CUTOFF FOR THE ISING MODEL ON THE LATTICE

Eyal Lubetzky
Microsoft Research

Joint work w. Allan Sly

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Ising model

- Underlying geometry: finite graph $G = (V, E)$.
- Set of possible configurations: $\Omega = \{\pm 1\}^V$
- Probability of a configuration $\sigma \in \Omega$ given by the Gibbs distribution
  
  $$\mu(\sigma) = \frac{1}{Z(\beta)} \exp \left( \beta \sum_{xy \in E} \sigma(x)\sigma(y) \right) \quad \text{[no external field]}$$

- Ferromagnetic $\leftrightarrow$ inverse-temperature $\beta \geq 0$.
- Phase transition as $\beta$ varies (in some geometries).
Glauber dynamics for Ising

- One of the most commonly used MC samplers for the Gibbs distribution:
  - Update sites via iid Poisson(1) clocks
  - Each update replaces a spin at $u \in V$ by a new one $\sim \mu$ conditioned on $V \setminus \{u\}$ (heat-bath version).

- Ergodic reversible MC with stationary measure $\mu$.

- Introduced by Glauber in 1963. Other versions of the dynamics include e.g. Metropolis.

- How fast does it converge to equilibrium?
Example: Glauber dynamics for critical Ising on the square lattice

- 256 x 400 square lattice w. boundary conditions: (+) at bottom, (−) elsewhere.
- Frame after $2^{20}$ steps, i.e. $\sim 10$ updates per site.
Rate of convergence to equilibrium

- **Mixing time**: standard measure of convergence:
  - The $L^1$ (total-variation) mixing time within $\varepsilon$ is
    \[ t_{\text{mix}}(\varepsilon) = \inf \left\{ t : \max_\sigma \| H_t(\sigma, \cdot) - \mu \|_{TV} \leq \varepsilon \right\} \]
    where $H$ is the heat-kernel.
  - “Mixing time” usually taken as $t_{\text{mix}}(1/4)$ by convention.

- **Spectral gap**: governs convergence in $L^2(\mu)$:
  
  \[
  \text{gap} = \text{smallest positive eigenvalue of the kernel } H.
  \]
General (believed) picture for Glauber dynamics

- Setting: Ising model on the lattice $(\mathbb{Z}/n\mathbb{Z})^d$.
- Belief: For some critical inverse-temperature $\beta_c$:
  - Low temperature: $(\beta > \beta_c)$, gap$^{-1}$ and $t_{mix}$ are exponential in the surface area.
  - Critical temperature: $(\beta = \beta_c)$, gap$^{-1}$ and $t_{mix}$ are polynomial in the surface area.
  - High temperature: $(\beta < \beta_c)$
    1. Rapid mixing: gap$^{-1} = O(1)$ and $t_{mix} \approx \log n$
    2. Mixing occurs abruptly, i.e., there is cutoff.
The Cutoff Phenomenon

- Describes a sharp transition in the convergence of finite ergodic Markov chains to stationarity.

Steady convergence
it takes a while to reach distance $\frac{1}{2}$ from stationarity then a while longer to reach distance $\frac{1}{4}$, etc.

Abrupt convergence
distance from equilibrium quickly drops from 1 to 0
Gap/mixing-time evolution for Ising on the complete graph (Curie-Weiss model)

\[ \text{gap}^{-1}, t_{\text{mix}} \approx \frac{1}{\beta - 1} \exp\left[\frac{3}{4} (\beta - 1)^2 n\right] \]

\[ \text{gap}^{-1}, t_{\text{mix}} \approx n^{1/2} \]

\[ \text{gap}^{-1} = \frac{1+o(1)}{1-\beta} \frac{1+o(1)}{2(1-\beta)} \log[(1-\beta)^2 n] \]

Above picture established in [Ding, L., Peres '09].
Mixing time for Ising on lattices: High temperature regime

- Mixing time of Ising on the lattice at high temp. was established in a series of seminal papers:
  - [Aizenman, Holley '84]
  - [Dobrushin, Shlosman '87]
  - [Holley, Stroock '87, '89]
  - [Holley '91]
  - [Stroock, Zegarlinski '92a, '92b, '92c]
  - [Zegarlinski '90, '92]
  - [Lu, Yau '93]
  - [Martinelli, Olivieri '94a, '94b]
  - [Martinelli, Olivieri, Schonmann '94]

- \( \Rightarrow \) Bounded log-Sobolev constant and \( O(\log n) \) mixing.

