Cutoff for Ising on the lattice

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Course plan

- Lecture 1: crash course on static & stochastic Ising
- Lecture 2: cutoff and two angles on the hypercube
- Lecture 3: reducing $L^1$ to $L^2$ mixing.
- Lecture 4: breaking the dependencies: update supports
- Lecture 5: existence of cutoff and summary.

Bibliography:

1. F. Martinelli, *Lectures on Glauber dynamics for discrete spin models*
   Lectures on probability theory and statistics, Saint-Flour, 1997
2. L. Saloff-Coste, *Lectures on finite Markov chains*
   Lectures on probability theory and statistics, Saint-Flour, 1996
3. D. Levin, Y. Peres & E. Wilmer, *Markov chains and mixing times*
   American Mathematical Society, 2008
Definition: the classical Ising model

- Underlying geometry: $\Lambda = \text{finite 2D grid}$.
- Set of possible configurations:
  $$\Omega = \{\pm 1\}^\Lambda$$
  (each site receives a plus/minus spin)
- Probability of a configuration $\sigma \in \Omega$
given by the Gibbs distribution:

$$\mu(\sigma) = \frac{1}{Z(\beta)} \exp \left( \beta \sum_{x \sim y} \sigma(x)\sigma(y) + h \sum_x \sigma(x) \right)$$

- Partition function
- Inverse temperature $\beta \geq 0$
- External field
The classical Ising model

\[ \mu(\sigma) \propto \exp\left( \beta \sum_{x \sim y} \sigma(x)\sigma(y) \right) \quad \text{for } \sigma \in \Omega = \{\pm 1\}^{\Lambda} \]

- Larger \( \beta \) favors configurations with aligned spins at neighboring sites.
- Spin interactions \( \approx \) local, justified by the rapid decay of magnetic force with distance.

- The magnetization is the (normalized) sum of spins:
  \[ M(\sigma) = |\Lambda|^{-1} \sum_{x \in \Lambda} \sigma(x) \]
  - Distinguishes between disorder \( (M \approx 0) \) and order.
The Ising phase-transition

- Ferromagnetism in this setting: [recall $M(\sigma) = \frac{1}{|\Lambda|} \sum \sigma(x)$]
  - Condition on the boundary sites all having *plus* spins.
  - Let the system size $|\Lambda|$ tend $\to \infty$ ($\approx$ a magnetic field with effect $\to 0$).
- What is the typical $M(\sigma)$ for large $|\Lambda|$? Does the effect of *plus* boundary vanish in the limit?
The Ising phase-transition (ctd.)

- Ferromagnetism in this setting: [recall $M(\sigma) = \frac{1}{|\Lambda|} \sum \sigma(x)$]
  - Condition on the boundary sites all having plus spins.
  - Let the system size $|\Lambda|$ tend $\rightarrow \infty$

- Expect: phase-transition at some critical $\beta_c$:

$$\lim_{|\Lambda| \to \infty} \mathbb{E}^+ [M(\sigma)] = \begin{cases} 0 & \text{if } \beta < \beta_c \\ c_\beta > 0 & \text{if } \beta > \beta_c \end{cases}$$

- **all-plus boundary**
- **spontaneous magnetization**
High temperatures

Def. ([Martinelli & Olivieri ’94]) property $\text{SM}(\Lambda, c, C)$ holds for a set $\Lambda \subset \mathbb{Z}^d$ and $c, C > 0$ iff $\forall \Delta \subset \Lambda$:

$$\sup_{\tau \in \{\pm 1\}^\Lambda} \left\| \mu^\tau_\Lambda \big|_\Delta - \mu^\tau_y \big|_\Delta \right\|_{TV} \leq C e^{-c \text{dist}(y, \Delta)}$$

Strong spatial mixing holds iff $\exists c, C, L > 0$ so that $\text{SM}(Q, c, C)$ holds for all cubes $Q$ of side length $L$.

- Here $\left\| \varphi - \nu \right\|_{TV} = \sup_{A \subset \Omega} \left[ \varphi(A) - \nu(A) \right]$.
- On $\mathbb{Z}^2$ strong spatial mixing holds for all or whenever $h \neq 0$.
- Implies $\mathbb{E}^+ [ M(\sigma) ] \to 0$ (no spontaneous mag)
Low temperatures

- Ingenious combinatorial argument due to [Peierls ‘36].
- Key idea: represent Ising configurations as contours in the dual graph: the edges are dual to disagreeing edges.
Peierls' phase transition argument

- When all boundary spins are +'s the Peierls contours are all closed [marking "islands" containing of −'s ].

