c\beta N}\,,$$

where in the last inequality, $c>0$ is a constant independent of N and β large, and the bound follows similarly to Eq. (3.4). For any $|\gamma|<(1+\delta)N$, Eq. (3.4) yields an upper bound on $\mathcal{G}_{\mathbb{H}}(\mathsf{x})$ of $\exp(-(\beta-3(1+\delta)e^{-(\beta-\beta_0)})N)$. This upper bound, the lower-bound on $\mathcal{G}_{\bar{W}}(\mathsf{x})$ in (3.5), and the previous display imply the following for any $|\gamma|<(1+\delta)N$:

$$\Big|\widehat{\pi}_W^{\texttt{legs}}(\gamma)-\mathbf{P}_{\mathbb{H}}^{\mathsf{x}}(\gamma)\Big|\leq e^{-c'\beta N}\,,$$

where $c'>0$ is a constant independent of β and N. Since the number of connected paths in $(\mathbb{Z}^2)^*$ rooted at o^* of length at most $(1+\delta)N$ is trivially bounded $4^{(1+\delta)N}$, the previous display along with Eq. (3.3) imply that the total variation distance between $\widehat{\pi}_W^{\texttt{legs}}$ viewed as a probability measure on $\mathcal{P}_{\mathbb{H}}(\mathsf{o}^*,\mathsf{x})$ and $\mathbf{P}_{\mathbb{H}}^{\mathsf{x}}$ is bounded by $e^{-c''\beta N}$. ☐

Now, from Eq. (A.8), we have

$$\pi_{\tilde{R}_0}^{\texttt{legs}}(\gamma)\propto\widehat{\pi}_{\tilde{R}_0}^{\texttt{legs}}(\gamma)\exp\left(-\frac{\lambda^{(n)}}{L}A(\gamma)+o(1)\right)\,, \tag{3.7}$$

where $A(\gamma)$ denotes the area under γ in W. From Eq. (3.1) and Eq. (3.2), the claim will be proven if we can show that for any $\epsilon>0$, there exists $\delta>0$ such that

$$\pi_W^{\texttt{legs}}(\min_{x\in I_0}\underline{\rho}^\gamma(x)<\delta L^{1/3})<\epsilon \tag{3.8}$$

for L sufficiently large. Theorem 1.1 and Claim 3.2 imply that, under horizontal rescaling by $|\mathsf{x}|_1=\lfloor 4L^{2/3}\rfloor$ and vertical rescaling by $\lfloor 4L^{2/3}\rfloor^{1/2}$, γ under $\widehat{\pi}_W^{\texttt{legs}}$ converges weakly to a Brownian excursion as $L\to\infty$. In particular, $\widehat{\pi}_W^{\texttt{legs}}(\min_{x\in I_0}\underline{\rho}^\gamma(x)<\delta L^{1/3})$ can be made arbitrarily close to 0 by taking δ small enough. Letting $\tilde{\mathbb{E}}$ denote expectation with respect to $\widehat{\pi}_W^{\texttt{legs}}$, we have

$$\pi_W^{\texttt{legs}}(\min_{x\in I_0}\underline{\rho}^\gamma(x)<\delta L^{1/3})=\frac{\tilde{\mathbb{E}}\Big[\mathbb{1}_{\{\min_{x\in I_0}\underline{\rho}^\gamma(x)<\delta L^{1/3}\}}e^{-\frac{\lambda^{(n)}}{L}A(\gamma)+o(1)}\Big]}{\tilde{\mathbb{E}}\Big[e^{-\frac{\lambda^{(n)}}{L}A(\gamma)+o(1)}\Big]}\,.$$

The denominator is bounded away from 0 because $\lambda^{(n)}$ is bounded for fixed β and the Brownian excursion limit implies that $A(\gamma)$ is unlikely to be much bigger than $L^{2/3}\cdot L^{1/3}=L$ (making the expectation strictly positive). Hence, since the exponential in the numerator is bounded above by $1+o(1)$, we get

$$\pi_W^{\mathtt{legs}}\big(\min_{x\in I_0}\underline{\rho}^\gamma(x)<\delta L^{1/3}\big)\le C\widehat{\pi}_W^{\mathtt{legs}}\big(\min_{x\in I_0}\underline{\rho}^\gamma(x)<\delta L^{1/3}\big)\,,$$

for some C depending only on β and L large enough. Since the right-hand side can be made arbitrarily small by taking δ small enough, as discussed above, this implies Eq. (3.8), thereby concluding the proof. ☐

4. The effective random walk model, free Ising polymers, and limit theorems

The factorization of $q(\Gamma)$ from (2.27) and the input from Ornstein-Zernike theory (Proposition 2.14) naturally leads one to consider the two-dimensional effective random walk defined in Section 4.1. In that subsection, we further expose the connection between this effective random walk and the cone-points of the *free* Ising polymer model in $\mathbb{H}$. A key result there is Proposition 4.1, which enables a quantitative comparison between the partition function $\mathcal{G}(\mathsf{x}\mid\gamma\subset\mathbb{H})$ and the probability that the random walk stays in $\mathbb{H}$. In particular, we will be able to rule out bad events for the cone-points of the free Ising polymer in $\mathbb{H}$ using random walk estimates. In the next Section, we develop comparison results between free and modified Ising polymers such that bad events for the cone-points of modified Ising polymers can also be ruled out using random walk estimates. Eventually, this will lead to a coupling between the cone-points of the modified Ising polymer and our random walk.

In the rest of this Section, we develop various limit theorems for our random walk that will be used in the analysis of (modified) Ising polymers in Section 5, culminating in the proof of Theorem 2.8 (and thus Theorem 1.2).

4.1. The random walk model and free Ising polymers.

Consider the random walk $(\mathsf{S}(i))_{i\in\mathbb{Z}_{\ge0}}$, whose law will be denoted by $\mathbb{P}$, with i.i.d. increments $\{X(i)=(X_1(i),X_2(i))\}_{i\ge1}$ of step distribution X, where for any $\vec{v}\in\mathbb{Z}^2$,

$$\mathbb{P}(X=\vec{v})=\sum_{\Gamma\in\mathsf{A}}\mathbb{P}^{\mathsf{h}_\mathsf{x}}(\Gamma)\mathbb{1}_{\{\mathsf{X}(\Gamma)=\vec{v}\}}\,. \tag{4.1}$$

For $\mathsf{u}\in\mathbb{Z}^2$, we will write $\mathbb{P}_\mathsf{u}$ to denote the law of $\mathsf{S}(\cdot)$ started from $\mathsf{S}(0)=\mathsf{u}$. From Eq. (2.25), we have $\mathbb{E}[X]=\alpha\mathsf{x}$, and from Eq. (2.26), we inherit exponential tail decay: for all β sufficiently large, there exists a constant $C':=C'(\beta)>0$ such that for all $k\ge1$, we have

$$\sum_{\vec{v}\in\mathbb{Z}^2:\|\vec{v}\|_1\ge k}\mathbb{P}(X=\vec{v})\le C'e^{-\nu_g\beta k}\,. \tag{4.2}$$

We remark that the step-distribution of X need not be symmetric in the y-coordinate X_2 due to the northeast splitting rule (this is the key difference between our random walk and the random walk considered in [31], and why we cannot use their random walk results here). Also note that $\mathbb{P}(X\in\mathcal{Y}^\blacktriangleleft)=1$, and thus the x-coordinate $X_1>0$ a.s.

We will write H_E^A to denote the first hitting time of a set $E\subset\mathbb{R}^2$ by a stochastic process $A_\cdot$, omitting the A from the notation when the stochastic process is clear. For singleton sets, we will simply write H_u^A rather than $H_{\{\mathsf{u}\}}^A$. Let $\mathbb{H}_-:=\mathbb{R}\times\mathbb{R}_{<0}$. In light of the factorization (2.27), for fixed $\Gamma^{(L)}$ and $\Gamma^{(R)}$, we will be interested in our random walk $\mathsf{S}(\cdot)$ started from $\Gamma^{(L)}$ and conditioned on the event

$$\{H_{\mathsf{x}-X(\Gamma^{(R)})}<H_{\mathbb{H}_-}\}\,.$$

It will therefore be helpful to obtain an expression for $\mathbb{P}_{\mathsf{u}}(H_{\mathsf{v}} < H_{\mathbb{H}_-})$ in terms of animal weights, for any $\mathsf{u}, \mathsf{v} \in \mathbb{Z}^2$. Towards this end, define the sets

$$\begin{aligned} \mathsf{V}^+_{\mathsf{u},\mathsf{v}} &:= \{(\vec{v}_1, \dots, \vec{v}_n) \in (\mathbb{Z}^2)^n : n \geq 1, \mathsf{u} + \vec{v}_1 + \dots + \vec{v}_m = \mathsf{v}, \mathsf{u} + \vec{v}_1 + \dots + \vec{v}_i \in \mathbb{H} \text{ for all } 1 \leq i \leq n\} \\ \mathsf{A}^+_{\mathsf{u},\mathsf{v}} &:= \{(\Gamma^{(1)}, \dots, \Gamma^{(n)}) \in \mathsf{A}^n : n \geq 1, (X(\Gamma^{(1)}), \dots, X(\Gamma^{(n)})) \in \mathsf{V}^+_{\mathsf{u},\mathsf{v}}\}, \end{aligned} \tag{4.3}$$

and, for a set of animals E, introduce

$$\mathcal{A}^+(\mathsf{u}, \mathsf{v}; E) = \sum_{n \geq 1} \sum_{(\Gamma^{(1)}, \dots, \Gamma^{(n)}) \in \mathsf{A}^+_{\mathsf{u},\mathsf{v}}} \prod_{i=1}^n \mathbb{P}^{\mathsf{h}_{\mathsf{x}}}(\Gamma^{(i)}) \mathbb{1}_{\{\Gamma^{(1)} \circ \dots \circ \Gamma^{(n)} \in E\}}.$$

When E is taken to be the set of all possible animals, we'll simply write $\mathcal{A}^+(\mathsf{u}, \mathsf{v}; E) = \mathcal{A}^+(\mathsf{u}, \mathsf{v})$. Observe that

$$\mathbb{P}_{\mathsf{u}}(H_{\mathsf{v}} < H_{\mathbb{H}_-}) = \sum_{n \geq 1} \sum_{(\vec{v}_1, \dots, \vec{v}_n) \in \mathsf{V}^+_{\mathsf{u},\mathsf{v}}} \prod_{i=1}^n \mathbb{P}(X = \vec{v}_i) = \mathcal{A}^+(\mathsf{u}, \mathsf{v}). \tag{4.4}$$

In words, the above equation states that the probability of the random walk hitting v before leaving $\mathbb{H}_-$ is equal to a sum over all possible products of weights of irreducible animals whose concatenation has all cone-points in $\mathbb{H}$.

Recall $\mathcal{A}(\cdot, \cdot; \cdot)$ from (2.28), and note that for any $\mathsf{u}, \mathsf{v} \in \mathbb{Z}^2$ and any set of animals E,

$$\mathcal{A}(\mathsf{u}, \mathsf{v}; E) \leq \mathcal{A}^+(\mathsf{u}, \mathsf{v}; E), \tag{4.5}$$

since the (only) difference between $\mathcal{A}^+(\mathsf{u}, \mathsf{v}; E)$ and $\mathcal{A}(\mathsf{u}, \mathsf{v}; E)$ is that the sum defining $\mathcal{A}(\mathsf{u}, \mathsf{v}; E)$ restricts to tuples of irreducible animals such that the *entire concatenation* $\mathsf{u} + \Gamma^{(1)} \circ \dots \circ \Gamma^{(n)}$ is contained in $\mathbb{H}$, while the sum defining $\mathcal{A}^+(\mathsf{u}, \mathsf{v}; E)$ requires only the *cone-points* to stay in $\mathbb{H}$. In particular, $\mathsf{Cpts}(\Gamma)$ (viewed as an ordered tuple, see Eq. (2.20)) under the conditioned law $\mathbf{P}^{\mathsf{x}}(\cdot \mid \Gamma^{(L)}, \Gamma^{(R)}, \gamma \subset \mathbb{H})$ and the trajectory of the random walk $\mathbb{P}_{\mathsf{X}(\Gamma^{(L)})}(\cdot \mid \mathbb{H}_{\mathsf{x}-\mathsf{X}(\Gamma^{(R)})} < \mathbb{H}_{\mathbb{H}_-})$ only differ due to these requirements.

Ultimately, we will achieve a coupling between $\mathsf{Cpts}(\Gamma)$ with the trajectory of the random walk because of an entropic repulsion result (Proposition 5.11): the cone-points of Γ in the "bulk" of the strip $[0, N] \times [0, \infty)$ stay far away from the boundary of $\mathbb{H}$ with high probability, so that the aforementioned difference between $\mathcal{A}^+$ and $\mathcal{A}$ becomes negligible. A major step towards showing entropic repulsion is is the following crucial result from [32].[9]

Proposition 4.1 ([32, Theorem 7])**.** *Recall ν_g from Proposition 2.14. There exists $\bar{\delta} \in (0, \nu_g/4)$ and $\beta_0 > 0$ such that for all $\beta > \beta_0$, there exists a constant $C := C(\beta) > 0$ such that*

$$\sup_{\substack{\mathsf{u}, \mathsf{x}-\mathsf{v} \in \mathcal{Y} \\ \mathsf{u}, \mathsf{v} \in \mathbb{H}}} e^{-\bar{\delta}\beta(\|\mathsf{u}\|_1 + \|\mathsf{x}-\mathsf{v}\|_1)} \mathbb{P}_{\mathsf{u}}(H_{\mathsf{v}} < H_{\mathbb{H}_-}) \leq C e^{\tau_\beta(\mathsf{x})} \mathcal{G}(\mathsf{x} \mid \gamma \subset \mathbb{H}).$$

Proposition 4.1 combined with the exponential tail decay of $\Gamma^{(L)}$ and $\Gamma^{(R)}$ (2.26) (note the significance of $\bar{\delta} < \nu_g/4$) will allow us to eliminate bad events for $\mathsf{Cpts}(\Gamma)$ under the free Ising polymer law via estimates for random walks in a half-space, which are significantly easier to obtain compared to directly analyzing the polymer law.

[9] The notation in [32] differs from ours: note that their quantity $\mathbb{P}^{\mathsf{h}_{\mathsf{x}}}(\mathsf{u}, \mathsf{v})$ (defined in [32, Eq. 5.10]) is exactly our $\mathcal{A}(\mathsf{u}, \mathsf{v})$. We have also used Eqs. (4.4) and (4.5) to replace $\mathcal{A}(\mathsf{u}, \mathsf{v})$ by $\mathbb{P}_u(H_{\mathsf{v}} < H_{\mathbb{H}_-})$.

4.2. **Limit theorems for our random walk, confined to the half-space.** We now state two important limit theorems, Theorems 4.2 and 4.3, for our random walk $\mathsf{S}(\cdot)$ confined to positive half-space, though our results apply to a more general class of two-dimensional random walks. Our random walk $\mathsf{S}(\cdot)$ is on $\mathbb{Z}^2$ and has i.i.d. increments X with mean $(\mu, 0)$ for some $\mu > 0$, exponential tails, and satisfies $X_1 > 0$ a.s. Let us also write $\mathrm{Var}(X) = (\sigma_1^2, \sigma_2^2)$.

Let V_1 denote the unique positive harmonic function for the one-dimensional random walk S_2 killed upon leaving $(0,\infty)$ satisfying

$$\lim_{a\to\infty} \frac{V_1(a)}{a} = 1\,.$$

The uniqueness of V_1 was established by Doney in [22]. Similarly, let V_1' denote the analogous harmonic function for the random walk $-\mathsf{S}_2$.

Theorem 4.2. *Fix any $A > 0$ and any $\delta \in (0, 1/2)$. Uniformly over $k \in [N/\mu - A\sqrt{N}, N/\mu + A\sqrt{N}]$ and $u, v \in (0, N^{1/2-\delta}] \cap \mathbb{N}$, we have the following results.*

(1) There exist constants $\mathbf{C} > 0$ and $\kappa := \kappa(X) > 0$ such that

$$\mathbb{P}_{(0,u)}\Big(\mathsf{S}(k) = (N, v)\,,\, H_{\mathbb{H}_-} > k\Big) \sim \mathbf{C}\kappa \frac{V_1(u)V_1'(v)}{k^2} \exp\Big(-\frac{(N-k\mu)^2}{2k\sigma_1^2}\Big)\,. \tag{4.6}$$

Furthermore, if

$$p_{N,A} := \mathbb{P}_{(0,u)}\Big(H_{(N,v)} \in [\tfrac{N}{\mu} - A\sqrt{N}, \tfrac{N}{\mu} + A\sqrt{N}] \mid H_{(N,v)} < H_{\mathbb{H}_-}\Big)$$

then

$$\lim_{A\to\infty} \liminf_{N\to\infty} p_{N,A} = 1\,. \tag{4.7}$$

(2) Let $\widehat{\mathsf{S}}_2(\cdot)$ denote the denote the linear interpolation of the points $(n, \mathsf{S}_2(n))_{n\in\mathbb{N}}$, viewed as an element of $C[0,\infty)$. The family of conditional laws

$$\mathbb{Q}^k_{u,v}(\cdot) := \mathbb{P}_{(0,u)}\Big(\Big(\frac{\widehat{\mathsf{S}}_2(tk)}{\sigma_2\sqrt{k}}\Big)_{t\in[0,1]} \in \cdot \mid \mathsf{S}(k) = (N, v)\,,\, H_{\mathbb{H}_-} > k\Big)$$

converges as $k \to \infty$ to the law of the standard Brownian excursion in $C[0, 1]$.

Next, we describe the diffusive limit of our random walk. For $\ell, n \in \mathbb{N}$ and any ordered ℓ-tuple of points $\mathcal{S}$ in $\mathbb{R}^2$, enumerated as

$$\mathcal{S} = \big((s_1(1), s_2(1)), (s_1(2), s_2(2)), \ldots, (s_1(\ell), s_2(\ell))\big)$$

and satisfying $s_1(i) < s_1(i+1)$ for each $i \in [1, \ell-1]$, we define $\mathfrak{I}_n(\mathcal{S})$ to be the linear interpolation through the diffusively-rescaled points

$$\mathfrak{I}_n(\mathcal{S}) := \Big(\frac{1}{n}s_1(i), \frac{1}{\sigma\sqrt{n}}s_2(i)\Big)_{i=1}^k \tag{4.8}$$

where $\sigma^2 := \sigma_2^2/\mu$ (c.f. Remark 2.9). Consider now the linear interpolation corresponding to our random walk $\mathsf{S}(\cdot)$ up to time $H_{(N,v)}$:

$$\mathfrak{e}_N^{\mathsf{S},v} := \mathfrak{I}_N\Big((\mathsf{S}(i))_{i=0}^{H_{(N,v)}}\Big)\,,$$

Viewing $\mathfrak{e}_N^{\mathsf{S},v}$ as a random element of $C[0, 1]$, Theorem 4.3 below gives a convergence result to the Brownian excursion on $[0, 1]$.

Theorem 4.3. *Fix any $\delta \in (0, 1/2)$. Uniformly over $u, v \in (0, N^{1/2-\delta}] \cap \mathbb{N}$, we have the following. The family of conditional laws*

$$\mathbf{Q}^N_{u,v}(\cdot) := \mathbb{P}_{(0,u)}\Big((\mathfrak{e}_N^{\mathsf{S},v}(t))_{t\in[0,1]} \in \cdot \mid H_{(N,v)} < H_{\mathbb{H}_-}\Big)\,,$$

converges weakly as $N \to \infty$ to the law of the standard Brownian excursion in $(C[0, 1], \|\cdot\|_\infty)$.

Theorems 4.2 and 4.3 are proved in Section 6.3. These results are modifications of a ballot-type theorem from [17] and an invariance principle from [23], respectively. Those important results were proved for a much broader class of random walks in very general cones; however, they were only proved for fixed start-points and end-points. The two results below have been modified to hold uniformly in appropriate ranges of start-points and end-points, which is crucial for our application. We remark that [31, Section 5] states and proves results that similarly modify the results of [17] and [23]. However, first, the random walk they consider is symmetric in the y-coordinate, which simplifies their analysis; and second, their proofs do not always make explicit the aforementioned uniformity, in particular for the x-coordinate of the end-point of their random walk. In Section 6, where our random walk estimates are proved, we take care to describe explicitly how we modify the proofs in [17] and [23] to handle uniformity in a broad range of start- and end-points.

5. Proof of Theorem 1.2 via random walk coupling

Throughout this Section, set $\mathsf{x} := \mathsf{x}(N) = (N, 0) \in \mathbb{Z}^2$. Recall the notation set forth in Section 2.5.

In this section we prove Theorem 2.8, thereby proving Theorem 1.2, via a certain coupling between $\mathsf{Cpts}(\Gamma)$ under the modified Ising polymer law $\mathbf{P}_D^{\mathsf{x}}$ and the effective random walk, where $D = Q$ or $\mathbb{H}$. This is accomplished as follows.

In Section 4.1, we described the particular role played by the free Ising polymer model in the half-space due to its direct connection with an effective random walk. It is then crucial for our analysis to be able to compare the modified Ising polymers we are interested in with free Ising polymers. In Section 5.1, we state our needed comparison results, Theorem 5.1 (the main theorem of [32]) and Proposition 5.2, the latter being an extension of the former that allows us to handle the $D = Q$ case. The key consequence of these comparisons is that contour events of small probability in one model are still small in the other model (Corollary 5.3).

In Section 5.2, we prove Proposition 5.2, showing along the way that Γ under the modified Ising polymer law in D typically has many cone-points (Lemma 5.7). In Section 5.3, we prove Proposition 2.5 in a similar way to Proposition 5.2.

In Section 5.4, we show that each (left/right-)irreducible piece $\Gamma^{(i)}$ has size bounded by $(\log N)^2$, $i \in \{L, 1, \dots, n, R\}$, w.h.p. under $\mathbf{P}_D^{\mathsf{x}}$.

In Section 5.5, we show the key result enabling a coupling between $\mathsf{Cpts}(\Gamma)$ and the effective random walk, Proposition 5.11. This result, which is an expression of entropic repulsion, states that, with high $\mathbf{P}_D^{\mathsf{x}}$-probability, each cone-point of Γ stays above height N^δ in the large interval $[N^{4\delta}, N - N^{4\delta}]$, for any $\delta \in (0, 1/4)$. This is achieved via Corollary 5.3, which reduces the result to the result in the free polymer model; and Proposition 4.1, which will allow us to reduce the result to an entropic repulsion result for the random walk conditioned to stay in $\mathbb{H}$.

The entropic repulsion, the shape of the cones $\mathcal{Y}^{\blacktriangleleft}$ and $\mathcal{Y}^{\blacktriangleright}$, and the bound $|\Gamma^{(i)}| \le (\log N)^2$ implies that the entire animal stays bounded away from $\partial\mathbb{H}$, the boundary of the half-plane, in the strip $[N^{4\delta}, N - N^{4\delta}] \times [0, \infty)$. The first consequence of this is that the weight modifications play no role in this strip; that is, for each cluster $\mathcal{C}$ of Γ in this strip, we have $\Phi_D(\mathcal{C}; \gamma) = \Phi(\mathcal{C}; \gamma)$. Thus, the portion of Γ contained in this strip behaves like a free Ising polymer, which can be related to the random walk. The second consequence is that the requirement that γ stays in D in this strip becomes trivial. In particular, following the discussion after Eq. (4.5), we will be able to couple the cone-points of Γ in this strip with an effective random walk. This is achieved in Section 5.6.

Theorem 2.8 is finally proved in Section 5.7.

5.1. **Comparing Ising polymers with modified and unmodified weights.** We begin with the main theorem of [32], which states that the partition functions of the Ising polymers in $\mathbb{H}$, with and without modifications, are equivalent up to a β-dependent constant.

Theorem 5.1 ([32, Theorem 1]). *For all $\beta > 0$ large enough, we have*

$$\tau_\beta(\mathsf{x}) = -\lim_{N\to\infty} \frac{1}{\|\mathsf{x}\|} \log \mathcal{G}_{\mathbb{H}}(\mathsf{x})\,. \tag{5.1}$$

In fact, there exist constants $C_1 := C_1(\beta), C_2 := C_2(\beta) > 0$ such that for all β large enough,

$$C_1\, \mathcal{G}(\mathsf{x} \mid \gamma \subset \mathbb{H}) \le \mathcal{G}_{\mathbb{H}}(\mathsf{x}) \le C_2\, \mathcal{G}(\mathsf{x} \mid \gamma \subset \mathbb{H})\,. \tag{5.2}$$

Though we only consider $\mathbb{H}$ in this article, [32, Theorem 1] is stated more generally for half-spaces whose interior normal has argument lying in $[-\pi/4, 3\pi/4]$.

The next proposition can be seen as an extension of Theorem 5.1 to address polymers in Q.

Proposition 5.2. *For all $\beta > 0$ sufficiently large, there exists a constant $C := C(\beta) > 0$ such that for any $N \in \mathbb{N}$,*

$$C^{-1}\, \mathcal{G}(\mathsf{x} \mid \gamma \subset \mathbb{H}) \le \mathcal{G}(\mathsf{x} \mid \gamma \subset Q) \le C\, \mathcal{G}(\mathsf{x} \mid \gamma \subset \mathbb{H})\,, \tag{5.3}$$

and

$$C^{-1}\, \mathcal{G}(\mathsf{x} \mid \gamma \subset Q) \le \mathcal{G}_Q(\mathsf{x}) \le C\mathcal{G}(\mathsf{x} \mid \gamma \subset Q)\,. \tag{5.4}$$

The proof of Proposition 5.2 is given in Section 5.2.