- In two dimensions this is known for all \( \beta < \beta_c \).
Mixing on the square lattice

- High temperature: $O(1) \log$-Sobolev constant and $O(\log n)$ mixing for all $\beta < \beta_c = \frac{1}{2} \log(1 + \sqrt{2})$.
- Dynamics conjectured to exhibit cutoff [Peres’04].

- Low temperature: for $\beta > \beta_c$, both gap$^{-1}$ and the mixing time are $\exp[(c(\beta) + o(1))n]$.
  [Schonmann ’87], [Chayes, Chayes, Schonmann’87],
  [Martinelli ’94], [Cesi, Guadagni, Martinelli, Schonmann’96].

- Critical temperature: No known sub-exponential upper bounds at $\beta = \beta_c$ for mixing or gap$^{-1}$ ...
Cutoff: formal definition

- A family of chains \((X^n_t)\) is said to have cutoff if:
  \[
  \lim_{n \to \infty} \frac{t_{\text{mix}}(\varepsilon)}{t_{\text{mix}}(1 - \varepsilon)} = 1 \quad \forall \ 0 < \varepsilon < 1.
  \]
  i.e., \(t_{\text{mix}}(\alpha) = (1 + o(1))t_{\text{mix}}(\beta)\) for any \(0 < \alpha, \beta < 1\).

- A sequence \((w_n)\) is called a cutoff window if
  \[
  w_n = o\left(t_{\text{mix}}(\frac{1}{4})\right),
  t_{\text{mix}}(\varepsilon) - t_{\text{mix}}(1 - \varepsilon) = O_{\varepsilon}(w_n) \quad \forall \ 0 < \varepsilon < 1.
  \]
Basic examples

Lazy discrete-time simple random walk

On the hypercube $\{0,1\}^n$:
- Exhibits cutoff at $\frac{1}{2} \log n + O(n)$
  [Aldous '83]

On the $n$-cycle:
- No cutoff.
The importance of cutoff

- Suppose we run Glauber dynamics for the Ising Model satisfying $t_{\text{mix}} \asymp f(n)$ for some $f(n)$.
- Cutoff $\Leftrightarrow \exists$ some $c_0 > 0$ so that:
  - Must run the chain for at least $\sim c_0 \cdot f(n)$ steps to even reach distance $(1 - \varepsilon)$ from $\mu$.
  - Running it any longer than that is essentially redundant.
- Proofs usually require (and thus provide) a deep understanding of the chain (its reasons for mixing).
- Many natural chains are believed to have cutoff, yet proving cutoff can be extremely challenging.
Cutoff History

- Random walks on graphs and groups:
  - Discovered:
    - Random transpositions on $S_n$ [Diaconis, Shahshahani ’81]
    - RW on the hypercube, Riffle-shuffle [Aldous ‘83]
  - Named “Cutoff Phenomenon” in the top-in-at-random shuffle analysis [Diaconis, Aldous ‘86]
  - RWs on finite groups [Saloff-Coste ‘04]
  - RWs on random regular graphs [L., Sly ’10+]

- One-dimensional Markov chains:
  - Birth-and-Death chains
    [Diaconis, Saloff-Coste ’06], [Ding, L., Peres ’09]

- No proofs of cutoff except when stationary distribution is completely understood and has many symmetries.
Cutoff for the Glauber dynamics

- So far only spin-systems where cutoff was verified are Ising and Potts models on the complete graph [Levin, Luczak, Peres ’10], [Ding, L., Peres ’09], [Cuff, Ding, L., Louidor, Peres, Sly]

