Critical slowdown for the Ising model on the 2D lattice

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Ising model

- Underlying geometry: finite graph $G = (V,E)$.
- Set of possible configurations:
  \[ \Omega = \{ \pm 1 \}^V \]
- Probability of a configuration $\sigma \in \Omega$ given by the Gibbs distribution
  \[ \mu(\sigma) = \frac{1}{Z(\beta)} \exp \left( \beta \sum_{xy \in E} \sigma(x)\sigma(y) \right) \] [no external field]
- Ferromagnetic $\leftrightarrow$ inverse-temperature $\beta \geq 0$.
- Phase transition as $\beta$ varies (in some geometries).
Glauber dynamics for Ising

- One of the most commonly used MC samplers for the Gibbs distribution:
  - Update sites via iid Poisson(1) clocks
  - Each update replaces a spin at $u \in V$ by a new one $\sim \mu$ conditioned on $V \setminus \{u\}$ (heat-bath version).

- Ergodic reversible MC with stationary measure $\mu$.

- Introduced by Glauber in 1963. Other versions of the dynamics include e.g. Metropolis.

- How fast does it converge to equilibrium?
Example: Glauber dynamics for critical Ising on the square lattice

- 256 x 400 square lattice w. boundary conditions: (+) at bottom (-) elsewhere.
- Frame every $\sim 2^{30}$ steps, i.e. $\sim 2^{13}$ updates/site.
Rate of convergence to equilibrium

- **Spectral gap** in the spectrum of the generator:
  \[
  \text{gap} = \text{smallest positive eigenvalue of the heat-kernel } H_t \text{ of the dynamics.}
  \]
  - Governs convergence in \( L^2(\mu) \).
  - Dirichlet-form characterization:
    \[
    \text{gap} = \inf_f \frac{\mathcal{E}(f)}{\text{Var}(f)}
    \]
    where
    \[
    \mathcal{E}(f) = \langle \mathcal{L}f, f \rangle_{L^2(\mu)} = \frac{1}{2} \sum_{\sigma, x} \mu(\sigma) c(x, \sigma) [f(\sigma^x) - f(\sigma)]^2.
    \]

- **Mixing time**: standard measure of convergence:
  - The \( L^1 \) (total-variation) mixing time within \( \varepsilon \) is
  \[
  t_{\text{mix}}(\varepsilon) = \inf \left\{ t : \max_{\sigma} \| H_t(\sigma, \cdot) - \mu \|_{TV} \leq \varepsilon \right\}.
  \]
General (believed) picture for Glauber dynamics

- Setting: Ising model on the lattice \((\mathbb{Z} / n\mathbb{Z})^d\).

- Belief: For some critical inverse-temperature \(\beta_c\):
  - Low temperature: \((\beta > \beta_c)\)
    \(\text{gap}^{-1}\) and \(t_{\text{mix}}\) are exponential in the surface area.
  - Critical temperature: \((\beta = \beta_c)\)
    \(\text{gap}^{-1}\) and \(t_{\text{mix}}\) are polynomial in the surface area.
  - High temperature: \((\beta < \beta_c)\)
    1. Rapid mixing: \(\text{gap}^{-1} \approx O(1)\) and \(t_{\text{mix}} \approx \log n\)
    2. Mixing occurs abruptly (cutoff phenomenon).
Gap/mixing-time evolution for Ising on mean-field

(Curie-Weiss model)

\[ \text{gap}^{-1}, t_{\text{mix}} \approx \frac{1}{\beta - 1} \exp\left[\frac{3}{4} (\beta - 1)^2 n\right] \]

\[ \text{gap}^{-1}, t_{\text{mix}} \approx n^{1/2} \]

\[ \text{gap}^{-1} = \frac{1 + o(1)}{1 - \beta} \]
\[ t_{\text{mix}} = \frac{1 + o(1)}{2(1 - \beta)} \log[(1 - \beta)^2 n] \]

Above picture established in [Ding, L., Peres ’09].
Mixing time for Ising on lattices: High temperature regime

- Mixing time of Ising on the lattice at high temp. was established in a series of seminal papers:
  - [Aizenman, Holley '84]
  - [Dobrushin, Shlosman '87]
  - [Holley, Stroock '87, '89]
  - [Holley '91]
  - [Stroock, Zegarlinski '92a, '92b, '92c]
  - [Zegarlinski '90, '92]
  - [Lu, Yau '93]
  - [Martinelli, Olivieri '94a, '94b]
  - [Martinelli, Olivieri, Schonmann '94]

- Bounded log-Sobolev constant and $O(\log n)$ mixing.
- In two dimensions this is known for all $\beta < \beta_c$. 
High temperature: $\text{gap}^{-1}$ is uniformly bounded, $O(\log n)$ mixing for all $\beta < \beta_c = \frac{1}{2} \log(1 + \sqrt{2})$.
- Dynamics conjectured to exhibit cutoff [Peres’04].
- Recently confirmed [L., Sly]: $t_{\text{mix}} = \frac{1 + o(1)}{\lambda_\infty} \log n$

Low temperature: for $\beta > \beta_c$ both $\text{gap}^{-1}$ and the mixing time are $\exp[(c(\beta) + o(1))n]$.