One can readily imply from Eqs. (5.2) and (5.4) the following, relating $\mathbf{P}^{\mathsf{x}}(\cdot \mid \gamma \subset D)$ and $\mathbf{P}^{\mathsf{x}}_D$, exactly as [32] derived its analogue from Eq. (5.2) (see the discussion below Eq. (2.7) in that paper):

Corollary 5.3. *Let $D = \mathbb{H}$ or Q. There exists a constant $C := C(\beta) > 0$ such that for any set of contours A contained in D,*

$$\mathbf{P}^{\mathsf{x}}_D(A) \le C\sqrt{\mathbf{P}^{\mathsf{x}}(A \mid \gamma \subset D)} \quad \text{ and } \quad \mathbf{P}^{\mathsf{x}}(A \mid \gamma \subset D) \le C\sqrt{\mathbf{P}^{\mathsf{x}}_D(A)}\,.$$

For instance, if $Y_\gamma := \sum_{\mathcal{C}\cap\nabla_\gamma\neq\emptyset} \Phi'_D(\mathcal{C};\gamma) - \Phi'(\mathcal{C};\gamma)$ and $\mathbf{E}[\cdot]$ denotes expectation under $\mathbf{P}^{\mathsf{x}}(\cdot \mid \gamma \subset D)$, then $\mathbf{E}[e^{Y_\gamma}]$ is just $\mathcal{G}_D(\mathsf{x})/\mathcal{G}(\mathsf{x} \mid \gamma \subset D)$, and therefore $\mathbf{P}^{\mathsf{x}}_D(A) = \mathbf{E}[\mathbb{1}_A e^{Y_\gamma}]/\mathbf{E}[e^{Y_\gamma}]$. Notice $C^{-1} \le \mathbf{E}[e^{Y_\gamma}] \le C$ by Eq. (5.2) (for $D = \mathbb{H}$) and Eq. (5.4) (for $D = Q$). By Cauchy–Schwarz, $\mathbf{P}^{\mathsf{x}}_D(A) \le C\sqrt{\mathbf{P}^{\mathsf{x}}(A \mid \gamma \subset D)}\sqrt{\mathbf{E}[e^{2Y_\gamma}]}$, and the last expectation is uniformly bounded (as $\mathbf{E}[e^{2Y_\gamma}] = \tilde{\mathcal{G}}_D(\mathsf{x})/\mathcal{G}(\mathsf{x} \mid \gamma \subset D)$, where $\tilde{\mathcal{G}}_D(\mathsf{x})$ is defined w.r.t. $\tilde{\Phi}'_D(\mathcal{C};\gamma) := 2\Phi'_D(\mathcal{C};\gamma) - \Phi'(\mathcal{C};\gamma)$).

A particular consequence of Corollary 5.3 is that contour events that hold with probability tending to 1 in the unmodified models (which are much easier to study) still hold with probability tending to 1 in the modified models. The most significant application of this fact is in the proof of entropic repulsion under $\mathbf{P}^{\mathsf{x}}_Q(\cdot)$ (Proposition 5.11), which, as explained at the start of this Section, is a crucial step towards the random walk coupling achieved in Section 5.6.

Remark 5.4. In Corollary 5.3, we specified that the event A should be a set of contours. The reason this was emphasized is that in Eq. (2.15) we extended the measure $\mathbf{P}^{\mathsf{x}}_D$ from contours to animals, but Corollary 5.3 does not hold for general animal events A. Indeed, it is *not* true that small animal events in one model stay small in another model, as the ratio $|q(\Gamma)/q_D(\Gamma)|$ is not bounded away from 0 nor ∞ (as there are no such bounds on $|\Phi'_Q(\mathcal{C};\gamma)|/|\Phi'(\mathcal{C};\gamma)|$ in the generality of weight modifications that we must consider).

5.2. **Proof of Proposition 5.2.** We will need two auxiliary lemmas. The first, Lemma 5.5, states that in any model where at least two cone-points exist, the partition function is dominated by animals whose first (left) and last (right) irreducible pieces have size of order 1.

Lemma 5.5. *Let D be either Q or $\mathbb{H}$. For any $\epsilon \in (0,1)$, there exists $K_\epsilon := K_{\epsilon,\beta} > 0$ such that*

$$\mathcal{G}_D(\mathsf{x} \mid |\mathsf{Cpts}(\Gamma)| \ge 2\,,\, \max(|\Gamma^{(L)}|, |\Gamma^{(R)}|) > K_\epsilon) \le \epsilon\mathcal{G}_{\mathbb{H}}(\mathsf{x}) \tag{5.5}$$

and

$$\mathcal{G}(\mathsf{x} \mid \gamma \subset D\,,\, |\mathsf{Cpts}(\Gamma)| \ge 2\,,\, \max(|\Gamma^{(L)}|, |\Gamma^{(R)}|) > K_\epsilon) \le \epsilon\mathcal{G}(\mathsf{x} \mid \gamma \subset \mathbb{H})\,. \tag{5.6}$$

Remark 5.6. Observe that once Proposition 5.2 is proved, the domain $\mathbb{H}$ in the right-hand side of Eqs. (5.5) and (5.6) may be replaced by D, for D either Q or $\mathbb{H}$.

The second auxiliary result, Lemma 5.7, extends Lemma 2.11, and thus Proposition 2.12 thanks to Remark 2.13, to the modified models on Q and $\mathbb{H}$.

Lemma 5.7. *Let D be either Q or $\mathbb{H}$. Fix any $\epsilon, \delta \in (0,1)$. There exist positive constants β_0, δ_0, and $\nu_0 > 0$ such that for all $\beta \geq \beta_0$, there exists $C := C(\beta) > 0$ such that the following hold uniformly over all $N \in \mathbb{N}$ and $r \geq 1+\epsilon$:*

$$\mathcal{G}_D(\mathsf{x} \mid |\gamma| \geq r \|\mathsf{x}\|_1) \leq Ce^{-\nu_0 \beta r \|\mathsf{x}\|_1} \mathcal{G}_D(\mathsf{x}) \tag{5.7}$$

$$\mathcal{G}_D(\mathsf{x} \mid \mathsf{Cpts}(\gamma) < 2\delta_0 \|\mathsf{x}\|_1) \leq Ce^{-\nu_0 \beta \|\mathsf{x}\|_1} \mathcal{G}_D(\mathsf{x}) \tag{5.8}$$

In particular, there exists $\nu > 0$ such that for all $\beta \geq \beta_0$, there exists $C := C(\beta) > 0$ such that the following holds uniformly over all $N \in \mathbb{N}$:

$$\mathcal{G}_D(\mathsf{x} \mid \mathsf{Cpts}(\Gamma) < \delta_0 \|\mathsf{x}\|_1) \leq Ce^{-\nu\beta\|\mathsf{x}\|_1} \mathcal{G}_D(\mathsf{x})\,. \tag{5.9}$$

We postpone the proof of Lemma 5.5 to the end of the subsection. We prove Proposition 5.2 and Lemma 5.7 in tandem, proceeding as follows:

(a) We begin by proving Lemma 5.7 for $D = \mathbb{H}$.

(b) We then prove the first part of Proposition 5.2 (Eq. (5.3)), as well as its analog for $\mathcal{G}_{\mathbb{H}}$, i.e.,

$$C^{-1}\mathcal{G}_{\mathbb{H}}(\mathsf{x}) \leq \mathcal{G}_{\mathbb{H}}(\mathsf{x} \mid \gamma \subset Q) \leq C\mathcal{G}_{\mathbb{H}}(\mathsf{x})\,. \tag{5.10}$$

(c) These results are then used to prove Lemma 5.7 for $D = Q$.

(d) Finally, we prove Eq. (5.4), thereby completing the proof of Proposition 5.2.

Proof of Part (a). Recall Lemma 2.11, which states that the analogous bounds to Eq. (5.7) and Eq. (5.8) hold for $\mathcal{G}(\mathsf{x})$. Eq. (5.1) implies that $\log \mathcal{G}(\mathsf{x})$ and $\log \mathcal{G}(\mathsf{x} \mid \gamma \subset \mathbb{H})$ have the same leading order, and thus, for some $\nu_0' \in (0, \nu_0]$, we have

$$\mathcal{G}(\mathsf{x} \mid \gamma \subset \mathbb{H}\,,\, |\gamma| \geq r \|\mathsf{x}\|_1) \leq \mathcal{G}(\mathsf{x} \mid |\gamma| \geq r \|\mathsf{x}\|_1) \leq Ce^{-\nu_0' \beta r \|\mathsf{x}\|_1} \mathcal{G}(\mathsf{x} \mid \gamma \subset \mathbb{H})\,, \text{ and}$$

$$\mathcal{G}(\mathsf{x} \mid \gamma \subset \mathbb{H}\,,\, \mathsf{Cpts}(\gamma) < 2\delta_0 \|\mathsf{x}\|_1) \leq \mathcal{G}(\mathsf{x} \mid \mathsf{Cpts}(\gamma) < 2\delta_0 \|\mathsf{x}\|_1) \leq Ce^{-\nu_0' \beta \|\mathsf{x}\|_1} \mathcal{G}(\mathsf{x} \mid \gamma \subset \mathbb{H})\,.$$

Eq. (5.7) and Eq. (5.8) for $D = \mathbb{H}$ now follow from Corollary 5.3, and (5.9) from Remark 2.13. ☐

Proof of Part (b). The upper-bounds in both Eqs. (5.3) and (5.10) hold with $C = 1$, since $Q \subset \mathbb{H}$.

We now show the lower bound for Eq. (5.10). The lower bound in Eq. (5.3) is obtained simply by replacing $q_{\mathbb{H}}$ with q in what follows. By Lemmas 5.5 and 5.7 (for $D = \mathbb{H}$), we have the following: for any $\epsilon \in (0,1)$, there exists $K := K(\beta, \epsilon) > 0$ such that for $\|\mathsf{x}\|_1$ large,

$$(1-\epsilon)\mathcal{G}_{\mathbb{H}}(\mathsf{x}) \leq \mathcal{G}_{\mathbb{H}}(\mathsf{x} \mid |\mathsf{Cpts}(\Gamma)| \geq \delta_0 \|\mathsf{x}\|_1\,,\, \max(|\Gamma^{(L)}|, |\Gamma^{(R)}|) \leq K)\,, \tag{5.11}$$

Now, using the weight factorization of $q_{\mathbb{H}}(\Gamma)$ in Eq. (2.18), we may express the right-hand-side of the above (for $N = \|\mathsf{x}\|_1 \geq 2/\delta_0$) as

$$\sum_{\substack{\Gamma^{(L)} \in \mathsf{A_L} \\ \gamma^{(L)} \subset \mathbb{H}}} \sum_{\substack{\Gamma^{(R)} \in \mathsf{A_R} \\ \gamma^{(R)} \subset \mathbb{H}}} q_{\mathbb{H}}(\Gamma^{(L)})\; q_{\mathbb{H}}(\Gamma^{(R)}) \mathbb{1}_{\{|\Gamma^{(L)}|, |\Gamma^{(R)}| \leq K\}}$$

$$\times \sum_{n \geq \delta_0 \|\mathsf{x}\|_1} \sum_{\Gamma^{(1)}, \dots, \Gamma^{(n)} \in \mathsf{A}} \prod_{i=1}^{n} q_{\mathbb{H}}(\Gamma^{(i)}) \mathbb{1}_{\{\Gamma^{(1)} \circ \cdots \circ \Gamma^{(n)} \in \mathcal{P}_{\mathbb{H}}(\mathsf{X}(\Gamma^{(L)}), \mathsf{x} - \mathsf{X}(\Gamma^{(R)}))\}}\,.$$

For any $\mathsf{u} \in \mathcal{Y}^{\blacktriangleleft}$, let $\gamma^{(L)}(\mathsf{u})$ denote an arbitrary up-right path from 0 to u, and let $\gamma^{(R)}(\mathsf{u})$ denote an arbitrary down-right path from $\mathsf{x} - \mathsf{u}$ to x (if one exists). Since the backwards cone of $\Gamma^{(L)}$ must contain the origin and the forwards cone of $\mathsf{x} - \Gamma^{(R)}$ must contain x, it follows that

$$\gamma^{(L)}(\mathsf{X}(\Gamma^{(L)})) \circ \gamma_1 \circ \cdots \circ \gamma_m \circ \gamma^{(R)}(\mathsf{X}(\Gamma^{(R)})) \subset Q$$

for any collection $\{\Gamma^{(L)},\Gamma^{(1)},\dots,\Gamma^{(m)},\Gamma^{(R)}\}$ contributing to a nonzero term in the second-to-last display. In particular, for any $\Gamma^{(L)}\in\mathsf{A}_\mathsf{L}$ and $\Gamma^{(R)}\in\mathsf{A}_\mathsf{R}$ such that $\{|\Gamma^{(L)}|,|\Gamma^{(R)}|\le K\}$, we have

$$q_{\mathbb{H}}(\gamma^{(L)}(\mathsf{X}(\Gamma^{(L)})))q_{\mathbb{H}}(\gamma^{(R)}(\mathsf{X}(\Gamma^{(R)})))\sum_{m\ge 1}\sum_{\Gamma^{(1)},\dots,\Gamma^{(m)}\in\mathsf{A}}\prod_{i=1}^{m}q_{\mathbb{H}}(\Gamma^{(i)})\mathbb{1}_{\{\Gamma^{(1)}\circ\cdots\circ\Gamma^{(m)}\in\mathcal{P}_{\mathbb{H}}(\mathsf{X}(\Gamma^{(L)}),\mathsf{x}-\mathsf{X}(\Gamma^{(R)}))\}}$$
$$\le\mathcal{G}_{\mathbb{H}}(\mathsf{x}\mid\gamma\subset Q)\,. \quad (5.12)$$

Since $q_{\mathbb{H}}(\Gamma^{(L)})\le q_{\mathbb{H}}(\gamma^{(L)}(\mathsf{X}(\Gamma^{(L)})))$ and $q_{\mathbb{H}}(\Gamma^{(R)})\le q_{\mathbb{H}}(\gamma^{(R)}(\mathsf{X}(\Gamma^{(R)})))$, it follows from the last three displays that the right-hand side of Eq. (5.11) is bounded above by

$$\Big(\sum_{\substack{\Gamma^{(L)}\in\mathsf{A}_\mathsf{L}\\ \gamma^{(L)}\subset\mathbb{H}}}\frac{q_{\mathbb{H}}(\Gamma^{(L)})}{q_{\mathbb{H}}(\gamma^{(L)}(\mathsf{X}(\Gamma^{(L)})))}\mathbb{1}_{\{|\Gamma^{(L)}|\le K\}}\Big)\Big(\sum_{\substack{\Gamma^{(R)}\in\mathsf{A}_\mathsf{R}\\ \gamma^{(R)}\subset\mathbb{H}}}\frac{q_{\mathbb{H}}(\Gamma^{(R)})}{q_{\mathbb{H}}(\gamma^{(R)}(\mathsf{X}(\Gamma^{(R)})))}\mathbb{1}_{\{|\Gamma^{(R)}|\le K\}}\Big)\mathcal{G}_{\mathbb{H}}(\mathsf{x}\mid\gamma\subset Q)$$
$$\le\Big(\sum_{\mathsf{u}\in\mathbb{Z}^2:\|\mathsf{u}\|_1\le K}e^{\beta\|\mathsf{u}\|_1}\mathcal{G}_{\mathbb{H}}(\mathsf{u})\Big)^2\mathcal{G}_{\mathbb{H}}(\mathsf{x}\mid\gamma\subset Q)\,.$$

The pre-factor in the last display is a constant depending only $K:=K(\beta,\epsilon)$, finishing the proof. ☐

Proof of Part (c). In Part (a), we proved Eqs. (5.7) and (5.8) for $D=\mathbb{H}$. Bounding below the left-hand sides of these equations by adding the restriction $\gamma\subset Q$, and giving an upper bound on the right-hand sides by using Eq. (5.10) to replace $\mathcal{G}_{\mathbb{H}}(\mathsf{x})$ by $C\mathcal{G}_{\mathbb{H}}(\mathsf{x}\mid\gamma\subset Q)$ yields the same results for $\mathcal{G}_{\mathbb{H}}(\mathsf{x}\mid\gamma\subset Q,\cdot)$:

$$\mathcal{G}_{\mathbb{H}}(\mathsf{x}\mid\gamma\subset Q\,,\,|\gamma|\ge 1.1\,\|\mathsf{x}\|_1\,)\le Ce^{-\nu_0\beta\|\mathsf{x}\|_1}\mathcal{G}_{\mathbb{H}}(\mathsf{x}\mid\gamma\subset Q)\,,\ \text{and} \quad (5.13)$$
$$\mathcal{G}_{\mathbb{H}}(\mathsf{x}\mid\gamma\subset Q\,,\,\mathsf{Cpts}(\gamma)<2\delta_0\,\|\mathsf{x}\|_1\,)\le Ce^{-\nu_0\beta\|\mathsf{x}\|_1}\mathcal{G}_{\mathbb{H}}(\mathsf{x}\mid\gamma\subset Q) \quad (5.14)$$

uniformly in β large enough and x. Similar to Eq. (2.13), we have

$$\left|\log\frac{q_{\mathbb{H}}(\gamma)}{q_Q(\gamma)}\right|\le 6e^{-\chi\beta}|\gamma|\,. \quad (5.15)$$

Eq. (5.7) and Eq. (5.8) for $D=Q$ are then simple consequences of Eqs. (5.13) to (5.15). Once again, Eq. (5.9) for $D=Q$ follows by Remark 2.13. ☐

Proof of Part (d). Let us start by proving the upper bound in Eq. (5.4). As in Eq. (5.11), Lemmas 5.5 and 5.7 (now for $D=Q$) yield the following for any $\epsilon\in(0,1)$, some $K:=K(\beta,\epsilon)>0$, and all $\|\mathsf{x}\|_1$ large enough:

$$\mathcal{G}_Q(\mathsf{x})\le\epsilon\mathcal{G}_{\mathbb{H}}(\mathsf{x})+\mathcal{G}_Q(\mathsf{x}\mid|\mathsf{Cpts}(\Gamma)|\ge\delta_0\|\mathsf{x}\|_1\,,\,\max(|\Gamma^{(L)}|,|\Gamma^{(R)}|)\le K)\,. \quad (5.16)$$

The first term on the right-hand side, $\epsilon\mathcal{G}_{\mathbb{H}}(\mathsf{x})$, may be bounded by $C_0(\beta)\mathcal{G}(\mathsf{x}\mid\gamma\subset Q)$ via Eqs. (5.2) and (5.3). We now follow the proof of Part (b) to bound the second term: using the factorization Eq. (2.19) of $q_Q(\Gamma)$, expanding the right-hand side as in Part (b), and replacing $\Gamma^{(L)}$ and $\Gamma^{(R)}$ by cluster-less up-right paths $\gamma^{(L)}(\mathsf{X}(\Gamma^{(L)}))$ and $\gamma^{(R)}(\mathsf{X}(\Gamma^{(R)}))$, we obtain an upper bound by $C_1\mathcal{G}_{\mathbb{H}}(\mathsf{x}\mid\gamma\subset Q)$, which by Eqs. (5.2) and (5.3) is bounded above by $C_1'\mathcal{G}(\mathsf{x}\mid\gamma\subset Q)$.

For the lower bound in Eq. (5.4), start instead from $\mathcal{G}(\mathsf{x}\mid\gamma\subset Q)$. Bound it from above by $C_2\mathcal{G}(\mathsf{x}\mid\gamma\subset\mathbb{H})$ using Eq. (5.3), and bound the latter by $C_2'\mathcal{G}_{\mathbb{H}}(\mathsf{x})$ using Eq. (5.2). Now, use the inequality in Eq. (5.11), expand the right-hand side exactly as in the proceeding display, and again replace $\Gamma^{(L)}$ and $\Gamma^{(R)}$ by cluster-less, up-right paths. Observe that the bound in Eq. (5.12) holds with the right-hand side replaced with $\mathcal{G}_Q(\mathsf{x})$, since all animals on the left-hand side are contained in Q (and therefore the $q_{\mathbb{H}}$-weights of these animals are equal to the q_Q-weights). Following the remainder of the proof of Part (b), we obtain the lower bound in Eq. (5.4). ☐

Proof of Lemma 5.5. For any $\mathsf{u},\mathsf{v}\in\mathbb{Z}^2$ and for any set of animals E, define the partition functions

$$\mathcal{G}_D(\mathsf{u}\to\mathsf{v}) := \sum_{\substack{\gamma:\mathsf{u}\to\mathsf{v}\\ \gamma\subset D}} q_D(\gamma) \quad\text{and}\quad \mathcal{G}_D(\mathsf{u}\to\mathsf{v}\mid E) := \sum_{\Gamma\in\mathcal{P}_D(\mathsf{u},\mathsf{v})\cap E} q_D(\Gamma)\,. \tag{5.17}$$

We begin with three estimates pertaining to $\mathcal{G}_D(\mathsf{u}\to\mathsf{v})$.

First, we have the analogue of Eq. (2.26) for the modified animal weights: for all $\beta>0$ sufficiently large, there exists a constant $c' := c'(\beta)>0$ such that for all $k,N\geq 1$,

$$\sum_{\Gamma\in\mathsf{A}_\mathsf{L}\cup\mathsf{A}_\mathsf{R}} e^{\mathsf{h}_\mathsf{x}\cdot\mathsf{X}(\Gamma)} q_D(\Gamma)\mathbb{1}_{\{|\Gamma|\geq k\}} \leq c' e^{-\nu_g\beta k}\,. \tag{5.18}$$

This is proved at the end of Appendix B.

Next, below Theorem 3 of [32], it is noted that Eq. (5.2) in the current article actually holds if the end-points of the contour are not on the line $\partial\mathbb{H}$; that is, for any $\mathsf{u},\mathsf{v}\in\mathbb{H}$, we have

$$C_1(\beta)\mathcal{G}(\mathsf{u}\to\mathsf{v}\mid\gamma\subset\mathbb{H}) \leq \mathcal{G}_\mathbb{H}(\mathsf{u}\to\mathsf{v}) \leq C_2(\beta)\mathcal{G}(\mathsf{u}\to\mathsf{v}\mid\gamma\subset\mathbb{H})\,. \tag{5.19}$$

Similarly, in Sections 4.1 and 4.2 of [32], it is shown that contours with linear size and many cone-points dominate $\mathcal{G}(\mathsf{x}\mid\gamma\subset\mathbb{H})$, and the arguments there yield that the same is true when the end-points of the contour are not on $\partial\mathbb{H}$; that is, for any fixed $\epsilon\in(0,1)$, there exist constants $\delta_0,\nu,c>0$ such that the following bounds hold uniformly over $\beta>0$ sufficiently large, over $\mathsf{u},\mathsf{v}\in\mathbb{H}$ satisfying $\mathsf{v}\in\mathsf{u}+\mathcal{Y}^{\blacktriangleleft}_\delta\setminus\{0\}$, and over $r\geq 1+\epsilon$:

$$\mathcal{G}(\mathsf{u}\to\mathsf{v}\mid\gamma\subset\mathbb{H}\,,\,|\gamma|\geq r\,\|\mathsf{v}-\mathsf{u}\|_1) \leq ce^{-\nu\beta\|\mathsf{v}-\mathsf{u}\|_1}\mathcal{G}(\mathsf{u}\to\mathsf{v}\mid\gamma\subset\mathbb{H})\,.$$

and

$$\mathcal{G}(\mathsf{u}\to\mathsf{v}\mid\gamma\subset\mathbb{H}\,,\,|\mathsf{Cpts}(\gamma)|<2\delta_0\|\mathsf{v}-\mathsf{u}\|_1) \leq ce^{-\nu\beta\|\mathsf{v}-\mathsf{u}\|_1}\mathcal{G}(\mathsf{u}\to\mathsf{v}\mid\gamma\subset\mathbb{H})\,.$$

As a consequence of Remark 2.13, we have many cone-points for animals Γ as well:

$$\mathcal{G}(\mathsf{u}\to\mathsf{v}\mid\gamma\subset\mathbb{H}\,,\,|\mathsf{Cpts}(\Gamma)|<\delta_0\|\mathsf{v}-\mathsf{u}\|_1) \leq ce^{-\nu\beta\|\mathsf{v}-\mathsf{u}\|_1}\mathcal{G}(\mathsf{u}\to\mathsf{v}\mid\gamma\subset\mathbb{H})\,. \tag{5.20}$$

Now, towards Eq. (5.5), we only show $\mathcal{G}_D(\mathsf{x}\mid|\mathsf{Cpts}(\Gamma)|\geq 2\,,\,|\Gamma^{(L)}|>K_\epsilon)\leq\epsilon\mathcal{G}_D(\mathsf{x})$, as the analogous bound with $\Gamma^{(R)}$ instead of $\Gamma^{(L)}$ follows via the same argument. We begin by using Eqs. (2.18) and (2.19) to obtain the expansion

$$\begin{aligned}&\mathcal{G}_D(\mathsf{x}\mid|\mathsf{Cpts}(\Gamma)|\geq 2\,,\,|\Gamma^{(L)}|>K_\epsilon)\\ &= \sum_{\substack{\mathsf{u}\in\mathcal{Y}^{\blacktriangleleft}\\ \mathsf{x}-\mathsf{v}\in\mathcal{Y}^{\blacktriangleleft}}}\sum_{\substack{\Gamma^{(L)}\in\mathsf{A}_\mathsf{L}\\ \Gamma^{(L)}\in\mathcal{P}_D(\mathsf{u})}} q_D(\Gamma^{(L)})\mathbb{1}_{\{|\Gamma^{(L)}|>K_\epsilon\}}\sum_{\substack{\Gamma^{(R)}\in\mathsf{A}_\mathsf{R}\\ \Gamma^{(R)}\in\mathcal{P}_D(\mathsf{v},\mathsf{x})}} q_D(\Gamma^{(R)})\Big(\sum_{n\geq 1}\sum_{\substack{\Gamma^{(1)},\dots,\Gamma^{(n)}\in\mathsf{A}\\ \Gamma^{(1)}\circ\cdots\circ\Gamma^{(n)}\in\mathcal{P}_\mathbb{H}(\mathsf{u},\mathsf{v})}}\prod_{i=1}^n q_\mathbb{H}(\Gamma^{(i)})\Big)\,.\end{aligned} \tag{5.21}$$

The above double sum over n and irreducible animals is bounded by $\mathcal{G}_\mathbb{H}(\mathsf{u}\to\mathsf{v})$, which by Eq. (5.19) is bounded by $C_2(\beta)\mathcal{G}(\mathsf{u}\to\mathsf{v}\mid\gamma\subset\mathbb{H})$. It then suffices to show

$$\mathcal{G}(\mathsf{u}\to\mathsf{v}\mid\gamma\subset\mathbb{H}) \leq Ce^{\mathsf{h}_\mathsf{x}\cdot(\mathsf{u}+\mathsf{x}-\mathsf{v})+\bar\delta\beta(\|\mathsf{u}\|_1+\|\mathsf{x}-\mathsf{v}\|_1)}\mathcal{G}(\mathsf{x}\mid\gamma\subset\mathbb{H})\,. \tag{5.22}$$

where $C:=C(\beta)>0$ is a constant and $\bar\delta$ is as in Proposition 4.1. Indeed, substituting the bound in Eq. (5.22) into Eq. (5.21), using the exponential tail from Eq. (5.18), and recalling $\bar\delta<\nu_g/4$ yields

$$\mathcal{G}_D(\mathsf{x}\mid|\mathsf{Cpts}(\Gamma)|\geq 2\,,\,|\Gamma^{(L)}|>K_\epsilon) \leq C'e^{-\frac{\nu_g}{2}\beta K_\epsilon}\mathcal{G}(\mathsf{x}\mid\gamma\subset\mathbb{H})\,.$$

By Theorem 5.1, the right-hand side above is bounded by $\epsilon\mathcal{G}_\mathbb{H}(\mathsf{x})$ for $K_\epsilon := K_{\epsilon,\beta}>0$ large enough.