- Conjectured to believe at high temperatures for:
  - Ising on the lattice, e.g. with periodic or free boundary.
  - Potts model on the lattice.
  - Gas Hard-core model on lattices.
  - Colorings of lattices.
  - Arbitrary boundary conditions / external field.
  - Anti-ferromagnetic Ising/Potts models, Spin-glass, Other lattices / amenable transitive graphs,...
Theorem [L., Sly]:

Let $\beta_c = \frac{1}{2} \log(1 + \sqrt{2})$ be the critical inverse-temperature for the Ising model on $\mathbb{Z}^2$. Then the continuous-time Glauber dynamics for the Ising model on $(\mathbb{Z}/n\mathbb{Z})^2$ with periodic boundary conditions at $0 \leq \beta < \beta_c$ has cutoff at $(1/\lambda_\infty) \log n$, where $\lambda_\infty$ is the spectral gap of the dynamics on the infinite volume lattice.

Analogous result holds for any dimension $d \geq 1$:

- Cutoff at $(d/2\lambda_\infty) \log n$
- E.g., cutoff at $\left[2(1 - \tanh(2\beta))\right]^{-1} \log n$ for $d = 1$. 
Cutoff for Ising on the lattice

- Main result hinges on an $L^1-L^2$ reduction, enabling the application of log-Sobolev inequalities.
-Generic method gives further results on many other models conjectured to have cutoff:
  - Ising on the lattice, e.g. with periodic or free boundary.
  - Potts model on the lattice.
  - Gas Hard-core model on lattices.
  - Colorings of lattices.
  - Arbitrary boundary conditions / external field.
  - Anti-ferromagnetic Ising/Potts models, Spin-glass, Other lattices / amenable transitive graphs,...
Theorem [L., Sly]: Critical slowdown verified in $\mathbb{Z}^2$:

Consider the critical Ising model on a finite box $\Lambda \subset \mathbb{Z}^2$ of side-length $n$, i.e. at inverse-temperature $\beta_c = \frac{1}{2} \log (1 + \sqrt{2})$. Let $\text{gap}_{\Lambda}^{\tau}$ denote the spectral-gap in the generator of the corresponding Glauber dynamics under an arbitrary fixed boundary condition $\tau$. Then there exists an absolute $C > 0$ (independent of $\Lambda, \tau$) such that $(\text{gap}_{\Lambda}^{\tau})^{-1} \leq n^C$.

More on this in the next Harvard probability seminar Thursday (Mar 11) 3:10pm, Science Center 232.
Proving Cutoff for Ising: Toy example: cutoff at $\beta = 0$

- No interactions:
  - Stationary distribution is uniform.
  - Spins evolve via independent cont.-time MCs.
- Equivalent to the lazy RW on the hypercube $\{0,1\}^n$.
- [Aldous ’83]: Cutoff at $\frac{1}{2} \log n + O(1)$
  - Constant window
  - Twice faster than trivial upper bound.
Magnetization is a birth-and-death chain:
- By symmetry start at the all-plus state.
- \# of +’s at time $t$ is $\sim \text{Bin}(n, \frac{1}{2}(1+e^{-t}))$.
- \# of +’s under stationary measure $\sim \text{Bin}(n, \frac{1}{2})$ which has Gaussian fluctuations of $O(\sqrt{n})$.
- Mixing occurs when $\frac{1}{2} e^{-t} \approx \sqrt{n}$. 
$L^1$-$L^2$ reduction for product chains

- Setup: general family of ergodic product chains:
  \[ (X_{t(n)}^i) = \{X_{t(n)}^i : i = 1, \ldots, m(n) \} \]
  \[ \lim_{n \to \infty} \left\| \mathbb{P}(X_t^i \in \cdot) - \pi^i \right\|_{L^2(\pi^i)} = 0 \]

- Define:
  \[ M \triangleq \sum_{i=1}^{m} \left\| \mathbb{P}(X_t^i \in \cdot) - \pi^i \right\|_{L^2(\pi^i)}^2 \]

- The following then holds:
  \[ M \to 0 \implies \left\| \mathbb{P}(X_t \in \cdot) - \pi \right\|_{TV} \to 0 \]
  \[ M \to \infty \implies \left\| \mathbb{P}(X_t \in \cdot) - \pi \right\|_{TV} \to 1 \]