[Schonmann ‘87], [Chayes, Chayes, Schonmann’87], [Martinelli ’94], [Cesi, Guadagni, Martinelli, Schonmann’96].

Remains to verify power-law at critical $\beta = \beta_c$...
Glauber dynamics at criticality

- Polynomial lower bound on $\text{gap}^{-1}$ via the polynomial decay of spin-spin correlation whose asymptotics were established by [Onsager ’44] ([cf. Holley ’91]).

- Numerical experiments: $\exists$ universal exponent of $\sim 2.17$
  - [Ito ’93], [Wang, Hatano, Suzuki ’95], [Grassberger ’95], [Nightingale, Blöte ’96], [Wang, Hu ’97],...

- Compared to conjectured power-law behavior of $\text{gap}^{-1}$:
  - No known sub-exponential upper bounds ...

- Only geometries with proved power-law for critical Ising:
  - Mean-field [Ding, L., Peres ’09] (Curie-Weiss model)
  - Regular tree [Ding, L., Peres ’10] (Bethe lattice).
Scaling limit of critical Ising

- Understanding of the limit developed emerged with the advent of SLE ([Schramm '00]), CLE and tools to study conformally invariant systems.
- Recent breakthrough results due to [Smirnov] describe full scaling limit of the Ising cluster interfaces as CLE with parameter $\kappa = 3$.
  - cf. [Werner '03], [Lawler-Werner '04], [Sheffield '09].
- Important role in the analysis of critical Ising: its counterpart Fortuin-Kasteleyn representation.
Critical FK-Ising Model

- The FK-model is a measure over bond-percolation configurations also factoring in # of clusters.
- Scaling limits initially obtained for FK then converted to Ising.
  - E.g., full ensemble of FK cluster interfaces $\rightarrow \text{CLE}_{16/3}$.
- Recent development via the above theory & tools:
  - Russo-Seymour-Welsh type estimates for FK-Ising with various BC due to [Duminil-Copin, Hongler, Nolin ’09] [Camia, Newman ’09], [Chelkak, Smirnov ’09].
Main result: power-law at criticality

**Theorem [L., Sly]:** Critical slowdown verified in $\mathbb{Z}^2$:

Consider the critical Ising model on a finite box $\Lambda \subset \mathbb{Z}^2$ of side-length $n$, i.e. at inverse-temperature $\beta_c = \frac{1}{2} \log(1 + \sqrt{2})$. Let $\text{gap}_\tau^\Lambda$ denote the spectral-gap in the generator of the corresponding Glauber dynamics under an arbitrary fixed boundary condition $\tau$. Then there exists an absolute $C > 0$ (independent of $\Lambda, \tau$) such that $(\text{gap}_\tau^\Lambda)^{-1} \leq n^C$.

**Corollary:** Polynomial $L^1$ (total-variation) mixing time under any fixed boundary condition.
Further bounds on critical gap

- Analogous results for:
  - Free / periodic boundary conditions.
  - Critical anti-ferromagnetic Ising model.
- A new lower bound (previously known lower bound was nearly linear due to [Holley ’91]).

**Theorem**

Let $\text{gap}_\Lambda^\tau$ denote the spectral-gap of the Glauber dynamics for critical Ising on a finite box $\Lambda \subset \mathbb{Z}^2$ of side-length $n$ with an arbitrary boundary condition $\tau$. Then $\left(\text{gap}_\Lambda^\tau\right)^{-1} \geq cn^{7/4}$ for some absolute $c > 0$. 
Ramifications for sampling

- First rigorous efficient algorithm for approximated sampling of critical 2D Ising (\& its partition func.) for \textit{arbitrary} (e.g. mixed) boundary conditions.
  - For the \textit{free} boundary an efficient algorithm achieving this was given by [Jerrum Sinclair ’93].

- Perfect simulation:
  - Enabled by the [Propp-Wilson ’96] famous CFTP.
  - Applied to [JS ’93] algorithm by [Randall Wilson ’99] when boundary conditions are free/all-plus/all-minus.
  - New results allow \textit{rigorous efficient perfect simulation} under any arbitrary boundary via CFTP for Glauber dynamics.
Main techniques

- Common approach for analyzing the dynamics:
  - Control rate of mixing via a spatial-mixing result for the influence of individual boundary-spins on distant sites.
  - Use decay of correlation with distance.
- At criticality (Onsager’s work, also from “large” conformal loops with positive probability) there are long range correlations foiling this approach.
- Alternative approach:
  - Use conformal invariance to get a spatial-mixing result, combine it with classical ingredients from MC analysis.
  - Analyze effect of an entire face of the boundary on spins (just enough spatial mixing to push this program through... \(\ldots\))
Consider critical Ising model on a box $\Lambda \subset \mathbb{Z}^2$ of dimensions $m \times n$. Let $\text{gap}_\Lambda^\tau$ denote the spectral-gap in the generator of the corresponding Glauber dynamics under an arbitrary fixed boundary condition $\tau$. There exists an absolute $C > 0$ (independent of $\Lambda, \tau$) such that for any $m = m(n)$ we have 

$$(\text{gap}_\Lambda^\tau)^{-1} \leq n^C.$$ 

Only depends on the shorter side-length, e.g. on an extremely long rectangle of size $n \times \exp(\exp(n))$ we have the exact same $n^C$ bound of given for the square.
**Theorem**