To show Eq. (5.22), we use the existence of cone-points Eq. (5.20) (for which we write a $1+o(1)$, where the $o(1)$ term vanishes as $\|\mathsf{v}-\mathsf{u}\|_1$ tends to infinity) and the weight factorization in Eq. (2.27):

$$\begin{aligned}
&\mathcal{G}\Big(\mathsf{u}\to\mathsf{v}\mid\gamma\subset\mathbb{H}\Big)\\
&\le(1+o(1))\sum_{\substack{\mathsf{u}'\in\mathsf{u}+\mathcal{Y}^{\blacktriangleleft}\\ \mathsf{x}-\mathsf{v}-\mathsf{v}'\in\mathcal{Y}^{\blacktriangleleft}}}\sum_{\substack{\Gamma^{(L)}:\mathsf{u}\to\mathsf{u}'\in\mathsf{A}_\mathsf{L}\\ \Gamma^{(R)}:\mathsf{v}'\to\mathsf{v}\in\mathsf{A}_\mathsf{R}}} q(\Gamma^{(L)})q(\Gamma^{(R)})\sum_{n\ge1}\sum_{\Gamma^{(1)},\dots,\Gamma^{(n)}\in\mathsf{A}}\prod_{i=1}^n q(\Gamma^{(i)})\mathbb{1}_{\{\Gamma^{(1)}\circ\cdots\circ\Gamma^{(n)}\in\mathcal{P}(\mathsf{u}',\mathsf{v}')\}}\\
&=(1+o(1))e^{-\mathsf{h}_\mathsf{x}\cdot(\mathsf{v}-\mathsf{u})}\sum_{\substack{\mathsf{u}'\in\mathsf{u}+\mathcal{Y}^{\blacktriangleleft}\\ \mathsf{x}-\mathsf{v}-\mathsf{v}'\in\mathcal{Y}^{\blacktriangleleft}}}\sum_{\substack{\Gamma^{(L)}:\mathsf{u}\to\mathsf{u}'\in\mathsf{A}_\mathsf{L}\\ \Gamma^{(R)}:\mathsf{v}'\to\mathsf{v}\in\mathsf{A}_\mathsf{R}}}\mathbb{P}^{\mathsf{h}_\mathsf{x}}(\Gamma^{(L)})\mathbb{P}^{\mathsf{h}_\mathsf{x}}(\Gamma^{(R)})\mathcal{A}(\mathsf{u}',\mathsf{v}')\\
&\le(1+o(1))e^{\mathsf{h}_\mathsf{x}\cdot(\mathsf{u}+\mathsf{x}-\mathsf{v})}\mathcal{G}(\mathsf{x}\mid\gamma\subset\mathbb{H})\\
&\quad\times\Big(\sum_{\mathsf{u}'\in\mathsf{u}+\mathcal{Y}^{\blacktriangleleft}}\sum_{\Gamma^{(L)}:\mathsf{u}\to\mathsf{u}'\in\mathsf{A}_\mathsf{L}}\mathbb{P}^{\mathsf{h}_\mathsf{x}}(\Gamma^{(L)})e^{\bar\delta\beta\|\mathsf{u}'\|_1}\Big)\Big(\sum_{\mathsf{x}-\mathsf{v}-\mathsf{v}'\in\mathcal{Y}^{\blacktriangleleft}}\sum_{\Gamma^{(R)}:\mathsf{v}'\to\mathsf{v}\in\mathsf{A}_\mathsf{R}}\mathbb{P}^{\mathsf{h}_\mathsf{x}}(\Gamma^{(R)})e^{\bar\delta\beta\|\mathsf{x}-\mathsf{v}'\|_1}\Big)\\
&\le(1+o(1))e^{\mathsf{h}_\mathsf{x}\cdot(\mathsf{u}+\mathsf{x}-\mathsf{v})+\bar\delta\beta(\|\mathsf{u}\|_1+\|\mathsf{x}-\mathsf{v}\|_1)}\mathcal{G}(\mathsf{x}\mid\gamma\subset\mathbb{H})\\
&\quad\times\Big(\sum_{\Gamma^{(L)}\in\mathsf{A}_\mathsf{L}}\mathbb{P}^{\mathsf{h}_\mathsf{x}}(\Gamma^{(L)})e^{\bar\delta\beta\|\mathsf{X}(\Gamma^{(L)})\|_1}\Big)\Big(\sum_{\Gamma^{(R)}\in\mathsf{A}_\mathsf{R}}\mathbb{P}^{\mathsf{h}_\mathsf{x}}(\Gamma^{(R)})e^{\bar\delta\beta\|\mathsf{X}(\Gamma^{(R)})\|_1}\Big)
\end{aligned}$$

In the second-to-last line we used Eqs. (4.4) and (4.5) and then Proposition 4.1 to bound the $\mathcal{A}(\mathsf{u}',\mathsf{v}')$ term, as well as $\mathsf{h}_\mathsf{x}\cdot\mathsf{x}=\tau_\beta(\mathsf{x})$. In the third-line, we used the triangle inequality. Equation (5.22) then follows from the exponential tails of $\Gamma^{(L)}$ and $\Gamma^{(R)}$ in Eq. (2.26), again recalling $\bar\delta<\nu_g/4$. This yields Eq. (5.5). The proof of Eq. (5.6) is identical: just replace the left-hand side of Eq. (5.21) by $\mathcal{G}(\mathsf{x}\mid|\mathsf{Cpts}(\Gamma)|\ge2,|\Gamma^{(L)}|>K_\epsilon,\gamma\subset D)$, and then replace all modified weights q_D by q. □

5.3. **Proof of Proposition 2.5.** Fix a direction $\vec{u}\in\mathbb{S}^{d-1}$, and let $t_N^-(\vec{u}):=(N,y_N)\in\mathbb{Z}^2$ denote the lattice point whose x-coordinate is equal to N and is on or below the line span$(\vec{u})$. Let $\mathsf{o}^*:=(1/2,1/2)$ be the origin of the dual lattice $(\mathbb{Z}^2)^*$, and define $\mathsf{x}_N(\vec{u}):=t_N^-(\vec{u})+(-1/2,1/2)\in(\mathbb{Z}^2)^*$. Then we have the formula

$$\tau_\beta^{\mathrm{SOS}}(\vec{u})=\lim_{N\to\infty}\lim_{M\to\infty}-\frac{1}{\beta\,\|\mathsf{x}_N(\vec{u})\|}\log\mathcal{G}_{\mathcal{I}_{N,M}}(\mathsf{x}_N(\vec{u}))\,,$$

where $\mathcal{I}_{N,M}$ denotes the strip $[0,N]\times[-M,M]$. [20, Theorem 4.16] implies that the surface tension τ_β is equal to the surface tension of the model in $\mathcal{I}_{N,M}$ with the same (free) weights $q(\gamma)$, that is:[10]

$$\tau_\beta(\vec{u})=\lim_{N\to\infty}\lim_{M\to\infty}-\frac{1}{\beta\,\|\mathsf{x}_N(\vec{u})\|}\log\mathcal{G}(\mathsf{x}_N(\vec{u})\mid\gamma\subset\mathcal{I}_{N,M})\,.$$

Thus, the proof will be finished if we can compare to exponential order the partition function with modified weights $\mathcal{G}_{\mathcal{I}_{N,M}}(\mathsf{x}_N(\vec{u}))$ and the partition function with free weights $\mathcal{G}(\mathsf{x}_N(\vec{u})\mid\gamma\subset\mathcal{I}_{N,M})$. The proof of this comparison is extremely similar to the proof of Eq. (5.4) above, though much simpler, and so it is omitted for brevity's sake.[11] □

[10]It is well-known that this result of [20] contains a mistake, corrected in the appendix of [32]. However, the mistake has to do with a part of the result that is irrelevant to us, namely, a part that claims an unchanged surface tension even after modification of the decoration functions $\Phi(\mathcal{C};\Gamma)$. In our application of [20, Theorem 4.16], we are not making any modifications to the decoration functions.

[11]A crucial input for the proof of Eq. (5.4) was Proposition 2.12, which we proved using Lemma 2.11, cited from the paper of [32]. Note [32] was written for Ising polymers, so in particular, Property (P4) of the surface tension is always assumed in that paper. However, we use Proposition 2.5 in order to prove Property (P4) of τ_β, so this might appear to be circular reasoning. However, Lemma 2.11 is proved in [32] (Eq. (4.5) there) without assuming Property (P4), so there is no issue in using it. No other results from [32] are needed for the proof of Proposition 2.5.

5.4. **Boundedness of the irreducible pieces.** The main result of this subsection is Proposition 5.8, which states that each irreducible piece of Γ has size bounded by $(\log N)^2$ with high $\mathbf{P}^{\mathsf{x}}_D(\cdot)-$probability.

When Γ has at least two cone-points, recall from Eq. (2.17) that we write $\Gamma^{(1)},\dots,\Gamma^{(n)}$ to denote the irreducible components of Γ, where $n := |\mathsf{Cpts}(\Gamma)| - 1$.

Proposition 5.8. *For $D = Q$ or $D = \mathbb{H}$,*

$$\mathbf{P}^{\mathsf{x}}_D(\{|\mathsf{Cpts}(\Gamma)| \geq 2\}\,,\,\{\exists i \in \{L,1,\dots,|\mathsf{Cpts}(\Gamma)|-1,R\} : |\Gamma^{(i)}| \geq (\log N)^2\}) = o(1)\,.$$

We will need the following two results on the asymptotic size of the partition functions of interest. Recall the notation for asymptotic relations set out at the start of Section 2.

Lemma 5.9. *There exists some $C := C(\beta) > 0$*

$$\mathcal{G}(\mathsf{x}) \leq Ce^{-\tau_\beta(\mathsf{x})}\,. \tag{5.23}$$

Proof. Similar to Eq. (2.29), we have from Eq. (4.4)

$$\begin{aligned}e^{\tau_\beta(\mathsf{x})}\mathcal{G}(x \mid |\mathsf{Cpts}(\Gamma)| \geq 2) &= \sum_{\Gamma^{(L)}\in\mathsf{A_L}}\sum_{\Gamma^{(R)}\in\mathsf{A_R}} \mathbb{P}^{\mathsf{h_x}}(\Gamma^{(L)})\mathbb{P}^{\mathsf{h_x}}(\Gamma^{(R)})\mathbb{P}_{\mathsf{X}(\Gamma^{(L)})}(H_{\mathsf{x}-\mathsf{X}(\Gamma^{(R)})} < \infty) \\ &\leq \Big(\sum_{\Gamma^{(L)}\in\mathsf{A_L}} \mathbb{P}^{\mathsf{h_x}}(\Gamma^{(L)})\Big)\Big(\sum_{\Gamma^{(R)}\in\mathsf{A_R}} \mathbb{P}^{\mathsf{h_x}}(\Gamma^{(R)})\Big)\,.\end{aligned}$$

The above summations are finite by Claim B.1. The Lemma then follows from Eq. (2.22). $\square$

We remark that a precise first-order asymptotic for $\mathcal{G}(\mathsf{x})$ is given in [20, Equation 4.12.3]. The asymptotic can be proved by showing $\mathbb{P}_{\mathsf{X}(\Gamma^{(L)})}(H_{\mathsf{x}-\mathsf{X}(\Gamma^{(R)})} < \infty)$ is of order $|\mathsf{x}|^{-1/2}$ via random walk estimates, similar to the proof of Theorem 4.2.

Lemma 5.10. *We have*

$$\mathcal{G}(\mathsf{x} \mid \gamma \subset \mathbb{H}) \gtrsim e^{-\tau_\beta(\mathsf{x})}N^{-3/2}\,, \tag{5.24}$$

where the implied constant depends on β.

Proof. Item (1) of Theorem 4.2 shows that for any positive, fixed u and v in $\mathbb{N}$ (independent of N), we have

$$\mathbb{P}_{(0,\mathsf{u})}(H_{(N,\mathsf{v})} < H_{\mathbb{H}_-}) \sim \mathbf{C}\kappa\frac{V_1(u)V_1'(v)}{N^{3/2}}\,. \tag{5.25}$$

The lower-bound of Eq. (5.24) then follows immediately from Proposition 4.1 and Eq. (5.25). $\square$

It is not too hard to show a matching upper-bound, so that $\mathcal{G}(\mathsf{x} \mid \gamma \subset \mathbb{H}) \asymp e^{-\tau_\beta(\mathsf{x})}N^{-3/2}$. Again, we do not pursue this here as the lower-bound suffices.

Proof of Proposition 5.8. Define the set of animals

$$\mathcal{P}^{\mathrm{cp,len}}_D(\mathsf{x},K) := \{\Gamma \in \mathcal{P}_D(\mathsf{x}) : |\mathsf{Cpts}(\Gamma)| \geq 2\,,\,\max(|\Gamma^{(L)}|,|\Gamma^{(R)}|) \leq K\,,\,|\Gamma| \leq 1.1\,\|\mathsf{x}\|_1\}\,. \tag{5.26}$$

In Lemmas 5.5 and 5.7 and Remark 5.6, it was shown that this set of animals dominates $\mathbf{P}^{\mathsf{x}}_D$, i.e.,

$$\lim_{K\to\infty}\lim_{N\to\infty}\mathbf{P}^{\mathsf{x}}_D(\mathcal{P}^{\mathrm{cp,len}}_D(x,K)) = 1\,.$$

Therefore, in light of Lemma 5.10, the proof of Proposition 5.8 will be complete upon showing

$$\mathcal{G}_D(\mathsf{x} \mid \mathcal{P}^{\mathrm{cp,len}}_D(\mathsf{x},K)\,,\,\{\exists 1 \leq i \leq |\mathsf{Cpts}(\Gamma)|-1 : |\Gamma^{(i)}| \geq (\log N)^2\}) = o(e^{-\tau_\beta(\mathsf{x})}N^{-3/2}) \tag{5.27}$$

for any fixed K.

Define the sets of animals

$$\mathsf{A}_{\mathsf{L},K} := \{\Gamma \in \mathsf{A_L} : |\Gamma| < K\} \quad\text{and}\quad \mathsf{A}_{\mathsf{R},K} := \{\Gamma \in \mathsf{A_R} : |\Gamma| < K\}\,. \tag{5.28}$$

Using the factorization of weights (Eq. (2.18), and Eq. (2.19) for $D = Q$), we have

$$\begin{aligned}
&\mathcal{G}_D(\mathsf{x} \mid \mathcal{P}_D^{\mathrm{cp,len}}(\mathsf{x}, K)\,,\, \{\exists 1 \le i \le |\mathsf{Cpts}(\Gamma)| - 1 : |\Gamma^{(i)}| \ge (\log N)^2\}) \\
&\quad = \sum_{\substack{\Gamma^{(L)} \in \mathsf{A}_{\mathsf{L},K} \\ \gamma^{(L)} \subset D}} q_D(\Gamma^{(L)}) \sum_{\substack{\Gamma^{(R)} \in \mathsf{A}_{\mathsf{R},K} \\ \gamma^{(R)} \subset D}} q_D(\Gamma^{(R)}) \sum_{n \ge 1} \sum_{\Gamma^{(1)},\dots,\Gamma^{(n)} \in \mathsf{A}} \prod_{i=1}^n q_{\mathbb{H}}(\Gamma^{(i)}) \\
&\qquad \times \mathbb{1}_{\{\Gamma^{(1)} \circ \cdots \circ \Gamma^{(n)} \in \mathcal{P}_{\mathbb{H}}(\mathsf{X}(\Gamma^{(L)}), \mathsf{x} - \mathsf{X}(\Gamma^{(R)}))\}} \mathbb{1}_{\{\exists 1 \le k \le n : |\Gamma^{(k)}| \ge (\log N)^2\}} \\
&\quad \le \sum_{\substack{\Gamma^{(L)} \in \mathsf{A}_{\mathsf{L},K} \\ \gamma^{(L)} \subset D}} q_D(\Gamma^{(L)}) \sum_{\substack{\Gamma^{(R)} \in \mathsf{A}_{\mathsf{R},K} \\ \gamma^{(R)} \subset D}} q_D(\Gamma^{(R)}) \\
&\qquad \times \sum_{\substack{\mathsf{u} \in \mathsf{X}(\Gamma^{(L)}) + \mathcal{Y}^{\blacktriangleleft} \\ \mathsf{v} \in \mathsf{x} - \mathsf{X}(\Gamma^{(R)}) + \mathcal{Y}^{\blacktriangleright}}} \mathcal{G}_{\mathbb{H}}(\mathsf{X}(\Gamma^{(L)}) \to \mathsf{u}) \mathcal{G}_{\mathbb{H}}(\mathsf{v} \to \mathsf{x} - \mathsf{X}(\Gamma^{(R)})) \sum_{\Gamma \in \mathsf{A} \cap \mathcal{P}_{\mathbb{H}}(\mathsf{u},\mathsf{v})} q_{\mathbb{H}}(\Gamma) \mathbb{1}_{\{|\Gamma| \ge (\log N)^2\}}\,.
\end{aligned}$$

For any $\mathsf{a}, \mathsf{b} \in \mathbb{H}$, Eqs. (5.19) and (5.23) along with the trivial bound $\mathcal{G}(\mathsf{a} \to \mathsf{b} \mid \gamma \subset \mathbb{H}) \le \mathcal{G}(\mathsf{b} - \mathsf{a})$. implies $\mathcal{G}_{\mathbb{H}}(\mathsf{a} \to \mathsf{b}) \le C e^{\tau_\beta(\mathsf{b} - \mathsf{a})}$ for some constant $C := C(\beta) > 0$. Define the weights $W_{\mathbb{H}}^{\mathsf{h}}(\Gamma) := e^{\mathsf{h} \cdot \mathsf{X}(\Gamma)} q_{\mathbb{H}}(\Gamma)$, under which Γ has exponential tails by Eq. (5.18). Then

$$\begin{aligned}
&\mathcal{G}_D(\mathsf{x} \mid \mathcal{P}_D^{\mathrm{cp,len}}(\mathsf{x}, K)\,,\, \{\exists 1 \le i \le |\mathsf{Cpts}(\Gamma)| - 1 : |\Gamma^{(i)}| \ge (\log N)^2\}) \\
&\le C^2 \sum_{\substack{\Gamma^{(L)} \in \mathsf{A}_{\mathsf{L},K} \\ \gamma^{(L)} \subset D}} q_D(\Gamma^{(L)}) e^{\tau_\beta(\mathsf{X}(\Gamma^{(L)}))} \sum_{\substack{\Gamma^{(R)} \in \mathsf{A}_{\mathsf{R},K} \\ \gamma^{(R)} \subset D}} q_D(\Gamma^{(R)}) e^{\tau_\beta(\mathsf{X}(\Gamma^{(R)}))} \\
&\times \sum_{\substack{\mathsf{u} \in \mathsf{X}(\Gamma^{(L)}) + \mathcal{Y}^{\blacktriangleleft} \\ \mathsf{v} \in \mathsf{x} - \mathsf{X}(\Gamma^{(R)}) + \mathcal{Y}^{\blacktriangleright}}} \Big(\sum_{\Gamma \in \mathsf{A}} W_H^{\mathsf{h}_{\mathsf{v}-\mathsf{u}}}(\Gamma) \mathbb{1}_{\{|\Gamma| \ge (\log N)^2\}} \Big) \\
&\times \exp\Big(-\beta \Big(\tau_\beta(\mathsf{X}(\Gamma^{(L)})) + \tau_\beta(\mathsf{u} - \mathsf{X}(\Gamma^{(L)})) + \tau_\beta(\mathsf{v} - \mathsf{u}) + \tau_\beta(\mathsf{x} - \mathsf{X}(\Gamma^{(R)}) - \mathsf{v}) + \tau_\beta(\mathsf{X}(\Gamma^{(R)})) \Big) \Big)\,.
\end{aligned}$$

The exponential tails in Eq. (5.18) implies both

$$\sum_{\substack{\Gamma^{(L)} \in \mathsf{A}_{\mathsf{L},K} \\ \gamma^{(L)} \subset D}} q_D(\Gamma^{(L)}) e^{\tau_\beta(\mathsf{X}(\Gamma^{(L)}))} \quad \text{and} \quad \sum_{\substack{\Gamma^{(R)} \in \mathsf{A}_{\mathsf{R},K} \\ \gamma^{(R)} \subset D}} q_D(\Gamma^{(R)}) e^{\tau_\beta(\mathsf{X}(\Gamma^{(R)}))}$$

are bounded by a constant $C_K > 0$, while

$$\sum_{\Gamma \in \mathsf{A}} W_{\mathbb{H}}^{\mathsf{h}_{\mathsf{v}-\mathsf{u}}}(\Gamma) \mathbb{1}_{\{|\Gamma| \ge (\log N)^2\}} \le c' e^{-\nu_g \beta (\log N)^2}\,.$$

These bounds, the strong triangle inequality (Proposition 2.4), and the fact that there are at most $|0 + \mathcal{Y}^{\blacktriangleleft} \cap \mathsf{x} + \mathcal{Y}^{\blacktriangleright} \cap \mathbb{Z}^2| \le (N+1)^2$ possible values for u and for v yield

$$\mathcal{G}_Q(\mathsf{x} \mid \{\exists 1 \le i \le |\mathsf{Cpts}(\Gamma)| - 1 : |\Gamma^{(i)}| \ge (\log N)^2\}) \le C_K'(N+1)^4 e^{-\nu_g \beta (\log N)^2} e^{-\tau_\beta(\mathsf{x})}\,,$$

for some constant $C_K' > 0$. Thus, we have Eq. (5.27), finishing the proof. ☐

5.5. **Entropic repulsion of the animal.** Fix any $\delta \in (0, 1/4)$, and define the rectangles $\mathfrak{R}_0(\delta) := [N^{4\delta}, N - N^{4\delta}] \times [0, 2N^\delta]$ and $\mathfrak{R}(\delta) := [N^{4\delta}, N - N^{4\delta}] \times [0, N^\delta]$. The main result in this subsection is the entropic repulsion result Proposition 5.11, which states that, under $\mathbf{P}_D^{\mathsf{x}}(\cdot)$, the cone-points of Γ do not intersect $\mathfrak{R}(\delta)$ with probability tending to 1 as $N \to \infty$.

Proposition 5.11. *Let $D = Q$ or $D = \mathbb{H}$. For any $\delta \in (0, 1/4)$,*

$$\lim_{N \to \infty} \mathbf{P}_D^{\mathsf{x}}(\mathsf{Cpts}(\Gamma) \cap \mathfrak{R}(\delta) \ne \emptyset) = 0\,.$$

From the shape of $\mathcal{Y}^{\blacktriangleleft}$ and the $(\log N)^2$ bound on each $|\Gamma^{(i)}|$ (Proposition 5.8), Proposition 5.11 is enough to deduce that, with high $\mathbf{P}_D^{\mathsf{x}}$-probability, the entire animal (contour and clusters) stays away from $\partial\mathbb{H}$ in a slightly shorter rectangle (5.31). We will use this fact in the following subsection to couple $\mathsf{Cpts}(\Gamma)$ with the random walk. We begin with an analogous result in the free Ising polymer case, where we can exploit the connection with the random walk and associated estimates, à la Proposition 4.1.