- Goal: show that the fraction of sites inside such components is bounded away from $\frac{1}{2}$. 
Peierls’ phase transition argument

- Setting: $\Lambda \subset \mathbb{Z}^2$ is an $n \times n$ box with all-plus boundary.
- Fix a contour $C$ of length $\ell$.
- For each $\sigma$ containing $C$ flip all the spins of $C$ and its interior to arrive at a unique $\sigma'$:

Proof completed by a first moment argument.
Glauber dynamics / Stochastic Ising

- Glauber dynamics for the Ising model (also known as the Stochastic Ising model) introduced in 1963 by Roy J. Glauber (Nobel in Physics 2005).
  - finite ergodic Markov chain on $\Omega = \{\pm 1\}^A$
  - moves between states by flipping a single site.
  - converges to the stationary Ising measure $\mu$.
- Intensively studied over the last 30 years:
  - Natural efficient sampler for the Ising model.
  - Captures its stochastic evolution.
Glauber dynamics for Ising

- One of the most commonly used MC samplers for the Ising distribution $\mu$:

  Heat-bath version given by the generator

  $$\mathcal{L}_\Lambda^T(f)(\sigma) = \sum_{x \in \Lambda} [\mu_{\sigma,x}^T(f) - f(\sigma)]$$

  where $\mu_{\sigma,x}^T(f) = \mu_\Lambda^T(\cdot | \sigma_y, y \neq x)$

- Equivalent description:
  - Update sites via iid Poisson(1) clocks
  - Each update replaces a spin at $u \in V$ by a new spin $\sim \mu$ conditioned on all remaining spins at $V \setminus \{u\}$. 

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Glauber dynamics for Ising

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- The above is the heat-bath version. Other versions of the dynamics include e.g. Metropolis.

- To sample from the Ising model, start at an arbitrary state (e.g. all-plus) run the chain.
  - How long does it take it to converge to $\mu$?
Notions of convergence to equilibrium

- **Spectral gap** in the spectrum of the generator:
  \[\text{gap} = \text{smallest positive eigenvalue of } (-\mathcal{L}_t)\]
  associated with the heat-kernel \(H_t\).

- **Mixing time**: (according to a given metric).
  - Standard choice: \(L^1\) (total-variation) mixing time to within \(\varepsilon\) is defined as
    \[
    t_{\text{mix}}(\varepsilon) = \inf \left\{ t : \max_{\sigma} \left\| H_t(\sigma, \cdot) - \mu \right\|_{TV} \leq \varepsilon \right\}.
    \]
    where
    \[
    \left\| \mu - \nu \right\|_{TV} = \sup_{A \subseteq \Omega} \left[ \mu(A) - \nu(A) \right] = \frac{1}{2} \sum_{x \in \Omega} |\mu(x) - \nu(x)|.
    \]
The gap and mixing time

- Mixing time decays exponentially:
  \[
  \max_{\sigma} \left\| H_t (\sigma, \cdot) - \mu \right\|_{TV} \leq e^{-\frac{t}{t_{mix}} \left( \frac{1}{2e} \right)}
  \]

- Dirichlet form characterization for the spectral gap:
  \[
  \text{gap} = \inf_{f \in L^2 (\mu_\Lambda^T)} \frac{\mathcal{E}_\Lambda^T (f)}{\operatorname{Var}_\Lambda^T (f)}
  \]
  where
  \[
  \mathcal{E}_\Lambda^T (f) (\sigma) = \sum_{x \in \Lambda} \mu_\Lambda^T \left( \operatorname{Var}_{\sigma, x}^T (f) \right)
  \]

- Relating the gap to total-variation mixing:
  \[
  \text{gap}^{-1} \leq t_{mix} \left( \frac{1}{2e} \right) \leq \text{gap}^{-1} \log \left( \frac{2e}{\mu_{\text{min}}} \right)
  \]
Coupling

Well-known tool to bound total-variation distance:

$$\|\mu - \nu\|_{TV} \leq \mathbb{P}(X \neq Y)$$

for \( \forall \) coupling \((X, Y)\) with \( X \sim \mu \), \( Y \sim \nu \), and there \( \exists \) a maximal coupling achieving equality.