Let $\Lambda = [1, r] \times [1, r']$ for some integers $r, r'$ satisfying $r'/r \geq \alpha > 0$ with $\alpha$ fixed and let $\Lambda_T = [1, r] \times [\rho r, r']$ for some $\rho$ satisfying $\alpha \leq \rho < r'/r$. Let $\xi, \eta$ be two BC’s on $\Lambda$ that differ only on the bottom boundary $[1, r] \times \{0\}$. Then

$$\|\mu_\Lambda^\xi(\sigma(\Lambda_T) \in \cdot) - \mu_\Lambda^\eta(\sigma(\Lambda_T) \in \cdot)\|_{TV} \leq \exp(-\delta \rho),$$

Where $\delta > 0$ is a constant that depends only on $\alpha$.

**Proof** uses the RSW-estimate for critical crossing probabilities in a wired FK-Ising rectangle.
Single site vs. Block dynamics

- Classical tool in the analysis of Glauber dynamics:
  - Cover the sites using blocks $B = \{B_i\}$.
  - Each block updates via a rate-1 Poisson clock.
  - Updates are $\sim$ stationary given the rest of the system.

- **Proposition** (see, e.g. [Martinelli '97]):

\[
(gap_B^\tau)^{-1} \leq \sum_\sigma \mu_\Lambda^\tau(\sigma) \sum_{x \in \Lambda} N_x \frac{c(x, \sigma) \left[ f(\sigma^x) - f(\sigma) \right]^2}{\sum_\sigma \mu_\Lambda^\tau(\sigma) \sum_{x \in \Lambda} c(x, \sigma) \left[ f(\sigma^x) - f(\sigma) \right]^2} (gap_B^\tau)^{-1} \max_{i, \varphi} (gap_{B_i}^\varphi)^{-1}
\]

where $(gap_B^\tau)^{-1}$ is the gap of the block-dynamics and $N_x = \#\{i : B_i \ni x\}$
Consider the following choice of blocks:

\[
\Lambda_1(\ell) = [1, r] \times \left[ \frac{1}{3} r', \frac{\ell - 1}{3} \sqrt{rr'}, r' \right],
\]

\[
\Lambda_2(\ell) = [1, r] \times \left[ 1, \frac{1}{3} r' + \frac{\ell}{3} \sqrt{rr'} \right]
\]

for some \( \ell \in \{1, \ldots, \left\lfloor \sqrt{r'/r} \right\rfloor \} \).

The two blocks have a vertical overlap of height \( \frac{1}{3} \sqrt{rr'} \).

As a result of the spatial-mixing theorem:

For any boundary condition \( \xi \) on \( \Lambda \) we have

\[
(gap_{B}^{\xi})^{-1} \geq 1 - \exp \left( -c \sqrt{r'/r} \right)
\]

for an absolute \( c > 0 \).
Upper bound via spatial-mixing (ctd.)

- Block dynamics reverts to a smaller block size at the cost of $1/\left[1 - \exp(-c\sqrt{r'/r})\right]$.
- Average over the blocks to eliminate the contribution of $N_x$ and replace it by $1 + \frac{1}{\sqrt{r'/r}}$.
- Repeated applications yield $r' \leq \frac{2}{3} r$ at the cost of an absolute constant.
- Iterating $\log_{3/2} n$ steps completes the proof.
Suppose that the three identical boundaries are all-minus, and the bottom boundary is all-plus in one measure and all-minus in the other.

- Ising cluster adjacent to bottom in plus-measure converges to SLE$_3$, which does not climb past height $\rho r$ with positive probability.
- In that case, measures can be coupled.

Actual setting:
- Arbitrary (mixed) boundary conditions break this argument down...
Solution: reduce to FK Ising

- Ising and its FK counterpart are coupled by the Edwards-Sokal coupling:

  - Under an arbitrary boundary condition $\xi$ one can go from Ising $\rightsquigarrow$ FK $\rightsquigarrow$ Ising conditioned on some event $A_\xi$ which may have exponentially small probability in FK...
Carrying the proof

- Control crossing probabilities in the FK-Ising model conditioned on the event $A_\xi$.
- Utilize the recent RSW-type estimates with the FKG for the FK-model to derive the required coupling.
- Return to Ising via the Edwards-Sokal method to complete the proof.
Open problems

- Calculate the precise (universal) critical dynamical exponent.
- Establish power-law behavior on the lattice in 3 dimensions.
THANK YOU.