Lemma 5.12. *Let $D = Q$ or $D = \mathbb{H}$. For any $\delta \in (0, 1/8)$,*

$$\lim_{N\to\infty} \mathbf{P}^{\mathsf{x}}(\Gamma \cap \mathfrak{R}(\delta) \neq \emptyset \mid \gamma \subset D) = 0\,.$$

Proof. In light of Proposition 5.8 and the shape of $\mathcal{Y}^{\blacktriangleleft}$, it suffices to show that the cone-points of Γ avoid the larger rectangle $\mathfrak{R}_0(\delta)$ with high probability, i.e.,

$$\lim_{N\to\infty} \mathbf{P}^{\mathsf{x}}(E_\delta \mid \gamma \subset D) = 0\,, \qquad \text{where } E_\delta := \{\mathsf{Cpts}(\Gamma) \cap \mathfrak{R}_0(\delta) \neq \emptyset\}\,.$$

Recall $\mathcal{P}_D^{\mathrm{cp,len}}(\mathsf{x}, K)$ from (5.26), and observe that Lemmas 5.5 and 5.7 and Remark 5.6 imply

$$\lim_{K\to\infty} \lim_{N\to\infty} \mathbf{P}^{\mathsf{x}}(\mathcal{P}_D^{\mathrm{cp,len}}(\mathsf{x}, K) \mid \gamma \subset D) = 1\,.$$

From the previous two displays, the proof of the Lemma will be complete upon showing

$$\lim_{K\to\infty} \lim_{N\to\infty} \mathbf{P}^{\mathsf{x}}(\mathcal{P}_D^{\mathrm{cp,len}}(\mathsf{x}, K), E_\delta \mid \gamma \subset D) = 0\,. \tag{5.29}$$

Now, fix $K \geq 1$ and N large compared to K. Recall the sets $\mathsf{A}_{\mathsf{L},K}$ and $\mathsf{A}_{\mathsf{R},K}$ from (5.28). Using Eq. (2.29), we have

$$\begin{aligned}
&\mathbf{P}^{\mathsf{x}}(\mathcal{P}_D^{\mathrm{cp,len}}(\mathsf{x}, K), E_\delta \mid \gamma \subset D) \\
&\quad\leq \frac{1}{\mathcal{G}(\mathsf{x} \mid \gamma \subset D)} e^{-\tau_\beta(\mathsf{x})} \sum_{\substack{\Gamma^{(L)} \in \mathsf{A}_{\mathsf{L},K} \\ \gamma^{(L)} \subset D}} \mathbb{P}^{\mathsf{h}_{\mathsf{x}}}(\Gamma^{(L)}) \sum_{\substack{\Gamma^{(R)} \in \mathsf{A}_{\mathsf{R},K} \\ \gamma^{(R)} \subset D}} \mathbb{P}^{\mathsf{h}_{\mathsf{x}}}(\Gamma^{(R)})\; \mathcal{A}(\mathsf{X}(\Gamma^{(L)}), \mathsf{x} - \mathsf{X}(\Gamma^{(R)}); E_\delta)\,.
\end{aligned}$$

Using Eq. (5.3), Eq. (4.5) and Proposition 4.1, we may bound from above the right-hand side of the previous display

$$\begin{aligned}
&C^{-1} \sum_{\substack{\Gamma^{(L)} \in \mathsf{A}_{\mathsf{L},K} \\ \gamma^{(L)} \subset D}} \mathbb{P}^{\mathsf{h}_{\mathsf{x}}}(\Gamma^{(L)}) e^{\bar{\delta}\beta \|\mathsf{X}(\Gamma^{(L)})\|_1} \sum_{\substack{\Gamma^{(R)} \in \mathsf{A}_{\mathsf{R},K} \\ \gamma^{(R)} \subset D}} \mathbb{P}^{\mathsf{h}_{\mathsf{x}}}(\Gamma^{(R)}) e^{\bar{\delta}\beta \|\mathsf{X}(\Gamma^{(R)})\|_1} \frac{\mathcal{A}^+(\mathsf{X}(\Gamma^{(L)}), \mathsf{x} - \mathsf{X}(\Gamma^{(R)}); E_\delta)}{\mathbb{P}_{X(\Gamma^{(L)})}(H_{\mathsf{x}-\mathsf{X}(\Gamma^{(R)})} > H_{\mathbb{H}_-})} \\
&= C^{-1} \sum_{\substack{\Gamma^{(L)} \in \mathsf{A}_{\mathsf{L},K} \\ \gamma^{(L)} \subset D}} \mathbb{P}^{\mathsf{h}_{\mathsf{x}}}(\Gamma^{(L)}) e^{\bar{\delta}\beta \|\mathsf{X}(\Gamma^{(L)})\|_1} \sum_{\substack{\Gamma^{(R)} \in \mathsf{A}_{\mathsf{R},K} \\ \gamma^{(R)} \subset D}} \mathbb{P}^{\mathsf{h}_{\mathsf{x}}}(\Gamma^{(R)}) e^{\bar{\delta}\beta \|\mathsf{X}(\Gamma^{(R)})\|_1} \\
&\times \mathbb{P}_{\mathsf{X}(\Gamma^{(L)})}(\exists i < H_{\mathsf{x}-\mathsf{X}(\Gamma^{(R)})} : \mathsf{S}_1(i) \in [N^{4\delta}, N - N^{4\delta}]\,, \mathsf{S}_2(i) \in [0, 2N^{\delta}] \mid H_{\mathsf{x}-\mathsf{X}(\Gamma^{(R)})} < H_{\mathbb{H}_-}).
\end{aligned} \tag{5.30}$$

The ballot-type result Theorem 4.2(1) and standard random walk estimates yield

$$\lim_{N\to\infty} \mathbb{P}_{\mathsf{X}(\Gamma^{(L)})}(\exists i < H_{\mathsf{x}-\mathsf{X}(\Gamma^{(R)})} : \mathsf{S}_1(i) \in [N^{4\delta}, N - N^{4\delta}]\,, \mathsf{S}_2(i) \in [0, 2N^{\delta}] \mid H_{\mathsf{x}-\mathsf{X}(\Gamma^{(R)})} < H_{\mathbb{H}_-}) = 0$$

uniformly over $\Gamma^{(L)} \in \mathsf{A}_{\mathsf{L},K}$ and $\Gamma^{(R)} \in \mathsf{A}_{\mathsf{R},K}$. On the other hand, the exponential tails in Eq. (2.26) and the relation $4\bar{\delta} < \nu_g$ yield

$$C^{-1}\Big(\sum_{\substack{\Gamma^{(L)} \in \mathsf{A}_{\mathsf{L},K} \\ \gamma^{(L)} \subset Q}} \mathbb{P}^{\mathsf{h}_{\mathsf{x}}}(\Gamma^{(L)}) e^{\bar{\delta}\beta \|\mathsf{X}(\Gamma^{(L)})\|_1}\Big)\Big(\sum_{\substack{\Gamma^{(R)} \in \mathsf{A}_{\mathsf{R},K} \\ \gamma^{(R)} \subset Q}} \mathbb{P}^{\mathsf{h}_{\mathsf{x}}}(\Gamma^{(R)}) e^{\bar{\delta}\beta \|\mathsf{X}(\Gamma^{(R)})\|_1}\Big) \leq C_K\,,$$

for some constant $C_K := C_{K,\beta} > 0$. Taking the limit as N, then $K \to \infty$ in (5.30) yields (5.29). $\square$

Proof of Proposition 5.11. Lemma 5.12 and Corollary 5.3 imply that γ under $\mathbf{P}_D^{\mathsf{x}}(\cdot)$ stays above $\mathfrak{R}(\delta)$. Of course, all cone-points of Γ lie on γ, so we are done. ☐

5.6. **Coupling with the effective random walk.** Let $D = Q$ or $D = \mathbb{H}$, and fix $\delta \in (0, 1/8)$. The entropic repulsion result, Proposition 5.11, sets the stage for a coupling between the cone-points of Γ lying in the strip $\mathcal{S}_{N^{4\delta},N-N^{4\delta}} := [N^{4\delta}, N - N^{4\delta}] \times [0, \infty)$ and the random walk $\mathsf{S}(\cdot)$ defined in Section 4.1. Before defining this coupling explicitly, we set up some notation.

For $\mathsf{u}, \mathsf{v} \in \mathbb{Z}^2$, define

$$\begin{aligned}&\mathcal{P}^{\delta}_{D,*}(\mathsf{u},\mathsf{v})\\ &:= \Big\{\Gamma \in \mathcal{P}_D(\mathsf{u},\mathsf{v}) : |\mathsf{Cpts}(\Gamma)| \geq 2\,,\ \max_{i\in\{L,1,\dots,|\mathsf{Cpts}(\Gamma)|-1,R\}} |\Gamma^{(i)}| < (\log N)^2\,,\ \mathsf{Cpts}(\Gamma) \cap \mathfrak{R}(\delta) = \emptyset\Big\}\,,\end{aligned}$$

where, as usual, we write $\mathcal{P}^{\delta}_{D,*}(\mathsf{x}) := \mathcal{P}^{\delta}_{D,*}(0, \mathsf{x})$. Observe that the last two conditions in the definition, along with the shape of $\mathcal{Y}^{◂}$, ensure that

$$\Gamma \cap ([N^{4\delta}, N - N^{4\delta}] \times [0, N^{\delta/2}]) = \emptyset \text{ for all } \Gamma \in \mathcal{P}^{*}_{D,\delta}(\mathsf{u},\mathsf{v})\,, \tag{5.31}$$

where the intersection pertains to both contour and clusters of Γ. From Lemma 5.7 and Propositions 5.8 and 5.11, we know that

$$\mathbf{P}_D^{\mathsf{x}}(\mathcal{P}^{\delta}_{D,*}(\mathsf{x})) = 1 + o(1)\,. \tag{5.32}$$

Define the measure

$$\mathbf{P}^{\mathsf{x}}_{D,*}(\cdot) := \mathbf{P}^{\mathsf{x}}_D(\cdot \mid \mathcal{P}^{\delta}_{D,*}(\mathsf{x}))\,.$$

For $\Gamma \in \mathcal{P}^{\delta}_{D,*}$, let $\zeta^{(\mathsf{L}*)}$ and $\zeta^{(\mathsf{R}*)}$ denote the left-most and right-most cone-points of Γ in $\mathcal{S}_{N^{4\delta},N-N^{4\delta}}$, respectively. Note that, since $\Gamma \in \mathcal{P}^{\delta}_{D,*}$, we have

$$\zeta^{(\mathsf{L}*)} \in (N^{4\delta}, N^{4\delta} + (\log N)^2] \times [N^{\delta}, N^{4\delta}(\log N)^2] \tag{5.33}$$

and

$$\zeta^{(\mathsf{R}*)} \in [N - N^{4\delta} - (\log N)^2, N - N^{4\delta}] \times (N^{\delta}, N^{4\delta}(\log N)^2]\,. \tag{5.34}$$

Let $\Gamma^{(\mathsf{L}*)} \in \mathcal{P}^{\delta}_{D,*}(0, \zeta^{(\mathsf{L}*)})$ denote the portion of Γ connecting 0 to $\zeta^{(\mathsf{L}*)}$ (including all clusters connected to $\gamma^{(\mathsf{L}*)}$). Similarly, define $\Gamma^{(\mathsf{R}*)} \in \mathcal{P}^{\delta}_{D,*}(\zeta^{(\mathsf{R}*)}, \mathsf{x})$ as the portion of Γ connecting $\zeta_{\mathsf{R}*}$ to x, and define $\Gamma^* \in \mathcal{P}^{\delta}_{D,*}(\zeta^{(\mathsf{L}*)}, \zeta^{(\mathsf{R}*)})$ as the portion of Γ connecting $\zeta_{\mathsf{L}*}$ to $\zeta_{\mathsf{R}*}$ so that

$$\Gamma = \Gamma^{(\mathsf{L}*)} \circ \Gamma^* \circ \Gamma^{(\mathsf{R}*)}\,.$$

Note that Γ^* is a concatenation of irreducible animals, each of which are completely contained in Q by (5.31). Therefore, from (2.18), we have

$$q_D(\Gamma) = q_D(\Gamma^{(\mathsf{L}*)}) q(\Gamma^*) q_D(\Gamma^{(\mathsf{R}*)})\,.$$

Crucially, the q_D-weight of Γ^* is equal to its (free) q-weight.

For any fixed $\overline{\zeta}^{(\mathsf{L}*)}$ and $\overline{\zeta}^{(\mathsf{R}*)}$ satisfying (5.33) and (5.34) respectively, and for any fixed animals $\overline{\Gamma}^{(\mathsf{L}*)} \in \mathcal{P}_D(0, \overline{\zeta}^{(\mathsf{L}*)})$ and $\overline{\Gamma}^{(\mathsf{R}*)} \in \mathcal{P}_D(\overline{\zeta}^{(\mathsf{R}*)}, \mathsf{x})$, consider the measure

$$\mathbf{P}^{\overline{\zeta}^{(\mathsf{L}*)},\overline{\zeta}^{(\mathsf{R}*)}}_{D,*}(\cdot) := \mathbf{P}^{\mathsf{x}}_{D,*}(\,\cdot\, \mid \Gamma^{(\mathsf{L}*)} = \overline{\Gamma}^{(\mathsf{L}*)}, \Gamma^{(\mathsf{R}*)} = \overline{\Gamma}^{(\mathsf{R}*)})\,, \tag{5.35}$$

which describes the conditional law of Γ^* and defines a probability measure on the following set of animals:

$$\mathcal{P}^{*,\mathsf{A}}_{D,\delta}(\overline{\zeta}^{(\mathsf{L}*)}, \overline{\zeta}^{(\mathsf{R}*)}) := \{\overline{\Gamma}^{(1)} \circ \cdots \circ \overline{\Gamma}^{(n)} \in \mathcal{P}^{*}_{D,\delta}(\overline{\zeta}^{(\mathsf{L}*)}, \overline{\zeta}^{(\mathsf{R}*)}) \,:\, n \in \mathbb{N}\,, \overline{\Gamma}^{(i)} \in \mathsf{A} \text{ for each } i \in [1, n]\}\,.$$

Note that, for fixed $\overline{\zeta}^{(\mathsf{L}*)}$ and $\overline{\zeta}^{(\mathsf{R}*)}$, the choice of $\bar{\Gamma}^{(\mathsf{L}*)}$ and $\bar{\Gamma}^{(\mathsf{R}*)}$ does not change $\mathbf{P}_{D,*}^{\overline{\zeta}^{(\mathsf{L}*)},\overline{\zeta}^{(\mathsf{R}*)}}(\cdot)$. In particular, we have the formula

$$\mathbf{P}_{D,*}^{\overline{\zeta}^{(\mathsf{L}*)},\overline{\zeta}^{(\mathsf{R}*)}}(E) = \sum_{\Gamma\in E\cap\mathcal{P}^{*,A}_{D,\delta}(\overline{\zeta}^{(\mathsf{L}*)},\overline{\zeta}^{(\mathsf{R}*)})} q(\Gamma) \Big/ \sum_{\Gamma\in\mathcal{P}^{*,A}_{D,\delta}(\overline{\zeta}^{(\mathsf{L}*)},\overline{\zeta}^{(\mathsf{R}*)})} q(\Gamma)\,. \tag{5.36}$$

Again, note that only the free weight appears in the above formula.

Lastly, for any $c>0$, define the half-space $\mathbb{H}_c := \{(x,y)\in\mathbb{R}^2 : y \geq c\}$.

Proposition 5.13. *Let $D=Q$ or $D=\mathbb{H}$. There exists $\nu_2>0$ such that for all $N\in\mathbb{N}$, for all $\beta>0$ sufficiently large, and for all $\overline{\zeta}^{(\mathsf{L}*)}$ and $\overline{\zeta}^{(\mathsf{R}*)}$ satisfying (5.33) and (5.34) respectively, if we let $T=H_{\overline{\zeta}^{(\mathsf{R}*)}}$ and view $\mathsf{Cpts}(\Gamma^*)$ as an ordered tuple (see Eq. (2.20)),*

$$\Big\|\mathbf{P}_{D,*}^{\overline{\zeta}^{(\mathsf{L}*)},\overline{\zeta}^{(\mathsf{R}*)}}\Big(\mathsf{Cpts}(\Gamma^*)\in\cdot\Big) - \mathbb{P}_{\overline{\zeta}^{(\mathsf{L}*)}}\Big((\mathsf{S}(i))_{i=0}^T \in \cdot \mid T < H_{\mathbb{H}_{N^\delta}}\Big)\Big\|_{\mathrm{TV}} \leq Ce^{-\nu_2\beta(\log N)^2}$$

for some constant $C:=C(\beta)>0$.

Proof. Fix $\overline{\zeta}^{(\mathsf{L}*)}$ and $\overline{\zeta}^{(\mathsf{R}*)}$ satisfying (5.33) and (5.34) respectively. Recall $\mathsf{V}^+_{\mathsf{u},\mathsf{v}}$ from (4.3), and let

$$\mathsf{V}^\delta_{\overline{\zeta}^{(\mathsf{L}*)},\overline{\zeta}^{(\mathsf{R}*)}} := \Big\{(\vec{v}_1,\dots,\vec{v}_n)\in\mathsf{V}^+_{\overline{\zeta}^{(\mathsf{L}*)},\overline{\zeta}^{(\mathsf{R}*)}} : n\geq 1,\ \|\vec{v}_k\|_1 \leq (\log N)^2 \text{ and } \overline{\zeta}^{(\mathsf{L}*)} + \sum_{i=1}^k \vec{v}_i \geq N^\delta\ \forall k\in[1,n]\Big\}$$

$$\mathsf{A}^\delta_{\overline{\zeta}^{(\mathsf{L}*)},\overline{\zeta}^{(\mathsf{R}*)}} := \Big\{(\overline{\Gamma}^{(1)},\dots,\overline{\Gamma}^{(n)})\in\mathsf{A}^n :\ n\geq 1,\ (X(\overline{\Gamma}^{(1)}),\dots,X(\overline{\Gamma}^{(n)}))\in\mathsf{V}^\delta_{\overline{\zeta}^{(\mathsf{L}*)},\overline{\zeta}^{(\mathsf{R}*)}}\Big\}\,.$$

Note that the sets $\mathsf{A}^\delta_{\overline{\zeta}^{(\mathsf{L}*)},\overline{\zeta}^{(\mathsf{R}*)}}$ and $\mathcal{P}^{*,\mathsf{A}}_{D,\delta}(\overline{\zeta}^{(\mathsf{L}*)},\overline{\zeta}^{(\mathsf{R}*)})$ are in bijection with one another, since for any $(\overline{\Gamma}^{(1)},\dots,\overline{\Gamma}^{(n)})\in\mathsf{A}^\delta_{\overline{\zeta}^{(\mathsf{L}*)},\overline{\zeta}^{(\mathsf{R}*)}}$, the animal $\overline{\Gamma}^{(1)}\circ\cdots\circ\overline{\Gamma}^{(n)}$ is in $\mathcal{P}^{*,\mathsf{A}}_{D,\delta}(\overline{\zeta}^{(\mathsf{L}*)},\overline{\zeta}^{(\mathsf{R}*)})$. This observation along with (2.27) yields the following

$$\begin{aligned}
\sum_{\Gamma\in\mathcal{P}^{*,A}_{D,\delta}(\overline{\zeta}^{(\mathsf{L}*)},\overline{\zeta}^{(\mathsf{R}*)})} q(\Gamma) &= e^{\mathsf{h}_\mathsf{x}\cdot(\overline{\zeta}^{(\mathsf{L}*)}+\mathsf{x}-\overline{\zeta}^{(\mathsf{R}*)})} \sum_{n\geq 1}\ \sum_{(\overline{\Gamma}^{(1)},\dots,\overline{\Gamma}^{(n)})\in\mathsf{A}^\delta_{\overline{\zeta}^{(\mathsf{L}*)},\overline{\zeta}^{(\mathsf{R}*)}}}\ \prod_{i=1}^n \mathbb{P}^{\mathsf{h}_\mathsf{x}}(\overline{\Gamma}^{(i)}) \\
&= e^{\mathsf{h}_\mathsf{x}\cdot(\overline{\zeta}^{(\mathsf{L}*)}+\mathsf{x}-\overline{\zeta}^{(\mathsf{R}*)})} \sum_{n\geq 1}\ \sum_{(\vec{v}_1,\dots,\vec{v}_n)\in\mathsf{V}^\delta_{\overline{\zeta}^{(\mathsf{L}*)},\overline{\zeta}^{(\mathsf{R}*)}}}\ \prod_{i=1}^n \Big(\sum_{\overline{\Gamma}^{(i)}\in\mathsf{A}} \mathbb{P}^{\mathsf{h}_\mathsf{x}}(\overline{\Gamma}^{(i)})\mathbb{1}_{\{\overline{\mathsf{X}}(\overline{\Gamma}^{(i)})=\vec{v}_i\}}\Big) \\
&= e^{\mathsf{h}_\mathsf{x}\cdot(\overline{\zeta}^{(\mathsf{L}*)}+\mathsf{x}-\overline{\zeta}^{(\mathsf{R}*)})} \sum_{n\geq 1}\ \sum_{(\vec{v}_1,\dots,\vec{v}_n)\in\mathsf{V}^\delta_{\overline{\zeta}^{(\mathsf{L}*)},\overline{\zeta}^{(\mathsf{R}*)}}}\ \prod_{i=1}^n \mathbb{P}(\mathsf{S}(1)=\vec{v}_i) \\
&= e^{\mathsf{h}_\mathsf{x}\cdot(\overline{\zeta}^{(\mathsf{L}*)}+\mathsf{x}-\overline{\zeta}^{(\mathsf{R}*)})}\ \mathbb{P}_{\overline{\zeta}^{(\mathsf{L}*)}}(E^*_\delta)\,,
\end{aligned} \tag{5.37}$$

where we define the random walk event

$$E^*_\delta := \{H_{\overline{\zeta}^{(\mathsf{R}*)}} < H_{\mathbb{H}_{N^\delta}}\,,\ \|\mathsf{S}(i+1)-\mathsf{S}(i)\|_1 \leq (\log N)^2,\ \forall i\in[0, H_{\overline{\zeta}^{(\mathsf{R}*)}}-1]\}\,.$$

A nearly identical calculation but for the numerator of the right-hand side of (5.36) yields the following: for any $n\geq 1$ and for any $(\vec{v}_1,\dots,\vec{v}_n)\in(\mathbb{Z}^2)^n$,

$$\begin{aligned}
&\mathbf{P}_{D,*}^{\overline{\zeta}^{(\mathsf{L}*)},\overline{\zeta}^{(\mathsf{R}*)}}\Big(\mathsf{Cpts}(\bar{\Gamma}) = \Big(\overline{\zeta}^{(\mathsf{L}*)},\ \overline{\zeta}^{(\mathsf{L}*)}+v_1,\dots,\ \overline{\zeta}^{(\mathsf{L}*)}+\sum_{i=1}^{n-1}\vec{v}_i, \overline{\zeta}^{(\mathsf{R}*)}\Big)\Big) \\
&= \mathbb{P}_{\overline{\zeta}^{(\mathsf{L}*)}}\Big(H_{\overline{\zeta}^{(\mathsf{R}*)}} = n+1\,,\ \mathsf{S}^{(k)} = \overline{\zeta}^{(\mathsf{L}*)} + \sum_{i=1}^k \vec{v}_i\,,\ \forall k\in[1,n] \mid E^*_\delta\Big)\mathbb{1}_{\{(\vec{v}_1,\dots,\vec{v}_n)\in\mathsf{V}^\delta_{\overline{\zeta}^{(\mathsf{L}*)},\overline{\zeta}^{(\mathsf{R}*)}}\}}\,.
\end{aligned}$$

Thus, the law of $\mathsf{Cpts}(\overline{\Gamma})$ under $\mathbf{P}_{D,*}^{\overline{\zeta}^{(\mathsf{L}*)},\overline{\zeta}^{(\mathsf{R}*)}}$ is *equal to* the law of $(\mathsf{S}_i)_{i=0}^{H_{\overline{\zeta}^{(\mathsf{R}*)}}}$ under $\mathbb{P}_{\overline{\zeta}^{(\mathsf{L}*)}}(\cdot \mid E_\delta^*)$. Now, observe that Theorem 4.2 along with shift-invariance of the random walk (to shift the walk downwards by N^δ) give the following estimate, which holds uniformly over our ranges of $\overline{\zeta}^{(\mathsf{L}*)}$ and $\overline{\zeta}^{(\mathsf{R}*)}$:

$$\mathbb{P}_{\overline{\zeta}^{(\mathsf{L}*)}}\big(H_{\overline{\zeta}^{(\mathsf{R}*)}} < H_{\mathbb{H}_{N^\delta}}\big) = \mathbb{P}_{\overline{\zeta}^{(\mathsf{L}*)}-(0,N^\delta)}\big(H_{\overline{\zeta}^{(\mathsf{R}*)}-(0,N^\delta)} < H_{\mathbb{H}_-}\big) \gtrsim N^{-3/2}\,.$$

Exponential tails of the random walk increments (4.2) and the above estimate then imply that

$$\mathbb{P}_{\overline{\zeta}^{(\mathsf{L}*)}}\big(E_\delta^* \mid H_{\overline{\zeta}^{(\mathsf{R}*)}} < H_{\mathbb{H}_{N^\delta}}\big) \geq 1 - Ce^{-\nu\beta(\log N)^2} \tag{5.38}$$

for some constant $\nu > 0$ independent of β, and a constant $C := C(\beta) > 0$, finishing the proof. ☐

5.7. **Proof of Theorem 2.8.** We see from the shape of the cone $\mathcal{Y}^{\blacktriangleleft}$ and the restriction on $|\Gamma^{(i)}|$ that for all $\Gamma \in \mathcal{P}_{D,\delta}^*(\mathsf{x})$,

$$\frac{1}{\sqrt{N}} \max_{x\in[0,N]} |\overline{\gamma}(x) - \underline{\gamma}(x)| \leq \frac{2(\log N)^2}{\sqrt{N}}\,.$$

In addition to (5.32), this tells us that it suffices to show $\mathfrak{J}_N(\mathsf{Cpts}(\Gamma))$ under $\mathbf{P}_{D,*}^{\mathsf{x}}(\cdot)$ converges weakly to a Brownian excursion. Now, we have the following on $\mathcal{P}_{D,\delta}^*$:

$$\sup\{y : \exists x \in \mathbb{Z} \text{ s.t. } (x,y) \in \gamma^{(\mathsf{L}*)}\}\,, \sup\{y : \exists x \in \mathbb{Z} \text{ s.t. } (x,y) \in \gamma^{(\mathsf{R}*)}\} \leq N^{4\delta}(\log N)^2 \tag{5.39}$$

as well as the bounds in Eqs. (5.33) and (5.34). These bounds imply that $\Gamma^{(\mathsf{L}*)}$ and $\Gamma^{(\mathsf{R}*)}$ do not impact the diffusive scaling limit of $\mathsf{Cpts}(\Gamma)$. Thus, recalling the linear interpolation $\mathfrak{J}_N$ from (4.8), it suffices to show $\mathfrak{J}_N(\mathsf{Cpts}(\Gamma^*))$ under law $\mathbf{P}_{D,*}^{\overline{\zeta}^{(\mathsf{L}*)},\overline{\zeta}^{(\mathsf{R}*)}}(\cdot)$ converges weakly to a Brownian excursion, uniformly in $\overline{\zeta}^{(\mathsf{L}*)}$ and $\overline{\zeta}^{(\mathsf{R}*)}$ satisfying (5.33) and (5.34). But this is an immediate consequence of Proposition 5.13, the shift invariance of the random walk, and Theorem 4.3.

This concludes the proof of Theorem 2.8, and hence also Theorem 1.2. ☐

6. Random Walks in a half-space

This section is devoted to the proofs of Theorems 4.2 and 4.3 via the analysis of random walks in a half-space.

As mentioned after Theorem 4.3, we seek to extend many of the results from [17] and [23]. These papers are written for random walks whose coordinates are uncorrelated. Though $\mathsf{S}(\cdot)$ may not have this property, we can obtain such a random walk by rotating— this is done in Section 6.3.

As such, we must consider random walks in general half-spaces. Throughout the rest of this section, fix a unit vector $\vec{n} \in \mathbb{S}^1$, and consider the half-space through the origin with inward normal $\vec{n}$, and call this space $\mathbb{H}_{\vec{n}} \subset \mathbb{R}^2$. Let $\vec{n}^\perp \in \mathbb{S}^1$ be a unit vector orthogonal to $\vec{n}$ (the choice between $\vec{n}$ and $-\vec{n}$ does not matter). We will think of our random walk in terms of coordinates with respect to $\vec{n}$ and $\vec{n}^\perp$ instead of e_1 and e_2. Let $S(\cdot)$ denote a general 2D random walk with step distribution $X = (X_1, X_2)$ such that $\mathbb{E}X = \vec{0}$, $\operatorname{Cov} X = \mathrm{Id}$, and $\mathbb{P}(|X| > m) \leq c_1 e^{-c_2 m}$ for some constants $c_1, c_2 > 0$ and for all $m \geq 1$. Additionally, we impose a lattice assumption: assume that X takes values on a lattice $\mathcal{L}$ that is a non-degenerate linear transformation of $\mathbb{Z}^2$, and that the distribution of X is strongly aperiodic; that is, for each $\mathsf{u} \in \mathcal{L}$, $\mathcal{L}$ is the smallest subgroup of $\mathbb{Z}^2$ containing

$$\{\mathsf{v} : \mathsf{v} = \mathsf{u} + \mathsf{w} \text{ for some } \mathsf{w} \text{ such that } \mathbb{P}(X = \mathsf{w}) > 0\}\,.$$

The theory of random walks confined to a half-space is intimately related to the theory of harmonic functions for processes killed upon exiting a half-space, and so we recall the relevant facts from this theory before proceeding.