Monotone coupling for Glauber dynamics:

- If \( X_t \geq Y_t \) then \( X_{t+1} \geq Y_{t+1} \)
- Consequently:

$$\max_x \| \mathbb{P}_x(X_t \in \cdot) - \mu \|_{TV} \leq \mathbb{P}_{+,-}(X_t \neq Y_t)$$
Example: fast mixing at high temp

- When all $\Delta$ neighbors of a site are plus, probability of minus is
  \[ \frac{1}{2} (1 - \tanh(\beta \Delta)) = \frac{e^{-\beta \Delta}}{e^{\beta \Delta} + e^{-\beta \Delta}} = \frac{1}{2} - \varepsilon \]

- Run monotone coupling: given $k$ disagreements prior to an update:
  - Eliminate one of them with probability $\geq (1 - 2\varepsilon) \frac{k}{n}$
  - Introduce a new one with probability $\leq 2\varepsilon \frac{\Delta k}{n}$

- Contraction for small enough $\varepsilon$
  \[ \Rightarrow t_{\text{mix}} = O(\log n) \text{ in cont-time.} \]
Glauber dynamics for critical Ising

How fast does the dynamics converge?

- 256 x 400 square lattice with boundary conditions: (+) at bottom, (-) elsewhere.
- Frame every \( \sim 2^{30} \) steps, i.e. \( \sim 2^{13} \) updates/site.
General (believed) picture for the Glauber dynamics

- **Setting:** Ising model on the lattice \((\mathbb{Z} / n\mathbb{Z})^d\).
  - Belief: For some critical inverse-temperature \(\beta_c\):
    - **Low temperature:** \(\beta > \beta_c\)
      - \(\text{gap}^{-1}\) and \(t_{\text{mix}}\) are *exponential* in the surface area.
    - **Critical temperature:** \(\beta = \beta_c\)
      - \(\text{gap}^{-1}\) and \(t_{\text{mix}}\) are *polynomial* in the surface area.
        - Exponent of \(\text{gap}^{-1}\) is universal (the *dynamical critical exponent* \(z\)).
    - **High temperature:** \(\beta < \beta_c\)
      - *Rapid mixing:* \(\text{gap}^{-1} = O(1)\) and \(t_{\text{mix}} \asymp \log n\)
      - Mixing occurs abruptly, *i.e.* there is *cutoff*.
The Cutoff Phenomenon

- Describes a sharp transition in the convergence of finite ergodic Markov chains to stationarity.

Steady convergence: it takes a while to reach distance $\frac{1}{2}$ from stationarity, then a while longer to reach distance $\frac{1}{4}$, etc.

Abrupt convergence: distance from equilibrium quickly drops from 1 to 0.
Example: mixing picture for Ising on the complete graph

\[ \text{gap}^{-1}, t_{\text{mix}} \asymp \frac{1}{\beta - 1} \exp \left[ \frac{3}{4} (\beta - 1)^2 n \right] \]

\[ \text{gap}^{-1}, t_{\text{mix}} \asymp n^{1/2} \]

\[ \text{gap}^{-1} = \frac{1 + o(1)}{1 - \beta} \]

\[ t_{\text{mix}} = \frac{1 + o(1)}{2(1 - \beta)} \log[(1 - \beta)^2 n] \]

(Scaling window established in [Ding, L., Peres '09])

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Critical slowdown

- Intuition: low temperature
  - Exponential mixing due to a bottleneck between the “mostly-plus” and the “mostly-minus” states

- Intuition: high temperature
  - At $\beta = 0$ there is complete independence.
  - For very small $\beta > 0$ a spin is likely to choose the same update given 2 very different neighborhoods (weak “communication” between sites).
  - States can be coupled quickly, hence rapid mixing.