- For the hypercube $m = n$ and we want to drop the individual $L^2$ distances ($\asymp e^{-t}$) below $1/\sqrt{n}$. 
$L^1$-$L^2$ reduction for Ising

- **Framework:**
  - $(X_t)$: continuous-time Glauber dynamics for $\mathbb{Z}_n^d$
  - $(X'^*_t)$: continuous-time Glauber dynamics on a smaller lattice: $\mathbb{Z}_r^d$ for $r = 3 \log^3 n$.
  - $B$: smaller box within $\mathbb{Z}_r^d$ of side-length $2 \log^3 n$.

- **Define:**
  
  \[ m_t \triangleq \max_{x_0} \left\| \mathbb{P}_{x_0} (X'^*_t(B) \in \cdot) - \mu_B^* \right\|_{L^2(\mu_B^*)} \]

  measuring the $L^2$ convergence of the projection of $(X'^*_t)$ onto the box $B$. 
$L^1$-$L^2$ reduction for Ising (ctd.)

- **Recall:**
  \[ m_t \triangleq \max_{x_0} \left\| \mathbb{P}_{x_0} (X^*_t(B) \in \cdot) - \mu^*_B \right\|_{L^2(\mu^*_B)} \]

- **Theorem:**
  Let \( s = s(n) \) and \( t = t(n) \) satisfy
  \[
  (10d / \alpha_s^*) \log \log n \leq s < \log^{4/3} n,
  \]
  \[
  (20d / \alpha_s^*) \log \log n \leq t < \log^{4/3} n,
  \]
  where \( \alpha_s^* \) is the infimum over log-Sobolev constants.
  \[
  \frac{n}{\log^5 n} d m_t^2 \to 0 \Rightarrow \limsup_{n \to \infty} \max_{x_0} \left\| \mathbb{P}_{x_0} (X_{t+s} \in \cdot) - \mu \right\|_{TV} = 0
  \]
  \[
  \frac{n}{\log^3 n} d m_t^2 \to \infty \Rightarrow \liminf_{n \to \infty} \max_{x_0} \left\| \mathbb{P}_{x_0} (X_t \in \cdot) - \mu \right\|_{TV} = 1
  \]

- Translates $L^1$ mixing to $L^2$ mixing (to within a finer scale) on projections in smaller boxes.
Existence of cutoff

- Recall that

\[ m_t \triangleq \max_{x_0} \left\| \mathbb{P}_{x_0} (X_t^*(B) \in \cdot) - \mu_B^* \right\|_{L^2(\mu_B^*)} \]

and choose:

\[ t^* \triangleq \inf \left\{ t : m_t^2 \leq \frac{\log^{3d+1} n}{n^d} \right\}. \]

- Log-Sobolev inequalities ensure that \( t^* = O(\log n) \).

- Take \( s = (10d / \alpha_s^*) \log \log n \).
  - \implies (n / \log^5 n)^d m_{t^*}^2 = \log^{1-2d} n = o(1)
  - Theorem implies \( L^1 \)-distance of \( o(1) \) by time \( t^* + s \).

- Since \( t^* \asymp \log n \implies t^* \geq (20d / \alpha_s^*) \log \log n \).
  - \implies (n / \log^3 n)^d m_{t^*}^2 = \log n \to \infty
  - Theorem implies \( L^1 \)-distance of \( 1-o(1) \) at time \( t^* \).
Ideas from the proof: $L^1 - L^2$ reduction & cutoff location

- Additional effort needed to establish cutoff location in terms of $\lambda_\infty$:
  - Express cutoff location in terms of the spectral-gaps on the smaller $\mathbb{Z}^d_r$ and show these converge to $\lambda_\infty$.

- Reduction is enabled by the following:
  - Information spreads at rate 1 while mixing is $O(\log n)$: No time for information to spread...
  - Consider the (random) “update support”: the smallest set of spins whose value at time $t$ is needed in order to determine the state at time $t+s$.
  - Geometric properties of support $\Rightarrow$ product chain.
Support is sparse
Volume decays exponentially
Components separated & small
THANK YOU.