6.1. **Harmonic functions for processes in a half-space.** The study of Markov processes conditioned to stay in a domain is intimately related to the study of their corresponding positive harmonic functions. Our domain of interest is a relatively simple one, the half-space $\mathbb{H}_{\vec{n}}$, and we are concerned with the harmonic functions of the Brownian motion and of the random walk $S(\cdot)$ killed upon exit from $\mathbb{H}_{\vec{n}}$.

The harmonic function of the Brownian motion killed at $\partial\mathbb{H}_{\vec{n}}$ is given by the minimal (up to a constant), strictly positive harmonic function on $\mathbb{H}_{\vec{n}}$ with zero boundary conditions. For us, such functions take a very explicit form. Consider the rotation that sends $\vec{n} \mapsto \mathsf{e}_2$, and thus $\mathbb{H}_{\vec{n}}$ to $\mathbb{H}$. This is a conformal mapping that induces a bijection between positive harmonic functions in $\mathbb{H}$ that vanish on $\partial\mathbb{H}$ to positive harmonic functions in $\mathbb{H}_{\vec{n}}$ that vanish on $\partial\mathbb{H}_{\vec{n}}$. Since[12] every such harmonic function in $\mathbb{H}$ takes the form $h(\mathsf{v}) = c\mathsf{v}\cdot\mathsf{e}_2$, for some constant $c > 0$, it follows that every harmonic h function in $\mathbb{H}_{\vec{n}}$ takes the form $h(\mathsf{v}) = c\mathsf{v}\cdot\vec{n}$. In particular, the harmonic function only depends on the projection of v onto $\vec{n}$. In what follows, we take $c = 1$ and define

$$u(\mathsf{v}) := \mathsf{v}\cdot\vec{n}\,.$$

We refer to [15] for a general discussion of the harmonic function of the Brownian motion in cones.

One of the many achievements of [17] is the construction of a positive harmonic function V for a wide class of random walks $S(\cdot)$ killed upon exit from K, i.e., V solves the equation[13]

$$\mathbb{E}_{\mathsf{v}}[V(S(1)), H_{K^c} > 1] = V(\mathsf{v})\,, \quad \text{for } \mathsf{v}\in K\,,$$

where K is an element of a wide class of cones.[14] Many of their limit theorems that we wish to extend were stated in terms of V. The function V was constructed as

$$V(\mathsf{v}) = \lim_{n\to\infty} \mathbb{E}_{\mathsf{v}}[h(S(n))\,,\, H_{K^c} > n]\,,$$

where h is a choice of the harmonic function for Brownian motion killed at ∂K. For $K := H_{\mathbb{H}_{\vec{n}}}$, $h := u$, and sowe see

$$V(\mathsf{v}) = V_1(\mathsf{v}\cdot\vec{n})\,, \text{ for all } \mathsf{v}\in\mathbb{H}_{\vec{n}}\,, \tag{6.1}$$

where $V_1 := V_1^{\vec{n}}$ is the unique positive harmonic function for the one-dimensional random walk $\vec{n}\cdot S(\cdot)$ killed at leaving $(0,\infty)$ satisfying

$$\lim_{a\to\infty}\frac{V_1(a)}{a} = 1\,.$$

The uniqueness of V_1, as well as an exact formula for V_1, was established by Doney in [22].

6.2. **Limit theorems for general random walks in a half-space.** The following two results, Propositions 6.1 and 6.2, are modifications of a ballot-type theorem from [17] and an invariance principle from [23], respectively. We'll write $\mathfrak{e}(\cdot)$ to denote the standard Brownian excursion on $[0,1]$.

Proposition 6.1 (Modification of [17, Theorem 6])**.** *Fix any $A > 0$ and $\delta\in(0,1/2)$. Then there exists a constant $C_1 > 0$ such that, uniformly over sequences a_k, b_k and u_k satisfying*

$$a_k \in [-A\sqrt{k}, A\sqrt{k}]\,, \quad \frac{a_k}{\sqrt{k}} \to a\in[-A,A]\,, \quad b_k, u_k\in(0, k^{1/2-\delta}]\,, \text{ and } \quad \{u_k\vec{n}\,,\, a_k\vec{n}^\perp + b_k\}\subset\mathcal{L} \tag{6.2}$$

we have

$$\mathbb{P}_{u_k\vec{n}}\Big(S(k) = a_k\vec{n}^\perp + b_k\vec{n}\,,\, H^S_{\mathbb{H}_{-\vec{n}}} > k\Big) \sim C_1\kappa^2\frac{V_1(u_k)V_1'(b_k)}{k^2}e^{-a^2/2}\,, \tag{6.3}$$

[12] See, e.g., [2, Theorem 7.22].

[13] In [17] and [23], the authors consider random walks $(S_\cdot)$ from $S_0 = 0$, and study the law of $(\mathsf{v} + S_\cdot)$ for $\mathsf{v}\in K$. In particular, they write τ_{v} to the hitting time $H^{S_\cdot}_{K^c}$ under law $\mathbb{P}_{\mathsf{v}}$.

[14] See also [18], where such a harmonic function was constructed for a more general class of cones; as well as [19], which addressed a more general class of random walks.

where V_1' is the positive harmonic function for $-\vec{n}\cdot S_2(\cdot)$ killed upon leaving $(0,\infty)$.

Proposition 6.2 (Modification of [23, Theorem 6])**.** *Fix any $\delta\in(0,1/2)$. Uniformly over sequences*

$$a_k\in[-A\sqrt{k},A\sqrt{k}] \quad \text{and} \quad b_k,u_k\in(0,k^{1/2-\delta}] \quad \text{and} \quad \{u_k\vec{n}\,,\,a_k\vec{n}^\perp+b_k\}\subset\mathcal{L}\,, \tag{6.4}$$

the family of conditional laws

$$\mathbf{Q}^k_{u_k,a_k,b_k}(\cdot):=\mathbb{P}_{u_k\vec{n}}\Big(\Big(\frac{\vec{n}\cdot S(\lfloor tk\rfloor)}{\sqrt{k}}\Big)_{t\in[0,1]}\in\cdot\mid S(k)=a_k\vec{n}^\perp+b_k\vec{n}\,,\,H^S_{\mathbb{H}_{-\vec{n}}}>k\Big)$$

converge as $k\to\infty$ to the law of $\mathfrak{e}(\cdot)$ in the Skorokhod space $(D[0,1],\|\cdot\|_\infty)$.

For the next result, let $\mathbf{S}_\cdot$ denote a two-dimensional random walk with step distribution $\mathbf{X}=(\mathbf{X}_1,\mathbf{X}_2)$ satisfying the same lattice, covariance, and tail decay assumptions as X, but with mean $\mathbb{E}\mathbf{X}=\mu\vec{n}^\perp$ for some $\mu>0$.

Proposition 6.3. *Fix any $\delta\in(0,1/2)$. There exists a constant $\mathbf{C}':=\mathbf{C}'(\mathbf{X})>0$ such that, uniformly over $u,v\in(0,N^{1/2-\delta}]$ and $\{u\vec{n}\,,\,N\vec{n}^\perp+v\vec{n}\}\subset\mathcal{L}$, we have*

$$\mathbb{P}_{u\vec{n}}\Big(H^{\mathbf{S}}_{N\vec{n}^\perp+v\vec{n}}<H^{\mathbf{S}}_{\mathbb{H}_{-\vec{n}}}\Big)\sim\mathbf{C}'\frac{V_1(u)V_1'(v)}{N^{3/2}}\,, \tag{6.5}$$

where V_1' denotes the harmonic function for $-\vec{n}\cdot\mathbf{S}_2(\cdot)$ killed upon leaving $(0,\infty)$.

Furthermore, if

$$\mathbf{p}_{N,A}:=\mathbb{P}_{u\vec{n}}\Big(H^{\mathbf{S}}_{N\vec{n}^\perp+v\vec{n}}\in[N/\mu-A\sqrt{N},N/\mu+A\sqrt{N}]\mid H^{\mathbf{S}}_{N\vec{n}^\perp+v\vec{n}}<H^{\mathbf{S}}_{\mathbb{H}_{-\vec{n}}}\Big)$$

then

$$\lim_{A\to\infty}\liminf_{N\to\infty}\mathbf{p}_{N,A}=1\,. \tag{6.6}$$

Propositions 6.1 to 6.3 will be proved in Section 6.5. Note that when $\vec{n}=\mathsf{e}_2$ (so $\mathbb{H}_{\vec{n}}=\mathbb{H}$), Proposition 6.3 reduces to [31, Theorem 5.1].

6.3. **Proof of Theorems 4.2 and 4.3.** Recall that the random walk $\mathsf{S}(\cdot)$ is a two-dimensional random walk whose step distribution $X=(X_1,X_2)$ satisfies $\mathbb{E}[X]=(\mu,0)$, for some $\mu>0$, and exponential tail decay for $|X|$. Recall $\sigma_1^2:=\mathrm{Var}(X_1)$ and $\sigma_2^2:=\mathrm{Var}(X_2)$.

Proof of Theorem 4.2. Theorem 4.2 follows from Propositions 6.1 to 6.3 by linearly transforming the random walk $\mathsf{S}(\cdot)$ to satisfy the hypotheses of these results. We demonstrate this explicitly for item (2) of Theorem 4.2; the proof of item (1) follows readily.

Following from [17, Example 2], we derive the aforementioned linear transformation, . Consider the re-centered, normalized random walk

$$\bar{\mathsf{S}}(n):=\begin{pmatrix}\sigma_1^{-1} & 0\\ 0 & \sigma_2^{-1}\end{pmatrix}\left(\mathsf{S}(n)-n\begin{pmatrix}\mu\\0\end{pmatrix}\right).$$

This is a random walk with increments $\bar{X}:=(\bar{X}_1,\bar{X}_2)$ of mean 0 such that, for some $\rho\in(0,1)$, we have $C:=\mathrm{Cov}\,\bar{X}=\mathbb{E}\bar{X}\bar{X}^t=\begin{pmatrix}1&\rho\\ \rho&1\end{pmatrix}$. Let $C=ODO^t$. Since C is a covariance matrix, it is positive semi-definite; however, if C had 0 as an eigenvalue, then that would imply the random walk lives on a line, which we know is not true as X can be e_1 or $\mathsf{e}_1+\mathsf{e}_2$ with positive probability. So, C is positive-definite. Consider the transformation matrix $M:=O^t\sqrt{D^{-1}}O$. Then

$$\mathrm{Cov}(MX)=\mathbb{E}[M\bar{X}\bar{X}^tM^t]=\mathrm{Id}\,.$$

To be explicit, straight-forward calculations reveal that if $\theta\in[-\pi,\pi]$ solves $\sin 2\theta=\rho$, then we may take

$$M=\frac{1}{\sqrt{1-\rho^2}}\begin{pmatrix}\cos\theta & -\sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\,,$$

though we will not need this explicit form.

Now, observe that $\vec{n} := M^{-1}\mathsf{e}_2$ has norm 1 (explicitly, $\vec{n} = (\sin\theta, \cos\theta)$). Using the fact that M^{-1} is symmetric and $\vec{n}$ is outward normal to $M\mathbb{H}_- = \mathbb{H}_{-\vec{n}}$,

$$\{H^{\mathsf{S}}_{\mathbb{H}_-} > k\} = \{\mathsf{e}_2^t\mathsf{S}(i) > 0\,,\ \forall i \in [0,k]\} = \{\mathsf{e}_2^t\overline{\mathsf{S}}(i) > 0\,,\ \forall i \in [0,k]\} = \{M\overline{\mathsf{S}}(i)\cdot M^{-1}\mathsf{e}_2 > 0\,,\ \forall i \in [0,k]\}$$
$$= \{H^{M\overline{\mathsf{S}}}_{\mathbb{H}_{-\vec{n}}} > k\}\,,$$

In what follows, we write $M(a,b)$ to denote the matrix applied to the vector $a\mathsf{e}_1 + b\mathsf{e}_2$. Then for any measurable set B,

$$\mathbb{P}_{(0,u)}\Big(\mathsf{S}(\cdot) \in \mathrm{B} \mid S(k) = (N,v)\,,\ H^{\mathsf{S}(\cdot)}_{\mathbb{H}_-} > k\Big)$$
$$= \mathbb{P}_{(0,\sigma_2^{-1}u)}\Big(\overline{\mathsf{S}}(\cdot) \in \mathrm{B} - k(\mu,0) \mid \overline{\mathsf{S}}(k) = (\tfrac{N-k\mu}{\sigma_1}, \tfrac{v}{\sigma_2})\,,\ H^{\overline{\mathsf{S}}}_{\mathbb{H}_-} > k\Big)$$
$$= \mathbb{P}_{M(0,u)}\Big(M\overline{\mathsf{S}}(\cdot) \in M(\mathrm{B} - k(\mu,0)) \mid M\overline{\mathsf{S}}(k) = M(\tfrac{N-k\mu}{\sigma_1}, \tfrac{v}{\sigma_2})\,,\ H^{M\overline{\mathsf{S}}}_{\mathbb{H}_{-\vec{n}}} > k\Big)\,. \tag{6.7}$$

Let us now check the hypotheses of Proposition 6.2. Take $\vec{n}^\perp := M\mathsf{e}_1/\,\|M\mathsf{e}_1\|$. Note that $\vec{n}^\perp$ spans $\partial\mathbb{H}_{-\vec{n}}$ and $\vec{n}$ is the inward normal of $M\mathbb{H} = \mathbb{H}_{\vec{n}}$. $M\mathsf{S}(\cdot)$ is a random walk on the lattice $\mathcal{L}$ generated by $\|M\mathsf{e}_1\|\,\vec{n}^\perp$ and $M\mathsf{e}_2$, the latter of which can be expressed as

$$M\mathsf{e}_2 = (M\mathsf{e}_2\cdot\vec{n}^\perp)\vec{n}^\perp + \vec{n} = \frac{\mathsf{e}_1^t M^t M\mathsf{e}_2}{\|M\mathsf{e}_1\|}\vec{n}^\perp + \vec{n} = \frac{\mathsf{e}_1^t C^{-1}\mathsf{e}_2}{\|M\mathsf{e}_1\|}\vec{n}^\perp + \vec{n}\,.$$

Thus, for any $x, y \in \mathbb{R}$,

$$M(x,y) = \frac{x + y\mathsf{e}_2^t C^{-1}\mathsf{e}_1}{\|M\mathsf{e}_1\|}\vec{n}^\perp + y\vec{n}\,.$$

For any $k \in [N/\mu - A\sqrt{N}, N/\mu + A\sqrt{N}]$, for A arbitrarily large (but fixed with respect to N), and for any $u, v \in [1, N^{1/2-\delta}]$, we see that

$$u_k\vec{n} := M(0,u) \quad\text{ and }\quad a_k\vec{n}^\perp + b_k\vec{n} := M(\tfrac{N-k\mu}{\sigma_1}, \tfrac{v}{\sigma_2})$$

satisfies Eq. (6.2). With the hypotheses of Proposition 6.2 satisfied, it follows from Eq. (6.7) and

$$\sigma_2^{-1}\mathsf{S}_2(\cdot) = \overline{\mathsf{S}}(\cdot)\cdot\mathsf{e}_2 = M\overline{\mathsf{S}}(\cdot)\cdot M^{-1}\mathsf{e}_2 = M\overline{\mathsf{S}}(\cdot)\cdot\vec{n}$$

that item (2) of Theorem 4.2 is an immediate consequence of Proposition 6.2. ☐

We now use Theorem 4.2 to prove the Brownian excursion limit.

Proof of Theorem 4.3. Due to Eq. (4.7), it suffices to show that, for each fixed k, uniformly in the number of steps $k \in [N/\mu - A\sqrt{N}, N\mu + A\sqrt{N}]$, the family of conditional laws

$$\mathbf{Q}^{N,k}_{u,v}(\cdot) := \mathbb{P}_{(0,u)}\Big(\big(\mathfrak{e}^{\mathsf{S},v}(t)\big)_{t\in[0,1]} \in \cdot \;\Big|\; \mathsf{S}(k) = (N,v)\,,\ k < H^{\mathsf{S}}_{\mathbb{H}_-}\Big)$$

converges as $k \to \infty$ to the law of the standard Brownian excursion on $[0,1]$. We begin with Claim 6.4, which localizes the x-coordinate $\mathsf{S}_1(j)$ to an interval of size $o(N)$, for $j \in [0,k]$, and allows us to compare the linear interpolation $\mathfrak{e}^{\mathsf{S},v}$ with the linear interpolation in item (2) of Theorem 4.2.

Claim 6.4. *For any $\eta > 0$, uniformly over u and v as in the theorem statement, we have*

$$\lim_{N\to\infty}\mathbb{P}_{(0,u)}\Big(\max_{j\in[0,k]}|\mathsf{S}_1(j) - \mu j| > N^{1/2+\eta} \mid \mathsf{S}(k) = (N,v)\,,\ k < H^{\mathsf{S}}_{\mathbb{H}_-}\Big) = 1\,. \tag{6.8}$$

Proof. In light of the bound on $\mathbb{P}_{(0,u)}(\mathsf{S}(k) = (N,v)\,,\ k < H^{\mathsf{S}}_{\mathbb{H}_-})$ provided by Eq. (4.6), it suffices to show

$$\mathbb{P}_0\Big(\max_{j\in[0,N]}|\mathsf{S}_1(j) - \mu j| > N^{1/2+\eta}\Big) = o(N^{-3/2})\,. \tag{6.9}$$

This follows from a union bound over j, the exponential tail bound on the steps of $\mathsf{S}_1(\cdot)$ from Eq. (4.2), and Hoeffding's inequality:

$$\mathbb{P}_0\Big(\max_{j\in[0,N]}\big|\mathsf{S}_1(j)-\mu j\big|>N^{1/2+\eta}\Big)\le N\max_{j\in[0,N]}\mathbb{P}_0\Big(\big|\mathsf{S}_1(j)-\mu j\big|>N^{1/2+\eta}\Big)$$
$$\le N\max_{j\in[0,N]}\mathbb{P}_0\Big(\big|\mathsf{S}_1(j)-\mu j\big|>N^{1/2+\eta},\ \max_{i\in[0,j-1]}\big|\mathsf{S}(i)-\mathsf{S}(i+1)\big|\le(\log N)^2\Big)+c'N^2e^{-\nu_g\beta(\log N)^2}$$
$$\le 2Ne^{-\frac{2N^{2\eta}}{(\log N)^4}}+c'N^2e^{-\nu_g\beta(\log N)^2}=o(N^{-3/2})\,. \qquad \square$$

Note that Theorem 4.2(2) and Claim 6.4 are the equivalents of Equations (76) and (77) of [31] for our random walk. We therefore finish the proof of our Theorem 4.3 exactly as in the proof of [31, Theorem 5.3]. □

6.4. **Uniform estimates for the random walk in a half-space.** Before proving Propositions 6.1 to 6.3, we need to modify several key estimates of [17] to address the range of parameters in Eqs. (6.2) and (6.4). In what follows, we will repeatedly refer back to [17], explaining how their arguments can be adapted to give the desired uniformity. When stating or citing the results of [17] and [23], we will for the most part use their notation, pointing out any discrepancies explicitly.

6.4.1. *Needed inputs.* The results of [17] are often stated in terms of the dimension d and a positive constant p, where p is related to the asymptotic behavior of the relevant harmonic function in the cone. For us, $d=2$ and $p=1$.

Our first input is an extension of [17, Equation 7], which gives a tail estimate on the hitting time of $\mathbb{H}_{-\vec{n}}$.

Proposition 6.5 (Modification of [17, Equation 7])**.** *Fix any $\delta\in(0,1/2)$. The following estimate holds uniformly over all $a_k\in\mathbb{R}$ and $b_k\in(0,k^{1/2-\delta}]$ such that $a_k\vec{n}^\perp+b_k\vec{n}\in\mathcal{L}$:*

$$\mathbb{P}_{a_k\vec{n}^\perp+b_k\vec{n}}(H_{\mathbb{H}_{-\vec{n}}}>k)\sim\kappa V_1(b_k)k^{-1/2}\,. \tag{6.10}$$

Proof. Eq. (6.10), for $a_k:=\mathfrak{a}$ and $b_k:=\mathfrak{b}$ fixed, is proved in [17] as an immediate consequence of the lemmas of [17, Sections 3 and 4]. Of these, only [17, Lemma 21] is insufficient for the uniformity that we require.[15] That is, we must show that for all $\epsilon>0$ sufficiently small,

$$\mathbb{E}_{a_k\vec{n}^\perp+b_k\vec{n}}\Big[u(S(\nu_k));H^S_{\mathbb{H}_{-\vec{n}}}>\nu_k,\nu_k\le k^{1-\epsilon}\Big]=V(a_k\vec{n}^\perp+b_k\vec{n})(1+o(1))\,,$$

where ν_k is defined as the first hitting time of $k^{1/2-\epsilon}\vec{n}+\mathbb{H}_{\vec{n}}$ and $o(1)\to0$ as $k\to\infty$ uniformly over the ranges of interest for a_k and b_k. Uniformity in a_k is trivial. Uniformity of b_k was proven in [31, Section 5.6] (see their equation 60, and recall that $u(S(\nu_k))=S(\nu_k)\cdot\vec{n}$ and $V(a_k\vec{n}^\perp+b_k\vec{n})=V_1(b_k)$). Thus, the proof of [17, Equation 7] given at the end of [17, Section 4] extends to prove our proposition. □

Proposition 6.6 (Modification of [17, Theorem 3])**.** *Uniformly over sequences a_k and b_k satisfying Eq. (6.2), the family of measures*

$$\mathbb{P}_{a_k\vec{n}^\perp+b_k\vec{n}}\Big(\frac{S(\lfloor tk\rfloor)}{\sqrt{k}}\in\cdot\mid H^S_{\mathbb{H}_{-\vec{n}}}>k\Big)$$

converges weakly as $k\to\infty$ to the probability measure on $\mathbb{H}_{\vec{n}}$ with density $H_0(y\cdot\vec{n})e^{-\|a\vec{n}^\perp-y\|^2/2}\mathrm{d}y$, where $H_0>0$ is the normalizing constant.

[15] It may appear that the constant $C(x)$ in [17, Lemma 16] also poses an issue for uniformity, but here $C(x)$ can be taken to be $C(\epsilon)$ for ϵ in that lemma, as a consequence of [17, Lemma 14] in our special case of $u(\mathsf{v}):=\mathsf{v}\cdot\vec{n}$.

Proof. Let B denote a measurable subset of K. Theorem 3 of [17] gives us

$$\lim_{k\to\infty} \mathbb{P}\Big(\frac{S(\lfloor tk\rfloor)}{\sqrt{k}} \in B - a\vec{n}^\perp \mid H^S_{\mathbb{H}_{-\vec{n}}} > k\Big) = H_0 \int_{B-a\vec{n}^\perp} (y\cdot\vec{n}) e^{-|y|^2/2} \mathrm{d}y = H_0 \int_B (y\cdot\vec{n}) e^{-|a\vec{n}^\perp - y|^2/2} \mathrm{d}y\,.$$

A simple continuity argument then yields

$$\lim_{k\to\infty} \mathbb{P}\Big(\frac{S(\lfloor tk\rfloor) + a_k\vec{n}^\perp + b_k\vec{n}}{\sqrt{k}} \in B \mid H^S_{\mathbb{H}_{-\vec{n}}} > k\Big) = H_0 \int_B (y\cdot\vec{n}) e^{-|a\vec{n}^\perp - y|^2/2} \mathrm{d}y$$

as well. This concludes the proof. □

Proposition 6.7 (Modification of [17, Theorem 5])**.** *Recall H_0 from Proposition 6.6. Then uniformly over sequences a_k and b_k satisfying Eq. (6.2),*

$$\limsup_{k\to\infty} \sup_{\mathsf{y}\in\mathbb{H}_{\vec{n}}} \left| \frac{k^{3/2}}{V_1(b_k)} \mathbb{P}_{a_k\vec{n}^\perp + b_k\vec{n}}(S(k) = \mathsf{y}\,,\, H^S_{\mathbb{H}_{-\vec{n}}} > k) - \kappa H_0 \frac{\mathsf{y}\cdot\vec{n}}{\sqrt{k}} e^{-\|a\sqrt{k}\vec{n}^\perp - \mathsf{y}\|^2/2k} \right| = 0\,. \tag{6.11}$$

Proof of Proposition 6.7. Below, we adapt the proof of Theorem 5 given in [17, Section 6.2], beginning as in [17] by splitting $K := \mathbb{H}_{\vec{n}}$ into three parts:

$$\begin{aligned} K^{(1)} &:= \{\mathsf{y} \in \mathbb{H}_{\vec{n}} : \|\mathsf{y}\| > R\sqrt{k}\} \\ K^{(2)} &:= \{\mathsf{y} \in \mathbb{H}_{\vec{n}} : \|\mathsf{y}\| \le R\sqrt{n}\,,\, \mathsf{y}\cdot\vec{n} \le 2\epsilon\sqrt{n}\} \\ K^{(3)} &:= \{\mathsf{y} \in \mathbb{H}_{\vec{n}} : \|\mathsf{y}\| \le R\sqrt{k}\,,\, \mathsf{y}\cdot\vec{n} > 2\epsilon\sqrt{k}\}\,, \end{aligned}$$

for some $R > 0$ and $\epsilon > 0$. Below, we let $C > 0$ denote a constant independent of k, y, R, and ϵ that may change from line to line. Since

$$\lim_{R\to\infty} \lim_{\epsilon\to 0} \sup_{\mathsf{y}\in K^{(1)}\cup K^{(2)}} \frac{\mathsf{y}\cdot\vec{n}}{\sqrt{k}} e^{-\|a\sqrt{k}\vec{n}^\perp - \mathsf{y}\|^2/2k} = 0\,,$$

Proposition 6.7 will be proved if we can show

$$\lim_{R\to\infty} \lim_{\epsilon\to 0} \limsup_{k\to\infty} \frac{k^{3/2}}{V_1(b_k)} \sup_{\mathsf{y}\in K^{(1)}\cup K^{(2)}} \mathbb{P}_{a_k\vec{n}^\perp + b_k\vec{n}}(S(k) = \mathsf{y}\,,\, H^S_{\mathbb{H}_{-\vec{n}}} > k) = 0 \tag{6.12}$$

and

$$\lim_{\epsilon\to 0} \limsup_{k\to\infty} \sup_{\mathsf{y}\in K^{(3)}} \left| \frac{k^{3/2}}{V_1(b_k)} \mathbb{P}_{a_k\vec{n}^\perp + b_k\vec{n}}(S(k) = \mathsf{y}\,,\, H_{\mathbb{H}_{-\vec{n}}} > k) - \kappa H_0 \frac{\mathsf{y}\cdot\vec{n}}{\sqrt{k}} e^{-\frac{\|a\vec{n}^\perp - \mathsf{y}\|^2}{2k}} \right| = 0\,. \tag{6.13}$$

We begin with the more complicated Eq. (6.13).