- Intuition: critical power-law:
  - Doubling the box incurs a constant factor in mixing...
Mixing for Ising on the 2D lattice

- Fast mixing at high temperatures:
  - [Aizenman, Holley ’84]
  - [Dobrushin, Shlosman ’87]
  - [Holley, Stroock ’87, ’89]
  - [Holley ’91]
  - [Stroock, Zegarlinski ’92a, ’92b, ’92c]
  - [Zegarlinski ’90, ’92]
  - [Lu, Yau ’93]
  - [Martinelli, Olivieri ’94a, ’94b]
  - [Martinelli, Olivieri, Schonmann ’94]

- Slow mixing at low temperatures:
  - [Schonmann ’87]
  - [Chayes, Chayes, Schonmann ’87]
  - [Martinelli ’94]
  - [Cesi, Guadagni, Martinelli, Schonmann ’96].
The gap and log-Sobolev const

- Recall: \( \text{gap} = \inf_{f \in L^2(\mu^\Lambda)} \frac{\mathcal{E}_\Lambda^T(f)}{\text{Var}_\Lambda^T(f)} \), where

  \[
  \mathcal{E}_\Lambda^T(f)(\sigma) = \sum_{x \in \Lambda} \mu_\Lambda^T(\text{Var}_{\sigma,x}^T(f)).
  \]

- The **log-Sobolev constant** is given by

  \[
  \alpha_s = \inf_{f \in L^2(\mu^\Lambda)} \frac{\mathcal{E}_\Lambda^T(f)}{\text{Ent}_\Lambda^T(f)}
  \]

  where

  \[
  \text{Ent}_\Lambda^T(f) = \mathbb{E}_{\mu^\Lambda} \left[ f^2(\sigma) \log \left( \frac{f^2(\sigma)}{\mathbb{E}_{\mu^\Lambda} f^2(\sigma)} \right) \right].
  \]

- Relating the gap to \( L^2 \) mixing:

  \[
  \left\| P_x(X_s \in \cdot) - \nu \right\|_{L^2(\nu)} \leq \exp \left[ 1 - \text{gap} \left( s - \frac{1}{4 \alpha_s} \log + \log \frac{1}{\nu(x)} \right) \right]
  \]
Mixing on the square lattice

- High temperature regime: if there is strong spatial mixing (on $\mathbb{Z}^2$ this covers $\forall \beta < \beta_c$): 
  - $O(1)$ inverse spectral-gap constant.
  - $O(1)$ inverse log-Sobolev constant and as a result $O(\log n)$ total-variation mixing.
- Dynamics conjectured to exhibit cutoff [Peres’04].
Cutoff for the Glauber dynamics

- Till recently: *only* spin-systems where cutoff was verified are Ising and Potts models on the complete graph [Levin, Luczak, Peres ’10], [Ding, L., Peres ’09], [Cuff, Ding, L., Louidor, Peres, Sly]
- Conjectured to believe at high temperatures for:
  - Ising on the lattice, e.g. with periodic or free boundary.
  - Potts model on the lattice.
  - Gas Hard-core model on lattices.
  - Colorings of lattices.
  - Arbitrary boundary conditions / external field.
  - Anti-ferromagnetic Ising/Potts models, Spin-glass, Other lattices / amenable transitive graphs,...
Cutoff for Ising on lattices

**Theorem [L., Sly]:**

Let $\beta_c = \frac{1}{2} \log(1+\sqrt{2})$ be the critical inverse-temperature for the Ising model on $\mathbb{Z}^2$. Then the continuous-time Glauber dynamics for the Ising model on $(\mathbb{Z}/n\mathbb{Z})^2$ with periodic boundary conditions at $0 \leq \beta < \beta_c$ has cutoff at $(1/\lambda_\infty) \log n$ where $\lambda_\infty$ is the spectral gap of the dynamics on the infinite volume lattice.

Analogous result holds for any dimension $d \geq 1$:
- Cutoff at $(d/2\lambda_\infty) \log n$.
- *E.g.*, cutoff at $[2(1-\tanh(2\beta))]^{-1} \log n$ for $d = 1$. 
Cutoff for Ising on the lattice

- Main result hinges on an $L^1$-$L^2$ reduction, enabling the application of log-Sobolev inequalities.
- Generic method gives further results on many other models conjectured to have cutoff:
  - Ising on the lattice, e.g. with periodic or free boundary.
  - Potts model on the lattice.
  - Gas Hard-core model on lattices.
  - Colorings of lattices.
  - Arbitrary boundary conditions / external field.
  - Anti-ferromagnetic Ising/Potts models, Spin-glass, Other lattices / amenable transitive graphs, ...
Key tool: breaking dependencies...