Set $m := \lfloor \epsilon^3 k\rfloor$. Our starting point is [17, Equation 82], reproduced below:

$$\begin{aligned} &\mathbb{P}_{a_k\vec{n}^\perp + b_k\vec{n}}(S(k) = \mathsf{y}\,,\, H^S_{\mathbb{H}_{-\vec{n}}} > k) \\ &\qquad = \sum_{\mathsf{z}\in\mathbb{H}_{\vec{n}}} \mathbb{P}_{a_k\vec{n}^\perp + b_k\vec{n}}(S_{n-m} = \mathsf{z}\,,\, H^S_{\mathbb{H}_{-\vec{n}}} > k - m) \mathbb{P}_z(S_m = \mathsf{y}\,,\, H^S_{\mathbb{H}_{-\vec{n}}} > m)\,. \end{aligned} \tag{6.14}$$

Let $\mathbb{H}^{(1)}_{\vec{n}}(\mathsf{y}) := \{\mathsf{z} \in \mathbb{H}_{\vec{n}} : \|\mathsf{z} - \mathsf{y}\| < \epsilon\sqrt{k}\}$. Then we may follow the computations in [17, Equations 83, 84] exactly, yielding a constant $a > 0$ such that the following inequalities hold uniformly in a_k, b_k and y such that $\mathsf{y}\cdot\vec{n} > 2\epsilon\sqrt{k}$:

$$\frac{k^{3/2}}{V_1(b_k)} \sum_{\mathsf{z}\in\mathbb{H}_{\vec{n}}\setminus\mathbb{H}^{(1)}_{\vec{n}}(\mathsf{y})} \mathbb{P}_{a_k\vec{n}^\perp + b_k\vec{n}}(S_{k-m} = \mathsf{z}\,,\, H^S_{\mathbb{H}_{-\vec{n}}} > k - m) \mathbb{P}_z(S_m = \mathsf{y}\,,\, H^S_{\mathbb{H}_{-\vec{n}}} > m) \le C\epsilon^{-3} e^{-\frac{a}{\epsilon}} \tag{6.15}$$

and

$$\frac{k^{3/2}}{V_1(b_k)} \sum_{\mathsf{z}\in\mathbb{H}^{(1)}_{\vec n}(\mathsf{y})} \mathbb{P}_{a_k\vec n^\perp+b_k\vec n}(S_{k-m}=\mathsf{z}\,,\,H^S_{\mathbb{H}_{-\vec n}}>k-m)\mathbb{P}_z(S_m=\mathsf{y}\,,\,H^S_{\mathbb{H}_{-\vec n}}<k) \leq C\epsilon^{-3}e^{-\frac{a}{\epsilon}}\,. \quad (6.16)$$

Both right-hand sides go to 0 as $\epsilon\to 0$, and so we turn our attention to the following expression:

$$\Sigma(\mathsf{y}) = \sum_{\mathsf{z}\in\mathbb{H}^{(1)}_{\vec n}(\mathsf{y})} \mathbb{P}_{a_k\vec n^\perp+b_k\vec n}(S_{k-m}=\mathsf{z}\,,\,H^S_{\mathbb{H}_{-\vec n}}>k-m)\mathbb{P}_z(S_m=\mathsf{y})\,.$$

Equation 85 of [17] also holds uniformly over a_k, b_k, and $\mathsf{y}\cdot\vec n > 2\epsilon\sqrt{k}$, except that there should be a $V(\mathsf{x})$ factor in the big-Oh expression of Equation 85, where for us $\mathsf{x} := a_k\vec n^\perp + b_k\vec n$ and so $V(\mathsf{x}) = V_1(b_k)$.[16] Altogether, we find

$$\Sigma(\mathsf{y}) = (2\pi k\epsilon^3)^{-1}\mathbb{P}_{a_k\vec n^\perp+b_k\vec n}(H^S_{\mathbb{H}_{-\vec n}}>k-m)\Sigma_1(\mathsf{y}) + O\Big(\frac{V_1(b_k)}{k^{3/2}}\epsilon^{-3}e^{-a/\epsilon}\Big)\,, \quad (6.17)$$

where

$$\Sigma_1(\mathsf{y}) := \sum_{\mathsf{z}\in\mathbb{H}^{(1)}_{\vec n}(\mathsf{y})} \mathbb{P}_{a_k\vec n^\perp+b_k\vec n}(S_{k-m}=\mathsf{z}\mid H^S_{\mathbb{H}_{-\vec n}}>k-m)e^{-\frac{\|\mathsf{y}-\mathsf{z}\|^2}{2\epsilon^3 k}}\,.$$

From Proposition 6.6 and a compactness argument, we have

$$\limsup_{k\to\infty}\sup_{\mathsf{y}\in K^{(3)}}\left|\Sigma_1(\mathsf{y}) - H_0\int_{\|(1-\epsilon^3)^{1/2}r-\mathsf{y}/\sqrt{k}\|<\epsilon}(\mathsf{r}\cdot\vec n)e^{-\|a\vec n^\perp-\mathsf{r}\|^2/2}e^{-\|(1-\epsilon^3)^{1/2}\mathsf{r}-\mathsf{y}/\sqrt{k}\|/2\epsilon^3}\mathrm{d}\mathsf{r}\right| = 0\,.$$

We can follow the steps up to the display before [17, Equation 86], appealing this time to the uniform continuity of $(\mathsf{r}\cdot\vec n^\perp)e^{-\|a\vec n^\perp-\mathsf{r}\|^2/2}$ (instead of the function $u(\mathsf{r})e^{-\|r\|^2/2}$ as in [17]), to obtain

$$\limsup_{k\to\infty}\sup_{\mathsf{y}\in K^{(3)}}\left|\Sigma_1(\mathsf{y}) - H_0\frac{\mathsf{y}\cdot\vec n}{\sqrt{k}}e^{-\frac{\|a\vec n^\perp-\mathsf{y}\|^2}{2k}}\right| = o(\epsilon^3)\,.$$

Combining the above with Eq. (6.17) and applying Proposition 6.5, we find

$$\lim_{\epsilon\to 0}\limsup_{k\to\infty}\sup_{\mathsf{y}\in K^{(3)}}\left|\frac{k^{3/2}}{V_1(b_k)}\Sigma(\mathsf{y}) - \kappa H_0\frac{\mathsf{y}\cdot\vec n}{\sqrt{k}}e^{-\frac{\|a\vec n^\perp-\mathsf{y}\|^2}{2k}}\right| = 0\,.$$

Combining this with Eqs. (6.15) and (6.16) yields Eq. (6.13).

The modifications of [17, Section 6.2] required to obtain Eq. (6.12) are much simpler than those needed to obtain Eq. (6.13). Equation (6.12) follows very similarly to the proofs of Equations 77 and 81 in [17], and so we just highlight the main differences. The main technical modification comes from the following, which adapts their Lemmas 27 and 28:

$$\mathbb{P}_{u\vec n}(S(k)=a\vec n^\perp+b\vec n\,,\,H^S_{\mathbb{H}_{-\vec n}}>k) \leq C(1+V_1(u))n^{-3/2}\wedge C(1+V_1(u))(1+V_1(b))n^{-2}\,, \quad (6.18)$$

for all $u,a,b\geq 0$. Indeed, note that Proposition 6.5 gives the bound

$$\mathbb{P}_{a_k\vec n^\perp+b_k\vec n}(H^S_{\mathbb{H}_{-\vec n}}>k) \leq C\big(1+V_1(b_k)\big)k^{-1/2}\,.$$

Then the proofs of Lemmas 27 and 28 of [17] can be repeated to yield Eq. (6.18) (in particular, the $C(\mathsf{x})$ in Lemma 27 can be expressed as $CV_1(\mathsf{x}\cdot\vec n)$, and the $C(\mathsf{x},\mathsf{y})$ in Lemma 28 can be expressed as $CV_1(\mathsf{x}\cdot\vec n)V_1(\mathsf{y}\cdot\vec n)$). So all $C(\mathsf{x})$ terms from the proofs of [17, Equations 77, 81] should be replaced by $V_1(b_k)$, after which one finds that their work up to [17, Equation 81] yields Eq. (6.12). □

[16] The $V_1(b_k)$ factor comes from an application of Proposition 6.5 in the second line of [17, Equation 85]. This term was dropped in [17] because they consider x fixed, and so $V_1(b_k)$ is order 1.

6.5. **Proofs of Propositions 6.1 to 6.3.**

Proof of Proposition 6.1. In what follows, all estimates will be uniform over a_k, b_k, and u_k satisfying Eq. (6.2).

We begin by following the proof given in [17, Section 6.3]. In particular, we also set $m = \lfloor(1-t)k\rfloor$ for some $t \in (0,1)$ and write the decomposition

$$\begin{aligned}&\mathbb{P}_{u_k\vec{n}}\Big(S(k) = a_k\vec{n}^\perp + b_k\vec{n}\,,\ H^S_{\mathbb{H}_{-\vec{n}}} > k\Big)\\&= \sum_{\mathsf{z}\in\mathbb{H}_{\vec{n}}} \mathbb{P}_{u_k\vec{n}}\Big(S(k-m) = \mathsf{z}\,,\ H^S_{\mathbb{H}_{-\vec{n}}} > k-m\Big)\mathbb{P}_{a_k\vec{n}^\perp+b_k\vec{n}}\Big(S'(m) = \mathsf{z}\,,\ H^{S'}_{\mathbb{H}_{-\vec{n}}} > m\Big)\,, \qquad (6.19)\end{aligned}$$

where S' is distributed as $-S$. Letting $\mathsf{x} := u_k\vec{n}$ and $\mathsf{y} := a_k\vec{n}^\perp + b_k\vec{n}$, and recalling Eq. (6.18) as the needed modification of Lemmas 27 and 28 of [17], we can follow the proof given in [17, Section 6.3] up to their equation (89) to find

$$\lim_{R\to\infty}\lim_{k\to\infty}\Big(\frac{V_1(u_k)V'(b_k)}{k^2}\Big)^{-1}\Sigma_1(R,k) = 0\,,$$

where

$$\Sigma_1(R,k) := \sum_{\mathsf{z}\in\mathbb{H}_{\vec{n}}:|\mathsf{z}|>R\sqrt{k}} \mathbb{P}_{u_k\vec{n}}\Big(S(k-m) = \mathsf{z}\,,\ H^S_{\mathbb{H}_{-\vec{n}}} > k-m\Big)\mathbb{P}_{a_k\vec{n}^\perp+b_k\vec{n}}\Big(S'(m) = \mathsf{z}\,,\ H^{S'}_{\mathbb{H}_{-\vec{n}}} > m\Big)\,.$$

We are now in a position to apply Proposition 6.7 to the remainder term

$$\begin{aligned}\Sigma_2(R,k) &:= \sum_{\mathsf{z}\in\mathbb{H}_{\vec{n}}:\|\mathsf{z}\|\le R\sqrt{k}} \mathbb{P}_{u_k\vec{n}}\Big(S(k-m) = \mathsf{z}\,,\ H^S_{\mathbb{H}_{-\vec{n}}} > k-m\Big)\mathbb{P}_{a_k\vec{n}^\perp+b_k\vec{n}}\Big(S'(m) = \mathsf{z}\,,\ H^{S'}_{\mathbb{H}_{-\vec{n}}} > m\Big)\\
&= \frac{H_0^2\kappa^2 V_1(u_k)V_1(b_k)}{\big(t(1-t)\big)^{3/2}k^3}\sum_{\mathsf{z}\in\mathbb{H}_{\vec{n}}:\|\mathsf{z}\|\le R\sqrt{k}}\Big(\frac{\mathsf{z}\cdot\vec{n}}{\sqrt{tk}}\Big)\Big(\frac{\mathsf{z}\cdot\vec{n}}{\sqrt{(1-t)k}}\Big)\exp\Big(-\frac{|\mathsf{z}|^2}{2tk} - \frac{|a\sqrt{k}\vec{n}^\perp - \mathsf{z}|^2}{2(1-t)k}\Big)\\
&\qquad + o(V_1(u_k)V'(b_k)k^{-2})\\
&= \frac{H_0^2\kappa^2 V_1(u_k)V_1(b_k)}{\big(t(1-t)\big)^{3/2}k^3}e^{-a^2/2}\sum_{\mathsf{z}\in\mathbb{H}_{\vec{n}}:\|\mathsf{z}\|\le R\sqrt{k}}\Big(\frac{\mathsf{z}\cdot\vec{n}}{\sqrt{tk}}\Big)\Big(\frac{\mathsf{z}\cdot\vec{n}}{\sqrt{(1-t)k}}\Big)\exp\Big(-\frac{(\mathsf{z}\cdot\vec{n})^2}{2t(1-t)k} - \frac{(\mathsf{z}\cdot\vec{n}^\perp - ta\sqrt{k})^2}{2t(1-t)k}\Big)\\
&\qquad + o(V_1(u_k)V'(b_k)k^{-2})\,.\end{aligned}$$

Note that, compared to the first display after equation (89) of [17], the only different terms are those involving a and the little-oh terms: this is a consequence of the modifications in our Proposition 6.7 compared to their Theorem 5. From here, we can follow their proof step-by-step until the end, yielding Eq. (6.3). ☐

Proof of Proposition 6.2. We begin by showing convergence of the finite-dimensional distributions. For this, it is enough to consider sequences $a_k \in [-A\sqrt{k}, A\sqrt{k}]$ such that $a_k/\sqrt{k} \to a$, and show that for any $a \in [-A, A]$, the finite-dimensional distributions converge to the same limit (as then every subsequence has a further subsequence converging to the same distribution).

We proceed by following the arguments as in [23, Section 4], making adaptions as necessary. We begin with [23, Equation 40], which states that for any $t \in [0,1)$ and $B \in \sigma(\{S_i^{(2)}, i \le kt\})$, we have

$$\mathbb{P}_{u_k\vec{n}}(B \mid S(k) = a_k\vec{n}^\perp + b_k\vec{n}\,,\ H^S_{\mathbb{H}_{-\vec{n}}} > k) = \mathbb{E}\Big[h^{(k)}_{u_k,a_k,b_k}(t, X_{k,t})\mathbb{1}_B \mid H^S_{\mathbb{H}_{-\vec{n}}} > kt\Big]\,, \qquad (6.20)$$

where $X_{k,t} := S(\lfloor tk \rfloor)/\sqrt{k}$ and

$$h^{(k)}_{u_k,a_k,b_k}(t,\mathsf{w}) = \frac{\mathbb{P}_{u_k\vec{n}}(H_{\mathbb{H}_{-\vec{n}}} > kt)\mathbb{P}_{\mathsf{w}\sqrt{n}}\Big(S_{(1-t)k} = a_k\vec{n}^{\perp} + b_k\vec{n}\,,\ H^S_{\mathbb{H}_{-\vec{n}}} > (1-t)k\Big)}{\mathbb{P}_{u_k\vec{n}}\Big(S(k) = a_k\vec{n}^{\perp} + b_k\vec{n}\,,\ H^S_{\mathbb{H}_{-\vec{n}}} > k\Big)}\,.$$

From Propositions 6.1 and 6.5, we have

$$\frac{\mathbb{P}_{u_k\vec{n}}(H_{\mathbb{H}_{-\vec{n}}} > kt)}{\mathbb{P}_{u_k\vec{n}}\Big(S(k) = a_k\vec{n}^{\perp} + b_k\vec{n}\,,\ H^S_{\mathbb{H}_{-\vec{n}}} > k\Big)} \sim \frac{t^{-1/2}e^{a^2/2}}{C_1\kappa V_1'(b_k)}k^{3/2}\,.$$

Now, let $S'(\cdot)$ denote the random walk whose increments are independent copies of $-X$. Considering the walk $S(\cdot)$ in reversed time, we have

$$\begin{aligned}\mathbb{P}_{\mathsf{w}\sqrt{n}}&\Big(S((1-t)k) = a_k\vec{n}^{\perp} + b_k\vec{n}\,,\ H^S_{\mathbb{H}_{-\vec{n}}} > (1-t)k\Big)\\ &= \mathbb{P}_{a_k\vec{n}^{\perp}+b_k\vec{n}}\Big(S'((1-t)k) = \mathsf{w}\sqrt{k}\,,\ H^{S'}_{\mathbb{H}_{-\vec{n}}} > (1-t)k\Big)\,.\end{aligned}$$

Applying Proposition 6.7 yields

$$\begin{aligned}\limsup_{k\to\infty}\sup_{\mathsf{w}\in\mathbb{H}_{\vec{n}}}\Bigg|&\frac{(1-t)^{3/2}k^{3/2}}{V_1'(b_k)}\mathbb{P}_{\mathsf{w}\sqrt{k}}\Big(S_{(1-t)k} = a_k\vec{n}^{\perp} + b_k\vec{n}\,,\ H^S_{\mathbb{H}_{-\vec{n}}} > (1-t)k\Big)\\ &- \kappa H_0\frac{\mathsf{w}\cdot\vec{n}}{(1-t)^{1/2}}e^{-\frac{|a\vec{n}^{\perp}-\mathsf{w}|^2}{2(1-t)}}\Bigg| = 0\end{aligned}$$

uniformly over a_k and b_k. Altogether, we find

$$h^{(k)}_{u_k,a_k,b_k}(t,\mathsf{w}) = (1+o(1))h(a,t,\mathsf{w})\,,$$

uniformly over $\mathsf{w} \in \mathbb{H}_{\vec{n}}$,

$$h(a,t,\mathsf{w}) := \frac{H_0}{C_1}t^{-\frac{1}{2}}(1-t)^2(\mathsf{w}\cdot\vec{n})e^{-\frac{|\mathsf{w}\cdot\vec{n}|^2}{2(1-t)}}\exp\Big(-\frac{|\mathsf{w}\cdot\vec{n}^{\perp}|^2}{2(1-t)} + \frac{a\mathsf{w}\cdot\vec{n}^{\perp}}{1-t} - \frac{ta^2}{2(1-t)}\Big)\,.$$

Recall $D[0,t]$ the space of cadlag functions from $[0,t]$ to $\mathbb{R}$. For any bounded and continuous functional $g_t : D[0,t] \to \mathbb{R}$, Eq. (6.20) gives us

$$\begin{aligned}\mathbb{E}_{u_k\vec{n}}&[g_t(X_{k,\cdot}\cdot\vec{n}) \mid S(k) = a_k\vec{n}^{\perp} + b_k\vec{n}\,,\ H^S_{\mathbb{H}_{-\vec{n}}} > k]\\ &= (1+o(1))\mathbb{E}_{u_k\vec{n}}[g_t(X_{k,\cdot}\cdot\vec{n})h(a,t,X_{k,t}) \mid H^S_{\mathbb{H}_{-\vec{n}}} > kt]\,,\end{aligned} \tag{6.21}$$

where we have used that for fixed a and t, $h(a,t,\mathsf{w})$ is uniformly bounded in w. Applying the convergence result [23, Theorem 2] (see also Remark 1 in [23]), we find

$$\begin{aligned}\lim_{k\to\infty}\mathbb{E}_{u_k\vec{n}}&[g_t(X_{k,\cdot}\cdot\vec{n}) \mid S(k) = a_k\vec{n}^{\perp} + b_k\vec{n}\,,\ H^S_{\mathbb{H}_{-\vec{n}}} > k]\\ &= \mathbb{E}\Big[g_t(t^{\frac{1}{2}}\mathfrak{M}_{\mathbb{H}_{\vec{n}}}\cdot\vec{n})h(a,t,t^{1/2}\mathfrak{M}_{\mathbb{H}_{\vec{n}}}(1))\Big]\,,\end{aligned} \tag{6.22}$$

where $(\mathfrak{M}_{\mathbb{H}_{\vec{n}}}(s))_{s\in[0,1]}$ denotes the Brownian meander in $\mathbb{H}_{\vec{n}}$ started from the origin. Recall that $\mathfrak{M}_{\mathbb{H}_{\vec{n}}}(s) = W_s\vec{n}^{\perp} + M_s\vec{n}$, where $(W_s)_{s\in[0,1]}$ is a standard one-dimensional Brownian motion, $(M_s)_{s\in[0,1]}$ is a standard one-dimensional Brownian meander, and $W_\cdot$ and $M_\cdot$ are independent processes. Then the expectation over $W_\cdot$ factors out of the right-hand side of Eq. (6.22) as follows:

$$\mathbb{E}\Big[\frac{H_0}{C_1}(1-t)^{-\frac{3}{2}}M_1e^{-\frac{t|M_1|^2}{2(1-t)}}g_t(t^{1/2}M_\cdot)\Big]\mathbb{E}\Big[\exp\Big(-\frac{t\mathcal{N}^2}{2(1-t)} + \frac{a\sqrt{t}}{1-t}\mathcal{N} - \frac{ta^2}{2(1-t)}\Big)\Big]\,,$$

where $\mathcal{N} \sim N(0,1)$. That second expectation evaluates to $\sqrt{1-t}$ (in particular, there is no dependence on a in the above expression). So, letting $C_2 := H_0/C_1$, we've found

$$\lim_{k\to\infty} \mathbb{E}_{u_k\vec{n}}\big[g_t(X_{k,\cdot}\cdot\vec{n}) \mid S(k) = a_k\vec{n}^\perp + b_k\vec{n}\,,\, H^S_{\mathbb{H}_{-\vec{n}}} > k\big] = \mathbb{E}\Big[C_2(1-t)^{-\frac{3}{2}} M_1 e^{-\frac{t|M_1|^2}{2(1-t)}} g_t(t^{1/2}M_\cdot)\Big]\,.$$

Thus, for every fixed $t<1$, we have shown convergence in distribution on $D[0,t]$ for every fixed $t<1$ to the same limit for every a. This in particular implies convergence of all finite dimensional distributions to the same limit for every a, as well as tightness on $[0,1-\delta]$, for any $\delta\in(0,1)$. Tightness on $[1-\delta,1]$ follows by applying the exact same arguments for the random walk reversed in time, just as in the end of [23, Section 4].

Thus, we have the convergence of the family of laws $\mathbf{Q}^k_{u_k,a_k,b_k}$. The limit may be identified as that of the standard Brownian excursion by taking $S(i) := \hat{S}_1(i)\vec{n}^\perp + \hat{S}_2(i)\vec{n}$, where $\hat{S}_1(\cdot)$ and $\hat{S}_2(\cdot)$ are independent simple symmetric random walks on $\mathbb{Z}$. □

Proof of Proposition 6.3. Following the proof of Theorem 5.1 in [31] until their Equation 59 yields equation Eq. (6.6). Equation (6.5) is then an immediate consequence of Proposition 6.1. □

7. Proof of Theorem 1.3

This section is dedicated to the proof of Theorem 1.3. We prove a slightly more detailed result below. Recall the generator L of the relevant Ferrari–Spohn diffusion defined in Eq. (1.5).

Theorem 7.1. *Fix an integer $n\geq 0$. Suppose that a_L, the fractional part of $\frac{1}{4\beta}\log L$, converges to some limit a, and let $\lambda>0$ denote the $L\to\infty$ limit of $\lambda^{(n)}(L) := c_\infty e^{4\beta a_L}(1-e^{-4\beta})e^{4\beta n}$. Define $\mathsf{y} := (\lfloor KL^{2/3}\rfloor, 0)$, for some $K>0$, and the box $Q := [\![-KL^{2/3}, KL^{2/3}]\!]^2$. Consider the modified Ising polymer γ in $D=Q$ or $\mathbb{H}$ with start point $-\mathsf{y}$ and end point y with area tilt $\exp(-\frac{\lambda^{(n)}(L)A(\gamma)}{L})$. Let $\overline{\gamma}(x)$ denote the maximum vertical distance of γ at $x\in\mathbb{R}$. Let $\sigma>0$ be the constant from Theorem 1.2, Part (b). For any fixed $T>0$, the diffusively-rescaled interface $\sigma^{-\frac{1}{2}}L^{-\frac{1}{3}}\overline{\gamma}(L^{\frac{2}{3}}x)$ converges weakly in $(D[-T,T],\|\cdot\|_\infty)$ as first $L\to\infty$ then $K\to\infty$ to the stationary Ferrari–Spohn diffusion on $(0,\infty)$ with generator L and Dirichlet boundary condition at 0. The same holds for $\underline{\gamma}(x)$, the minimum vertical distance at x.*

Remark 7.2. Fix β sufficiently large, and consider the SOS model with a floor on the box Q with boundary conditions $H-n$ everywhere, except on the bottom where they are $H-n-1$. Eq. (A.8) and Observation 2.7 imply that the law of the $H-n$-level line (connecting the bottom corners of Q) is given by a modified Ising polymer with area tilt as in Theorem 7.1. Thus, Theorem 7.1 implies Theorem 1.3.

Remark 7.3. The argument used to prove the above theorem holds mutatis mutandis for $K\in(0,L^\epsilon)$ for, say, $0<\epsilon<\frac{1}{20}$, where the restriction on ϵ is due to the fact that we are able to control the effect of the area tilt term $\exp[-\frac{\lambda}{L}A(\gamma)]$ on boxes of side-length $L^{2/3+\epsilon}$ (via [14, Prop. A.1]).

7.1. **Proof of Theorem 7.1.** Let $\mathbf{P}^{\mathsf{u},\mathsf{v}}_D$ be the modified Ising polymer in $D=Q$ or $\mathbb{H}$ with start-point $\mathsf{u}\in\mathbb{Z}^2$ and end-point $\mathsf{v}\in\mathbb{Z}^2$, i.e.,

$$\mathbf{P}^{\mathsf{u},\mathsf{v}}_D(\cdot) := \frac{\mathcal{G}_D(\mathsf{u}\to\mathsf{v}\mid \gamma\in\cdot)}{\mathcal{G}_D(\mathsf{u}\to\mathsf{v})}\,,$$

where we recall the partition functions $\mathcal{G}_D(\mathsf{u} \to \mathsf{v})$ from Eq. (5.17).[17] Let $\mathbf{E}_D^{\mathsf{u},\mathsf{v}}$ denote expectation with respect to $\mathbf{P}_D^{\mathsf{u},\mathsf{v}}$. Next, define the modified Ising polymer with area-tilt

$$\tilde{\mathbf{P}}_{D,\lambda}^{-\mathsf{y},\mathsf{y}}(\cdot) := \frac{\mathbf{E}_D^{-\mathsf{y},\mathsf{y}}\big[\mathbb{1}_{\{\gamma\in\cdot\}}e^{-\frac{\lambda}{L}A(\gamma)}\big]}{\mathbf{E}_D^{-\mathsf{y},\mathsf{y}}[e^{-\frac{\lambda}{L}A(\gamma)}]} \tag{7.1}$$

(we have replaced $\lambda^{(n)}(L)$ with λ, as the difference is a $o(1)$ term that plays no role in what follows). As explained above Eq. (2.15), we will also view $\mathbf{P}_{D,\lambda}^{-\mathsf{y},\mathsf{y}}$ as a measure on animals Γ (with $A(\Gamma) := A(\gamma)$).

We prove Theorem 7.1 by first coupling the cone-points of $\Gamma \sim \mathbf{P}_{D,\lambda}^{-\mathsf{y},\mathsf{y}}$ that lie inside some strip (those cone-points for which we will have entropic repulsion) with the trajectory of a random walk with area-tilt. We then fit this random walk into the framework of [30, Section 6], where a Ferrari–Spohn limit was proved for a broad class of directed, 2D random walks with area tilt.

7.1.1. *Coupling with an area-tilted random walk.* The following claim controls the influence of the area tilt on our estimates.

Claim 7.4. *For each fixed $K > 0$, there exists $a_K > 0$ such that $\mathbf{E}_D^{\mathsf{x}_L,\mathsf{x}_R}\big[e^{-\frac{\lambda}{L}A(\gamma)}\big] \to a_K$ as $L \to \infty$.*

Proof. From Theorem 1.2, we know that both $L^{-1/3}\underline{\gamma}(\lfloor L^{-2/3}x\rfloor)$ and $L^{-1/3}\overline{\gamma}(\lfloor L^{-2/3}x\rfloor)$ converge weakly in $D([-K,K],\|\cdot\|_\infty)$ to the Brownian excursion from 0 to 0. Since the function

$$\begin{aligned} f : (D[-K,K],\|\cdot\|_\infty) &\to \mathbb{R} \\ g &\mapsto \exp\Big(-\int_{-K}^{K} g(x)\mathrm{d}x\Big) \end{aligned}$$

is a bounded, continuous function, it follows that

$$\mathbf{E}_D^{-\mathsf{y},\mathsf{y}}[e^{-\frac{\lambda}{L}A(\gamma)}] \to \mathbf{E}[e^{-\lambda\int_{-K}^{K}\xi(x)\mathrm{d}x}]\,,$$

where ξ denotes a Brownian excursion from 0 to 0 on $[-K,K]$. This concludes the proof. ☐

The rest of this subsection will closely follow the notation and work in Section 5.6, where the existence of a coupling between the cone-points of a modified Ising polymer (*without* area tilt) and the corresponding random walk was proved. The inputs to construct such a coupling were:

(a) existence of many cone-points and boundedness of the polymer length (Lemma 5.7);
(b) boundedness of the irreducible pieces (Proposition 5.8); and
(c) entropic repulsion (Proposition 5.11).

These results all held with probability tending to 1 as N (the side length of the box) tends to infinity. In the current situation, we take $N = 2\|\mathsf{y}\|_1 = 2\lfloor KL^{2/3}\rfloor$ and we shift x (as in the statement of the above results) to y so that these results hold under $\mathbf{P}_D^{-\mathsf{y},\mathsf{y}}$ with probability tending to 1 as L tends to ∞. Claim 7.4 states that $\mathbf{P}_D^{-\mathsf{y},\mathsf{y}}[e^{-\frac{\lambda}{L}A(\gamma)}]$ is bounded away from 0 uniformly in L, so that Items (a) and (c) above hold with probability tending to 1 as L tends to ∞ under $\mathbf{P}_{D,\lambda}^{-\mathsf{y},\mathsf{y}}$ as well. Thus, we are in good position to establish a coupling between the area-tilted Ising polymer and an area-tilted random walk that we define below.

Fix $\delta \in (0, 1/8)$. Analogous to Eq. (5.35), define the measure

$$\mathbf{P}_{D,\lambda,*}^{\overline{\zeta}^{(\mathsf{L}*)},\overline{\zeta}^{(\mathsf{R}*)}}(\,\cdot\,) := \mathbf{P}_{D,\lambda}^{-\mathsf{y},\mathsf{y}}(\,\cdot\,\mid \mathcal{P}_{D,\delta}^*(-\mathsf{y},\mathsf{y}), \Gamma^{(\mathsf{L}*)} = \overline{\Gamma}^{(\mathsf{L}*)}, \Gamma^{(\mathsf{R}*)} = \overline{\Gamma}^{(\mathsf{R}*)})\,,$$

[17]In Definition 2.6, we defined modified Ising polymers with start-point zero for ease of notation, and because until now, essentially all of our Ising polymers started from $0 \in \mathbb{Z}^2$.

where $\overline{\zeta}^{(\mathsf{L}*)}$ and $\overline{\zeta}^{(\mathsf{R}*)}$ are points in $\mathbb{Z}^2$ satisfying

$$\overline{\zeta}^{(\mathsf{L}*)} \in [-\tfrac{N}{2} + N^{4\delta}, -\tfrac{N}{2} + N^{4\delta} + (\log N)^2] \times (N^{\delta}, N^{4\delta}(\log N)^2] \tag{7.2}$$

and

$$\overline{\zeta}^{(\mathsf{R}*)} \in [\tfrac{N}{2} - N^{4\delta} - (\log N)^2, -\tfrac{N}{2} - N^{4\delta}] \times (N^{\delta}, N^{4\delta}(\log N)^2] \tag{7.3}$$

(these are the analogues of Eqs. (5.33) and (5.34), respectively). Recall that, thanks to Item (b) and Item (c), the first and last cone-points of Γ in the strip $\mathcal{S}_{-\frac{N}{2}+N^{4\delta},\frac{N}{2}-N^{4\delta}}$ satisfy Eqs. (7.2) and (7.3) with $\mathbf{P}^{-\mathsf{y},\mathsf{y}}_{D,\lambda}$-probability tending to 1 as L tends to ∞. It is between these cone-points that we couple with an area-tilted random walk.

Recall the random walk S and its law $\mathbb{P}$ defined in Section 4.1. Write $\mathbb{E}$ and $\mathbb{E}_{\mathsf{u}}$ for expectation under $\mathbb{P}$ and $\mathbb{P}_{\mathsf{u}}$, respectively. Define the area under S as follows: for an l-steps walk $\mathsf{S} = \{(\mathsf{S}_1(0), \mathsf{S}_2(0)), \dots, (\mathsf{S}_1(l), \mathsf{S}_2(l))\}$, we write

$$A(\mathsf{S}) := \sum_{i=1}^{l} (\mathsf{S}_1(i) - \mathsf{S}_1(i-1))\mathsf{S}_2(i)\,. \tag{7.4}$$

To indicate the law of the random walk S with area tilt, started from $\mathsf{S}(0) = \mathsf{u}$, we write

$$\mathbb{P}^{\mathsf{u}}_{\lambda}(\cdot) := \frac{\mathbb{E}_{\mathsf{u}}\big[\mathbb{1}_{\{\mathsf{S}\in\cdot\}} e^{-\frac{\lambda}{L}A(\mathsf{S})}\big]}{\mathbb{E}_{\mathsf{u}}\big[e^{-\frac{\lambda}{L}A(\mathsf{S})}\big]}\,. \tag{7.5}$$

Proposition 7.5. *There exists $\nu > 0$ such that for all $K > 0$ and L large enough with respect to K, for all $\beta > 0$ sufficiently large, and for all $\overline{\zeta}^{(\mathsf{L}*)}$ and $\overline{\zeta}^{(\mathsf{R}*)}$ satisfying Eqs. (5.33) and (5.34) respectively, if we let $T = H_{\overline{\zeta}^{(\mathsf{R}*)}}$ and view $\mathsf{Cpts}(\Gamma^*)$ as an ordered tuple (see Eq. (2.20)), then*

$$\Big\|\mathbf{P}^{\overline{\zeta}^{(\mathsf{L}*)},\overline{\zeta}^{(\mathsf{R}*)}}_{D,\lambda,*}\Big(\mathsf{Cpts}(\Gamma^*) \in \cdot\Big) - \mathbb{P}^{\overline{\zeta}^{(\mathsf{L}*)}}_{\lambda}\Big((\mathsf{S}(i))_{i=0}^{T} \in \cdot \mid T < H_{\mathbb{H}_{N^\delta}}\Big)\Big\|_{\mathrm{TV}} \le Ce^{-\nu\beta(\log N)^2}$$

for some constant $C := C(\beta) > 0$.

Proof. This follows from the coupling of the untilted measures given by Proposition 5.13. One just needs to check that the difference in the definitions of area for Γ^* and S results in a negligible difference in the tilts. This is easy to see: by Item (b), the distance between two consecutive cone-points is at most $(\log L)^2$, and hence one can enclose the diamond between them in a square of area $(\log L)^4$. Since the total area of such squares is an upper bound on $|A(\Gamma^*) - A(S)|$, we find

$$\left|\frac{A(\Gamma^*)}{L} - \frac{A(\mathsf{S})}{L}\right| \le 2K\frac{(\log L)^4}{L^{1/3}} = o(1)\,.$$

This shows that the tilts are equivalent up to a $o(1)$ factor. □

7.1.2. *Convergence to Ferrari–Spohn.* We have reduced to a two-dimensional random walk bridge with area-tilt conditioned to stay in $\mathbb{H}_{N^\delta}$, with start-point $\bar{\mathsf{u}} := \overline{\zeta}^{(\mathsf{L}*)}$ and end-point $\bar{\mathsf{v}} := \overline{\zeta}^{(\mathsf{R}*)}$ satisfying Eqs. (7.2) and (7.3), respectively. In [30, Sections 6.6 and 6.7], the Ferrari–Spohn diffusion limit is proved for a wide-class of such random walks, with the stronger condition ([30, Equation (6.10)]) on the start-point u and end-point v of the random walk:

$$\mathsf{u} \in [-\bar{K}L^{2/3} - L^{1/3+\epsilon}, \bar{K}L^{2/3} + L^{1/3+\epsilon}] \times [cL^{1/3}, CL^{1/3}] \tag{7.6}$$

$$\mathsf{v} \in [\bar{K}L^{2/3} - L^{1/3+\epsilon}, \bar{K}L^{2/3} + L^{1/3+\epsilon}] \times [cL^{1/3}, CL^{1/3}]\,, \tag{7.7}$$

where $\epsilon > 0$ is any small constant, $C > c > 0$ are fixed constants, and $\bar{K} \le K$ is a parameter tending to ∞ after L (these conditions are stronger than Eqs. (7.2) and (7.3) in the y-coordinates only). Lemma 7.6 shows that our random walk indeed passes through points satisfying (7.6) and (7.7) with high probability, thereby putting us in the same setting as [30, Section 6.6]. We prove it using the Brownian excursion limit Theorem 4.3.

For brevity, we will write the law of the area-tilted random walk bridge as

$$\mathbb{P}^{\bar{\mathsf{u}},\bar{\mathsf{v}}}_{\lambda,+} := \mathbb{P}^{\bar{\mathsf{u}}}_{\lambda}(\cdot \mid H_{\bar{\mathsf{v}}} < H_{\mathbb{H}_{N^\delta}})\,.$$

Similarly, we write the law of the un-tilted random walk bridge via

$$\mathbb{P}^{\bar{\mathsf{u}},\bar{\mathsf{v}}}_{+} := \mathbb{P}_{\bar{\mathsf{u}}}(\cdot \mid H_{\bar{\mathsf{v}}} < H_{\mathbb{H}_{N^\delta}})\,.$$

We will denote expectation under these measures by replacing $\tilde{\mathbb{P}}$ with $\tilde{\mathbb{E}}$ and $\mathbb{P}$ with $\mathbb{E}$.

Lemma 7.6. *Let $E_{c,C}$ denote the event that the random walk S passes through u and v satisfying Eqs. (7.6) and (7.7), for some $C > c > 0$. Then for some constant $K_0 > 0$, the following limit holds uniformly in $\bar{\mathsf{u}}$ satisfying Eq. (7.2) and $\bar{\mathsf{v}}$ satisfying Eq. (7.3):*

$$\lim_{c\to 0}\inf_{K\geq K_0}\liminf_{L\to\infty}\mathbb{P}^{\bar{\mathsf{u}},\bar{\mathsf{v}}}_{\lambda,+}(E_{c,C}) = 1\,.$$

Proof. We will show that S passes through a point u satisfying Eq. (7.2) with high probability; the same argument will show the same is true for v. Further, one only needs to check that $\mathsf{u}_2 \geq cL^{1/3}$ (the upper bound by $CL^{1/3}$ is not needed), since $\mathsf{S}_2(0) = \bar{\mathsf{u}}_2 < cL^{1/3}$, and so the first time S_2 rises above $cL^{1/3}$, it will also be below $CL^{1/3}$ for L large enough (recall from Eq. (5.38) that under the non-area-tilted measure, S has increments bounded by $(\log L)^2$ with probability tending to 1 as $L \to \infty$, and so by Claim 7.4, the same is true under the area-tilted measure).

Now, using the localization of S_1 (Claim 6.4) and the triviality-in-L of the area tilt (Claim 7.4), we have that for all $\epsilon > 0$ small, the following hold with $\mathbb{P}^{\bar{\mathsf{u}},\bar{\mathsf{v}}}_{\lambda,+}$-probability tending to 1 as $L \to \infty$:

$$\begin{aligned}\mathcal{E}_1(L^{2/3}) &:= \{\mathsf{S}_1(L^{2/3}) \in [-(K-\mu)L^{2/3} - L^{1/3+\epsilon}, -(K-\mu)L^{2/3} + L^{1/3+\epsilon}]\}\,, \text{ and}\\ \mathcal{E}_1(2L^{2/3}) &:= \{\mathsf{S}_1(2L^{2/3}) \in [-(K-2\mu)L^{2/3} - L^{1/3+\epsilon}, -(K-2\mu)L^{2/3} + L^{1/3+\epsilon}]\}\,. \end{aligned} \tag{7.8}$$

Thus, the lemma will be proved upon showing

$$\lim_{c\to 0}\sup_{K\geq K_0}\limsup_{L\to\infty}\mathbb{P}^{\bar{\mathsf{u}},\bar{\mathsf{v}}}_{\lambda,+}(\mathsf{S}_2(L^{2/3}) < cL^{1/3} \mid \mathsf{S}_2(2L^{2/3}) < cL^{1/3}\ ,\ \mathcal{E}_1(2L^{2/3})) = 0\,. \tag{7.9}$$

Write the above probability as

$$\frac{\mathbb{E}^{\bar{\mathsf{u}},\bar{\mathsf{v}}}_{+}\Big[\mathbb{E}^{\bar{\mathsf{u}},\mathsf{S}(2L^{2/3})}_{+}\Big[\mathbb{1}_{\{\mathsf{S}_2(L^{2/3})<cL^{1/3}\}}e^{-\frac{\lambda}{L}A(\mathsf{S})}\Big]F(\mathsf{S}(i), i \geq 2L^{2/3})\Big]}{\mathbb{E}^{\bar{\mathsf{u}},\bar{\mathsf{v}}}_{+}\Big[\mathbb{E}^{\bar{\mathsf{u}},\mathsf{S}(2L^{2/3})}_{+}\Big[e^{-\frac{\lambda}{L}A(\mathsf{S})}\Big]F(\mathsf{S}(i), i \geq 2L^{2/3})\Big]}\,, \tag{7.10}$$

where

$$F(\mathsf{S}(i), i \geq 2L^{2/3}) := e^{-\frac{\lambda}{L}A(\mathsf{S}(i), i\geq 2L^{2/3})}\mathbb{1}_{\{\mathsf{S}_2(2L^{2/3})<cL^{1/3},\ \mathcal{E}_1(2L^{2/3})\}}\,.$$

Due to the restriction on $\mathsf{S}(2L^{2/3})$ imposed by the indicator defining F, Eq. (7.9) will be shown if we can prove

$$\lim_{c\to 0}\sup_{K\geq K_0}\limsup_{L\to\infty}\frac{\mathbb{E}^{\bar{\mathsf{u}},\bar{\mathsf{w}}}_{+}\Big[\mathbb{1}_{\{\mathsf{S}_2(L^{2/3})<cL^{1/3}\}}e^{-\frac{\lambda}{L}A(\mathsf{S})}\Big]}{\mathbb{E}^{\bar{\mathsf{u}},\bar{\mathsf{w}}}_{+}\Big[e^{-\frac{\lambda}{L}A(\mathsf{S})}\Big]} = 0\,, \tag{7.11}$$

uniformly over

$$\bar{\mathsf{w}} \in [-(K-2\mu)L^{2/3} - L^{1/3+\epsilon}, -(K-2\mu)L^{2/3} + L^{1/3+\epsilon}] \times [0, cL^{1/3})\,.$$

Note that the sup over K is trivial: indeed, K plays no role in the limit in Eq. (7.11), since by horizontal shift-invariance, we may shift to the right by $KL^{2/3}$. The denominator may be bounded from below as a constant, by shifting both $\bar{\mathsf{u}}_2$ and $\bar{\mathsf{w}}_2$ up to $cL^{1/3}$, and then applying the Brownian excursion limit as in the proof of Claim 7.4. The area tilt in the numerator may be bounded from above by 1. Thus, we have reduced to showing the following:

$$\lim_{c\to 0}\limsup_{L\to\infty}\mathbb{P}_+^{\bar{\mathsf{u}},\bar{\mathsf{w}}}(\mathsf{S}_2(L^{2/3})<cL^{1/3})=0\,,$$

uniformly over $\bar{u}$ and $\bar{w}$. By monotonicity, we may assume $\bar{w}_2=0$. The result now follows from the Brownian excursion limit of S_2 under $\mathbb{P}_+^{\bar{\mathsf{u}},\bar{\mathsf{w}}}$ (Theorem 4.3). ☐

Following [30], we are now ready to prove Theorem 7.1.

Proof of Theorem 7.1. We have reduced to an area-tilted, directed 2D random walk bridge between u and v satisfying Eq. (7.2) and (7.3) respectively, conditioned to stay positive. That is, we have a random walk S under law

$$\mathsf{S}\sim\mathbb{P}_{\lambda,+}^{\mathsf{u},\mathsf{v}}\,.$$

This puts us in the framework of the proof of the Ferrari–Spohn limit for such random walks given in [30, Section 6.6 and 6.7]. Tightness follows exactly as in [30, Section 6.6]. The proof of finite-dimensional distributions in [30, Section 6.7] had just two additional inputs: their Proposition 6.2 and Lemma 6.3, and so we will be done as soon as we establish our analogues of these results.

Their Proposition 6.2 holds exactly the same for us. Their Lemma 6.3 does as well, with the exception that the constant in front of our area tilt is slightly different, and thus the resulting generator is slightly different as well. We re-state and prove that result in our setting.

For any $n\in\mathbb{N}$, we will write $\mathsf{S}[0,n]:=(\mathsf{S}(i))_{i\in[0,n]}$. Following [30, Eq.(6.6)], for any function $f:\mathbb{N}\to\mathbb{R}$ and $u\in\mathbb{H}$, define the n-step partition function

$$\mathcal{G}^n_{\lambda,L,+}[f](\mathsf{u}):=\mathbb{E}_{\mathsf{u}}\Big[e^{-\frac{\lambda}{N}A(\mathsf{S}[0,n])}f(\mathsf{S}_2(n))\mathbb{1}_{\{\mathsf{S}[0,n]\subset\mathbb{H}\}}\Big]\,.$$

Recall from Section 4.1 that S under law $\mathbb{P}$ has step-distribution $X=(X_1,X_2)$. Recall also $\sigma^2:=\mathrm{Var}(X_2)/\mu$ from the discussion above Theorem 4.3. Following [30, Section 6.5], define the following operator on smooth test functions f with compact support in $(0,\infty)$:

$$\mathbf{T}_Lf(r):=\mathbb{E}\Big[e^{-\frac{\lambda}{L}X_1rL^{1/3}\sigma}f\Big(r+\frac{X_2}{L^{1/3}\sigma}\Big)\mathbb{1}_{\{r+\frac{X_2}{L^{1/3}\sigma}\geq 0\}}\Big]\,.$$

Note that the indicator may be dropped, as f is supported above 0. By Taylor expanding f to second order and e^x-1 to first order (the errors are $o(L^{-2/3})$), we have

$$\begin{aligned}&\lim_{L\to\infty}\frac{\mathbf{T}_L-\mathrm{Id}}{L^{-2/3}}f(r)\\&\quad=\lim_{L\to\infty}L^{2/3}\mathbb{E}\Big[(e^{-\frac{\lambda}{L}X_1rL^{1/3}\sigma}-1)f(r)+e^{-\frac{\lambda}{L}X_1rL^{1/3}\sigma}\Big(f'(r)\frac{X_2}{L^{1/3}\sigma}+\tfrac12 f''(r)\frac{X_2^2}{L^{2/3}\sigma^2}\Big)\Big]\\&\qquad=\lim_{L\to\infty}L^{2/3}\mathbb{E}\Big[-\frac{\lambda}{L^{2/3}}X_1r\sigma f(r)+e^{-\frac{\lambda}{L}X_1rL^{1/3}\sigma}\Big(f'(r)\frac{X_2}{L^{1/3}\sigma}+\tfrac12 f''(r)\frac{X_2^2}{L^{2/3}\sigma^2}\Big)\Big]\,.\end{aligned}$$

Now use $\mu:=\mathbb{E}[X_1]$, $\mathbb{E}[X_2]=0$, and $\mathrm{Var}(X_2)=\mathbb{E}[X_2^2]$, so that the $f'(r)$ term vanishes and we find

$$\lim_{L\to\infty}\frac{\mathbf{T}_L-\mathrm{Id}}{\mu L^{-2/3}}f(r)=\tfrac12 f''(r)-\lambda\sigma rf(r)=:\mathcal{L}f(r)\,.$$

Kurtz's semigroup convergence theorem ([24, Theorem 1.6.5], see also [33, Eq.(2.34)]) then gives

$$\lim_{L\to\infty}\mathbf{T}_L^{\lfloor tL^{2/3}/\mu\rfloor}f=e^{t\mathcal{L}}f\,,$$

uniformly over t in bounded intervals of $\mathbb{R}_+$. Define the rescaling operator

$$\mathrm{Sc}_Lf(x)=f(xL^{-1/3}\sigma^{-1})\,.$$

Observing that $\mathcal{G}^k_{\lambda,L,+}[\mathrm{Sc}_L f] = \mathbf{T}^k_L f$, the second-to-last-display immediately yields

$$\lim_{L\to\infty} \mathcal{G}^{\lfloor tL^{2/3}/\mu\rfloor}_{\lambda,L,+}[\mathrm{Sc}_L f] = e^{t\mathcal{L}} f\,,$$

which is our version of [30, Lemma 6.3]. □

Appendix A. Cluster expansion and open contours in the SOS model

A.1. **Cluster expansion.** Consider the SOS model in a region $\Lambda \Subset \mathbb{Z}^2$ without floor, with boundary condition $j \in \mathbb{Z}$. Fix any subset U of the inner boundary of Λ, and condition on $\{\varphi|_U \geq j\}$. Let $\widehat{Z}_{\Lambda,U} := \widehat{Z}^{j,+}_{\Lambda,U}$ denote the partition function of this model. Note that the partition function has no dependence on j and is also unchanged by a replacement of the conditioning with $\{\varphi|_U \leq j\}$. [13] proves that there exists a constant $\beta_0 > 0$ such that for all $\beta > \beta_0$, one has

$$\log \widehat{Z}_{\Lambda,U} = \sum_{V\subset\Lambda} f_U(V) \tag{A.1}$$

for some function f_U. Equation (A.1) was proven using the main theorem from [36], which can actually be used to show (with the same proof as in [13]) that the formula holds simultaneously for all subsets $\Lambda' \subset \Lambda$ (with the same f_U):

$$\log \widehat{Z}_{\Lambda',U} = \sum_{V\subset\Lambda'} f_U(V) \tag{A.2}$$

(here, $Z_{\Lambda',U}$ is defined as before, with the understanding that the condition on U only needs to be satisfied on $\Lambda' \cap U$, or in other words $Z_{\Lambda',U} := Z_{\Lambda',U\cap\Lambda'}$). The latter observation leads, using Möbius inversion, to the following formula for f_U:

$$f_U(V) = \sum_{W\subset V} (-1)^{|V|-|W|} \log \widehat{Z}_{W,U}\,. \tag{A.3}$$

From Eq. (A.3), we can read off some properties of the function f_U:

(1) $f_U(V)$ only depends on U only through $U\cap V$, and in particular, if $U\cap V = \emptyset$, $f_U(V) = f_0(V)$ for some universal function f_0. Moreover, $f_0(V)$ only depends on the "shape" and size of the volume V, i.e., $f_0(\cdot)$ is invariant with respect to lattice symmetries.
(2) If V is not connected, $f_U(V) = 0$. Indeed, if $V = V_1 \sqcup V_2$, each log-partition function on the right-hand side of Eq. (A.2) splits into the sum of two terms, both of which appear in the global sum with the same frequency of plus and minus signs.
(3) There exists a constant $\beta_0 > 0$ such that

$$\sup_U |f_U(\mathcal{C})| \leq \exp\left((\beta - \beta_0) d(\mathcal{C})\right)\,, \tag{A.4}$$

where $d(\mathcal{C})$ denotes the cardinality of the smallest connected set of bonds of $\mathbb{Z}^2$ containing all boundary bonds of $\mathcal{C}$ (i.e., bonds connecting $\mathcal{C}$ to $\mathcal{C}^c$), see [20, Proposition 3.9] for details.

A.2. **Proof of Proposition 2.1.** Recall $\widehat{Z}^{j,+}_{\Lambda,U}$ from above Eq. (A.1), and define $\widehat{Z}^{j,-}_{\Lambda,U}$ similarly but with the conditioning $\{\varphi|_U \leq j\}$ (as noted above Eq. (A.1), $\widehat{Z}^{j,+}_{\Lambda,U} = \widehat{Z}^{j,-}_{\Lambda,U}$. The different notation is mostly for clarity in the calculations that follow). Recall also $\Lambda^+_\gamma, \Lambda^-_\gamma, \Delta^+_\gamma$, and Δ^-_γ from above Proposition 2.1.

Observe that

$$\widehat{Z}^\xi_\Lambda = \sum_\gamma e^{-\beta|\gamma|} \widehat{Z}^{1,+}_{\Lambda^+_\gamma,\Delta^+_\gamma} \widehat{Z}^{0,-}_{\Lambda^-_\gamma,\Delta^-_\gamma}\,, \tag{A.5}$$

This is a simple consequence of the following identity: for any $\mathsf{u},\mathsf{v} \in \mathbb{Z}^2$ such that $\varphi_{\mathsf{u}} \geq 1$ and $\varphi_{\mathsf{v}} \leq 0$, we have

$$|\varphi_{\mathsf{u}} - \varphi_{\mathsf{v}}| = \big|(\varphi_{\mathsf{u}} - 1) - (\varphi_{\mathsf{v}} - 0) + (1-0)\big| = (\varphi_{\mathsf{u}} - 1) + (0 - \varphi_{\mathsf{v}}) + 1\,.$$

Now, cluster-expanding each of the partition functions on the right-hand side of Eq. (A.5) yields

$$\begin{aligned}\widehat{Z}^1_{\Lambda^+_\gamma,\Delta^+_\gamma}\widehat{Z}^0_{\Lambda^-_\gamma,\Delta^-_\gamma} &= \exp\Big(\sum_{\substack{\mathcal{C}\subset\Lambda\\ \mathcal{C}\cap\Delta_\gamma=\emptyset}} f_0(\mathcal{C}) + \sum_{\substack{\mathcal{C}\subset\Lambda^+_\gamma\\ \mathcal{C}\cap\Delta_\gamma\neq\emptyset}} f_{\Delta^+_\gamma}(\mathcal{C}) + \sum_{\substack{\mathcal{C}\subset\Lambda^-_\gamma\\ \mathcal{C}\cap\Delta_\gamma\neq\emptyset}} f_{\Delta^-_\gamma}(\mathcal{C})\Big)\\ &= e^{\sum_{\mathcal{C}\subset\Lambda} f_0(\mathcal{C})}\exp\Big(\Psi_\Lambda(\gamma)\Big) = \widehat{Z}^0_\Lambda \exp\Big(\Psi_\Lambda(\gamma)\Big)\,,\end{aligned} \tag{A.6}$$

where

$$\Psi_\Lambda(\gamma) := -\sum_{\substack{\mathcal{C}\subset\Lambda\\ \mathcal{C}\cap\Delta_\gamma\neq\emptyset}} f_0(\mathcal{C}) + \sum_{\substack{\mathcal{C}\subset\Lambda^+_\gamma\\ \mathcal{C}\cap\Delta_\gamma\neq\emptyset}} f_{\Delta^+_\gamma}(\mathcal{C}) + \sum_{\substack{\mathcal{C}\subset\Lambda^-_\gamma\\ \mathcal{C}\cap\Delta_\gamma\neq\emptyset}} f_{\Delta^-_\gamma}(\mathcal{C})\,.$$

Defining

$$\phi(\mathcal{C};\gamma) := -f_0(\mathcal{C}) + f_{\Delta^+_\gamma}(\mathcal{C})\mathbb{1}_{\{\mathcal{C}\cap\Delta^-_\gamma=\emptyset\}} + f_{\Delta^-_\gamma}(\mathcal{C})\mathbb{1}_{\{\mathcal{C}\cap\Delta^+_\gamma=\emptyset\}}, \tag{A.7}$$

it follows from Eqs. (A.5) and (A.6) that

$$\widehat{Z}^\xi_\Lambda = \widehat{Z}^0_\Lambda \sum_\gamma \exp\Big(-\beta|\gamma| + \sum_{\substack{\mathcal{C}\subset\Lambda\\ \mathcal{C}\cap\Delta_\gamma\neq\emptyset}} \phi(\mathcal{C};\gamma)\Big)\,.$$

Equations (2.1) and (2.3) follow immediately. From the definition of ϕ in Eq. (A.7) and the properties of f_U from Appendix A.1, we obtain Properties (i) to (iv) of the proposition. ☐

A.3. **Law of an open contour in the SOS model with floor.** Consider the SOS model π^ξ_R in a domain R with floor at 0 and boundary condition $\xi \in \{h-1, h\}$ inducing a unique open h-contour γ. Consider $h = \lfloor \frac{1}{4\beta}\log L\rfloor - n$, for $n \in \mathbb{N}$ fixed relative to L, and assume that the domain R satisfies $|R| \leq L^{\frac{4}{3}+2\epsilon}$ and $|\partial R| \leq L^{\frac{2}{3}+2\epsilon}$, for some $\epsilon \in (0, 1/20)$. In this subsection, we derive an asymptotic expression for the law of the random contour γ in terms of the no-floor law $\widehat{\pi}^\xi_R$.

For a subset U of the inner boundary of R, recall $\widehat{Z}^{j,\pm}_{R,U}$ as in Appendix A.2, and define $Z^{j,\pm}_{R,U}$ similarly but with a floor at 0. As in Eq. (A.5), we can write the law of γ in terms of the partition functions above and below it:

$$\pi^\xi_R(\gamma) = e^{-\beta|\gamma|} Z^{h,+}_{R^+_\gamma,\Delta^+_\gamma} Z^{h-1,-}_{R^-_\gamma,\Delta^-_\gamma}\,.$$

The partition functions $Z^{j,\pm}_{R,U}$ can be related to the corresponding partition function $\widehat{Z}^{j,\pm}_{R,U}$ of the model without floor via the following (taken from [14, Prop. A.1]):

$$Z^{j,\pm}_{R,U} = \widehat{Z}^{j,\pm}_{R,U}\exp(-\widehat{\pi}(\varphi_o > j)|R| + o(1))\,,$$

where $\widehat{\pi}$ is the infinite volume measure obtained as the thermodynamic limit of $\widehat{\pi}^0_\Lambda$, see [3]. It follows that

$$\pi^\xi_R(\gamma) \propto e^{-\beta|\gamma|}\widehat{Z}^{h,+}_{R^+_\gamma,\Delta^+_\gamma}\widehat{Z}^{h-1,-}_{R^-_\gamma,\Delta^-_\gamma}\exp\Big(-(\widehat{\pi}(\varphi_o > h-1) - \widehat{\pi}(\varphi_o > h))|R^-_\gamma| + o(1)\Big)\,,$$

where we used $|R^+_\gamma| = |R| - |R^-_\gamma|$ and dropped the term $|R|\widehat{\pi}(\varphi_o > h)$, which is independent of γ. It was proved in [14, Lemma 2.4] that there exists a constant $c_\infty := c_\infty(\beta)$ such that, for $j \geq 1$,

$$\widehat{\pi}(\varphi_o \geq j) = c_\infty e^{-4\beta j} + O(e^{-6\beta j})\,.$$

Define $a_L := \frac{1}{4\beta}\log L - \lfloor\frac{1}{4\beta}\log L\rfloor$, $\lambda := \lambda(L) = c_\infty e^{4\beta a_L}(1-e^{-4\beta})$ and $\lambda^{(n)} := \lambda e^{4\beta n}$. We have

$$\begin{aligned}\widehat{\pi}(\varphi_o > h-1) - \widehat{\pi}(\varphi_o > h) &= c_\infty e^{-4\beta h}(1-e^{-4\beta}) + O(e^{-6\beta h}) \\ &= c_\infty e^{4\beta a_L - \log L + 4\beta n}(1-e^{-4\beta}) + O(e^{-6\beta h}) = \lambda^{(n)}/L + O(e^{-6\beta h})\,.\end{aligned}$$

Putting everything together, we find

$$\pi_R^\xi(\gamma) \propto \widehat{\pi}_R^\xi(\gamma)\exp\Big(-\frac{\lambda^{(n)}}{L}A(\gamma) + o(1)\Big)\,, \tag{A.8}$$

where $A(\gamma) = |R_\gamma^-|$ is the area under γ in R.

Appendix B. Proof of Proposition 2.14 and Eq. (5.18)

Proof of Proposition 2.14. Fix $\delta \in (0,1)$. We begin by showing that for any $\mathsf{y} \in \mathcal{Y}_\delta^{\blacktriangleleft} \setminus \{0\}$, we have $f(\mathsf{h}_\mathsf{y}) = 1$, where

$$f(\mathsf{h}) := \sum_{\Gamma\in\mathsf{A}} W^{\mathsf{h}}(\Gamma)\,.$$

Let

$$\mathcal{W}_{\text{in}}^{\blacktriangleleft} := \{s\mathsf{h}_\mathsf{y} : \mathsf{y} \in \mathcal{Y}_\delta^{\blacktriangleleft} \setminus \{0\},\ s \in [0,1]\}$$

and

$$\mathcal{W}_{\text{out}}^{\blacktriangleleft} := \{s\mathsf{h}_\mathsf{y} : \mathsf{y} \in \mathcal{Y}_\delta^{\blacktriangleleft} \setminus \{0\},\ s \in [1, 1+\tfrac{\beta\nu}{2}\|\mathsf{h}_\mathsf{y}\|_2^{-1}]\}\,,$$

where ν is the same as in Proposition 2.12. In words, $\mathcal{W}_{\text{in}}^{\blacktriangleleft}$ is the sector of the Wulff shape $\mathcal{W}$ where the sector boundary (part of $\partial\mathcal{W}$) is $\{\mathsf{h}_\mathsf{y} : \mathsf{y} \in \mathcal{Y}_\delta^{\blacktriangleleft}\}$, while $\mathcal{W}_{\text{out}}^{\blacktriangleleft}$ lies right outside of $\mathcal{W}$ along the continuation of such sector (note that $\mathcal{W}_{\text{in}}^{\blacktriangleleft} \subset \mathcal{W}$ comes from $0 \in \mathcal{W}$ and the convexity of $\mathcal{W}$).

In what follows, we let $C := C(\beta) > 0$ denote a constant that may depend on β and may change from line-to-line. We begin with two claims.

Claim B.1. *For all $\mathsf{h} \in \mathcal{W}_{\text{in}}^{\blacktriangleleft} \cup \mathcal{W}_{\text{out}}^{\blacktriangleleft}$, we have*

$$\sum_{\Gamma\in\mathsf{A_L}\cup\mathsf{A_R}} W^{\mathsf{h}}(\Gamma) < \infty\,,$$

and the series converges uniformly over h. In particular, $f(\mathsf{h})$ is continuous on $\mathcal{W}_{\text{in}}^{\blacktriangleleft} \cup \mathcal{W}_{\text{out}}^{\blacktriangleleft}$.

Proof of Claim B.1. We will use the following bound many times: from Eq. (2.16), we have that for any $\epsilon \in (0,1)$ and for all $\beta > 0$ sufficiently large, there exists a constant $C := C(\epsilon,\beta) > 0$ such that for all $\mathsf{y} \in \mathbb{Z}^2$

$$\mathcal{G}(\mathsf{y}) \le Ce^{-\tau_\beta(\mathsf{y}) + \epsilon\|\mathsf{y}\|_1}\,. \tag{B.1}$$

For $\mathsf{h} \in \mathcal{W}_{\text{in}}^{\blacktriangleleft} \cup \mathcal{W}_{\text{out}}^{\blacktriangleleft}$, we have $\mathsf{h}\cdot\mathsf{y} - \tau_\beta(\mathsf{y}) \le \frac{\beta\nu}{2}\|\mathsf{y}\|_2 \le \frac{\beta\nu}{2}\|\mathsf{y}\|_1$ for all $\mathsf{y} \in \mathbb{Z}^2 \setminus \{0\}$. Then Proposition 2.12 and Eq. (B.1) yield

$$\sum_{\substack{\Gamma\in\mathsf{A_L}\cup\mathsf{A_R}\\ \Gamma:0\to\mathsf{y}}} W^{\mathsf{h}}(\Gamma) \le e^{\mathsf{h}\cdot\mathsf{y}}\mathcal{G}(\mathsf{y} \mid |\mathsf{Cpts}(\Gamma)| = 0) \le C\exp\Big(-(\tfrac{\nu\beta}{2} - \epsilon)\,\|\mathsf{y}\|_1\Big)\,. \tag{B.2}$$

Take $\epsilon < \nu\beta/2$. Then since $\mathsf{A} \subset \mathsf{A_L}$, the sequence of continuous functions

$$f_N(\mathsf{h}) := \sum_{\substack{\mathsf{y}\in\mathbb{Z}^2\setminus\{0\}\\ \|\mathsf{y}\|_1\le N}} \sum_{\substack{\Gamma\in\mathsf{A}\\ \Gamma:0\to\mathsf{y}}} W^{\mathsf{h}}(\Gamma)$$

converges uniformly to $f(\mathsf{h})$, and so the claim follows. ☐

Claim B.2. *For $\mathsf{h} \in \mathcal{W}_{\text{in}}^{\blacktriangleleft}$, $f(\mathsf{h}) \le 1$. For $\mathsf{h} \in \mathcal{W}_{\text{out}}^{\blacktriangleleft}$, $f(\mathsf{h}) \ge 1$.*

Proof of Claim B.2. Similar to Eq. (B.2), we have the following for any $\mathsf{h} \in \mathcal{W}_{\text{in}}^{\blacktriangleleft} \cup \mathcal{W}_{\text{out}}^{\blacktriangleleft}$:

$$\sum_{\mathsf{y} \in \mathcal{Y}_\delta^{\blacktriangleleft}} \sum_{\substack{\Gamma = [\gamma, \underline{\mathcal{C}}] \\ \gamma : 0 \to \mathsf{y}}} W^{\mathsf{h}}(\Gamma) \mathbb{1}_{\{|\mathsf{Cpts}(\Gamma)| < 2\}} \leq C + C \sum_{\substack{\mathsf{y} \in \mathcal{Y}_\delta^{\blacktriangleleft} \\ \|\mathsf{y}\|_1 > 2\delta_0^{-1}}} e^{\mathsf{h} \cdot \mathsf{y} - \beta\nu\|\mathsf{y}\|} \mathcal{G}(\mathsf{y}) < \infty \,. \tag{B.3}$$

Using the factorization of $q(\Gamma)$ (2.18) and (B.3), we have

$$\begin{aligned}
&\sum_{\mathsf{y} \in \mathcal{Y}_\delta^{\blacktriangleleft}} e^{\mathsf{h} \cdot \mathsf{y}} \mathcal{G}(\mathsf{y}) \\
&\qquad \leq C + \sum_{\mathsf{y} \in \mathcal{Y}^{\blacktriangleleft}} \sum_{m \geq 1} \sum_{\Gamma^{(L)} \in \mathsf{A_L}} \sum_{\Gamma^{(1)}, \dots, \Gamma^{(m)} \in \mathsf{A}} \sum_{\Gamma^{(R)} \in \mathsf{A_R}} W^{\mathsf{h}}(\Gamma^{(L)}) \Big(\prod_{i=1}^m W^{\mathsf{h}}(\Gamma^{(i)}) \Big) W^{\mathsf{h}}(\Gamma^{(R)}) \mathbb{1}_{\mathsf{X}(\Gamma) = \mathsf{y}} \\
&\qquad = C + \Big(\sum_{\Gamma^{(L)} \in \mathsf{A_L}} W^{\mathsf{h}}(\Gamma^{(L)}) \Big) \Big(\sum_{\Gamma^{(R)} \in \mathsf{A_R}} W^{\mathsf{h}}(\Gamma^{(R)}) \Big) \sum_{m \geq 1} f(\mathsf{h})^m \\
&\qquad =: C + B_L(\mathsf{h}) B_R(\mathsf{h}) \sum_{m \geq 1} f(\mathsf{h})^m \,,
\end{aligned}$$

where in the inequality we replaced $\mathcal{Y}_\delta^{\blacktriangleleft}$ by the full cone $\mathcal{Y}^{\blacktriangleleft}$ and in the next line we exchanged the sums and used the fact that an animal with at least one cone-point must necessarily satisfy $\mathsf{X}(\Gamma) \in \mathcal{Y}^{\blacktriangleleft}$. Hence, we have the chain of inequalities

$$\sum_{\mathsf{y} \in \mathcal{Y}_\delta^{\blacktriangleleft}} e^{\mathsf{h} \cdot \mathsf{y}} \mathcal{G}(\mathsf{y}) \leq C + B_L(\mathsf{h}) B_R(\mathsf{h}) \sum_{m \geq 1} f(\mathsf{h})^m \leq C + \sum_{\mathsf{y} \in \mathbb{Z}^2} e^{\mathsf{h} \cdot \mathsf{y}} \mathcal{G}(\mathsf{y}) \,. \tag{B.4}$$

Since $B_L(\mathsf{h}), B_R(\mathsf{h}) < \infty$ by Claim B.1, finiteness of the middle term in (B.4) depends only on the convergence of $\sum_{m \geq 1} f(\mathsf{h})^m$.

Now, if h is in the interior of $\mathcal{W}_{\text{in}}^{\blacktriangleleft}$ (and thus in the interior of $\mathcal{W}$), the rightmost term in (B.4) converges due to the second definition of $\mathcal{W}$, and thus $\sum_{m \geq 1} f(\mathsf{h})^m < \infty$, and thus $f(\mathsf{h}) < 1$. By continuity of f, we get $f(\mathsf{h}) \leq 1$ on $\mathcal{W}_{\text{in}}^{\blacktriangleleft}$. On the other hand, if h is in the interior of $\mathcal{W}_{\text{out}}^{\blacktriangleleft}$, the leftmost term in (B.4) diverges (by the same argument that showed the equivalence of the two definitions of $\mathcal{W}$ in subsection 2.6), and thus $\sum_{m \geq 1} f(\mathsf{h})^m = \infty$, and thus $f(\mathsf{h}) \geq 1$. □

By continuity of f, it follows that $f(\mathsf{h}) = 1$ on $\mathcal{W}_{\text{in}}^{\blacktriangleleft} \cap \mathcal{W}_{\text{out}}^{\blacktriangleleft}$, i.e., $f(\mathsf{h}_\mathsf{y}) = 1$ for $\mathsf{y} \in \mathcal{Y}_\delta^{\blacktriangleleft}$. This proves that $\mathbb{P}^{\mathsf{h}_\mathsf{y}}$ defines a probability distribution on A for $\mathsf{y} \in \mathcal{Y}_\delta^{\blacktriangleleft}$.

Next, we show (2.25). For any $\mathsf{y} \in \mathcal{Y}_\delta^{\blacktriangleleft} \setminus \{0\}$, define

$$F(\mathsf{y}) := f(\mathsf{h}_\mathsf{y}) = \sum_{\Gamma \in \mathsf{A}} \mathbb{P}^{\mathsf{h}_\mathsf{y}}(\Gamma) \,.$$

From above, we know that $F(\mathsf{y}) \equiv 1$. Since τ_β is analytic outside of the origin (Proposition 2.4), it is in particular twice-differentiable, and thus so is $\mathbb{P}^{\mathsf{h}_\mathsf{y}}(\Gamma)$ with $\nabla \mathbb{P}^{\mathsf{h}_\mathsf{y}}(\Gamma) = J_{\mathsf{h}_\mathsf{y}} \mathsf{X}(\Gamma) \mathbb{P}^{\mathsf{h}_\mathsf{y}}$, where $J_{\mathsf{h}_\mathsf{y}}$ is the Jacobian matrix of $\mathsf{y} \mapsto \mathsf{h}_\mathsf{y}$. Using the same argument as in Claim B.1 and boundedness of $J_{\mathsf{h}_\mathsf{y}}$, we find that $\sum_{\Gamma \in \mathsf{A}} J_{\mathsf{h}_\mathsf{y}} \mathsf{X}(\Gamma) \mathbb{P}^{\mathsf{h}_\mathsf{x}}(\Gamma)$ converges uniformly for $\mathsf{y} \in \mathbb{Z}^2 \setminus \{0\}$, and thus we can differentiate $F(\mathsf{y})$ under the sum to get

$$0 = \nabla F(\mathsf{y}) = \sum_{\Gamma \in \mathsf{A}} J_{\mathsf{h}_\mathsf{y}} \mathsf{X}(\Gamma) \mathbb{P}^{\mathsf{h}_\mathsf{y}}(\Gamma) = J_{\mathsf{h}_\mathsf{y}} \sum_{\Gamma \in \mathsf{A}} \mathsf{X}(\Gamma) \mathbb{P}^{\mathsf{h}_\mathsf{y}}(\Gamma) = J_{\mathsf{h}_\mathsf{y}} \mathbb{E}^{\mathsf{h}_\mathsf{y}}[\mathsf{X}(\Gamma)] \,,$$

i.e., $\mathbb{E}^{\mathsf{h}_\mathsf{y}}[\mathsf{X}(\Gamma)] \in \mathrm{Ker}(J_{\mathsf{h}_\mathsf{y}})$. Differentiating the relation $\mathsf{h}_\mathsf{y} \cdot \mathsf{y} = \tau_\beta(y)$, we see that $\mathsf{y} \in \mathrm{Ker}(J_{\mathsf{h}_\mathsf{y}})$, and so if $J_{\mathsf{h}_\mathsf{y}}$ is non-degenerate we get that $\mathbb{E}^{\mathsf{h}_\mathsf{y}}[\mathsf{X}(\Gamma)]$ is collinear to y, i.e., $\mathbb{E}^{\mathsf{h}_\mathsf{y}}[\mathsf{X}(\Gamma)] = \alpha \mathsf{y}$ for some scalar $\alpha = \alpha(\mathsf{y})$. Finally, the non-degeneracy condition can be removed: since τ_β is analytic (outside of the origin), the points at which $J_{\mathsf{h}_\mathsf{y}}$ is fully degenerate form a discrete set, and since collinearity must hold outside of such set and $\mathbb{E}^{\mathsf{h}_\mathsf{y}}[\mathsf{X}(\Gamma)]$ is continuous, we get the result for any $\mathsf{y} \in \mathcal{Y}_\delta^{\blacktriangleleft} \setminus \{0\}$.

It remains to show the exponential tail decay (2.26), which actually holds for the weights W^{h} for any $\mathsf{h} \in \mathcal{W}$. Below, $\nu_1 > 0$ and $\nu_2 > 0$ denote constants independent of h and β, while $C_1 > 0$ and $C_2 > 0$ may depend on β but not h. We express the left-hand side of (2.26) as follows:

$$\sum_{\substack{\Gamma \in \mathsf{A_L} \cup \mathsf{A_R} \\ |\Gamma| \geq k}} W^{\mathsf{h}}(\Gamma) \leq \sum_{\|\mathsf{y}\|_1 \geq k/2} \sum_{\substack{\Gamma \in \mathsf{A_L} \cup \mathsf{A_R} \\ |\Gamma| \geq k \\ \Gamma: 0 \to \mathsf{y}}} W^{\mathsf{h}}(\Gamma) + \sum_{\|\mathsf{y}\|_1 < k/2} \sum_{\substack{|\Gamma| \geq k \\ \Gamma: 0 \to \mathsf{y}}} W^{\mathsf{h}}(\Gamma)\,, \tag{B.5}$$

The first term in the above sum can be bounded using (B.1), Proposition 2.12, and the bound $\mathsf{h} \cdot \mathsf{y} \leq \tau_\beta(\mathsf{y})$ for any $\mathsf{h} \in \mathcal{W}$:

$$\sum_{\|\mathsf{y}\|_1 \geq k/2} \sum_{\substack{\Gamma \in \mathsf{A_L} \cup \mathsf{A_R} \\ |\Gamma| \geq k \\ \Gamma: 0 \to \mathsf{y}}} W^{\mathsf{h}}(\Gamma) \leq \sum_{\|\mathsf{y}\|_1 \geq k/2} e^{\mathsf{h} \cdot \mathsf{y}} \mathcal{G}(\mathsf{y} \mid |\mathsf{Cpts}(\Gamma)| = 0) \leq \sum_{\|\mathsf{y}\|_1 \geq k/2} e^{-(\nu\beta - \epsilon)\|\mathsf{y}\|_1} \leq C_1 e^{-\nu_1 \beta k}\,.$$

The second term in (B.5) is bounded by

$$\begin{aligned} \sum_{\|\mathsf{y}\|_1 < k/2} e^{\mathsf{h} \cdot \mathsf{y}} \mathcal{G}(\mathsf{y} \mid |\gamma| \geq k) &\leq \sum_{\|\mathsf{y}\|_1 < k/2} e^{\tau_\beta(\mathsf{y})} \Big(c e^{-\nu_0 \beta (k/\|\mathsf{y}\|_1)\|\mathsf{y}\|_1} \Big) \mathcal{G}(\mathsf{y}) \\ &\leq C_2 e^{-\nu_0 \beta k} \sum_{\|\mathsf{y}\|_1 < k/2} e^{\epsilon \|\mathsf{y}\|_1} \leq C_2 e^{-\nu_2 \beta k} \end{aligned}$$

where the first inequality uses (2.21) and $\mathsf{h} \cdot \mathsf{y} \leq \tau_\beta(\mathsf{y})$, and the second inequality uses (B.1). ☐

Proof of Eq. (5.18). Define the partition function over contours $\gamma \subset \mathbb{Z}^2$ such that $\gamma : 0 \to \mathsf{y}$:

$$\mathcal{G}_D^{\mathtt{full}}(\mathsf{y}) := \sum_{\gamma: 0 \to \mathsf{y}} q_D(\gamma)\,.$$

Eq. (2.13) gives us $q_D(\gamma) \leq \exp(6e^{-\chi\beta}|\gamma|) q(\gamma)$, from which we find

$$\mathcal{G}_D^{\mathtt{full}}(\mathsf{y}) \leq \sum_{\gamma: 0 \to \mathsf{y}} e^{6e^{-\chi\beta}|\gamma|} q(\gamma) \leq e^{6.6e^{-\chi\beta}\|\mathsf{y}\|_1} \mathcal{G}(\mathsf{y}) + \sum_{\substack{\gamma: 0 \to \mathsf{y} \\ |\gamma| \geq 1.1\|\mathsf{y}\|_1}} e^{6e^{-\chi\beta}|\gamma|} q(\gamma) \leq e^{7e^{-\chi\beta}\|\mathsf{y}\|_1} \mathcal{G}(\mathsf{y})\,,$$

where in the final inequality we used Eq. (2.21). We then obtain the analogue of Eq. (B.1) by applying Eq. (B.1) to the above: for any $\epsilon \in (0,1)$, there exists $\beta_0 := \beta_0(\epsilon) > 0$ such that for all $\beta \geq \beta_0$ and for all $\mathsf{y} \in \mathbb{Z}^2$,

$$\mathcal{G}_D^{\mathtt{full}}(\mathsf{y}) \leq C e^{-\tau_\beta(\mathsf{y}) + \epsilon \|\mathsf{y}\|_1} \tag{B.6}$$

for some $C := C(\epsilon, \beta) > 0$. Similarly, we obtain the length and contour cone-points bound Lemma 2.11 and the animal cone-points bound Proposition 2.12 with $\mathcal{G}_D^{\mathtt{full}}(\mathsf{y})$ replacing $\mathcal{G}(\mathsf{y})$ there. With these bounds, we may re-run the proof of Eq. (2.26), replacing $W^{\mathsf{h}}(\Gamma)$ with $e^{\mathsf{h} \cdot \mathsf{X}(\Gamma)} q_D(\Gamma)$. ☐

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