Cutoff for Ising on the lattice

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Recap: product chains $L^1 \to L^2$ reduction

**Proposition:**

Let $X_t = (X_t^1, \ldots, X_t^n)$ be a product chain where each $X_t^i$ is ergodic with stationary measures $\pi_i$ and $\pi = \prod_i \pi_i$. Let

$$
M_t = \sum_{i=1}^n m_t \quad \text{where} \quad m_t = \left\| \mathbb{P}(X_t^i \in \cdot) - \pi_i \right\|^2_{L^2(\pi_i)}.
$$

For $\forall \delta > 0$ there $\exists \varepsilon > 0$ so that if for some $t > 0$

$$
\max_i \left\| \mathbb{P}(X_t^i \in \cdot) - \pi_i \right\|_{L^\infty(\pi_i)} < \varepsilon
$$

then

$$
\left| \left\| \mathbb{P}(X_t \in \cdot) - \pi \right\|_{TV} - (2\Phi\left(\frac{1}{2}\sqrt{M_t}\right) - 1) \right| < \delta.
$$
**Corollary:**

Let $X_t$ be a product chain made of $n$ i.i.d. copies of a finite ergodic chain $Y_t$ with spectral-gap and log-Sobolev constant $\alpha_s$ resp. and stationary measure $\varphi$. If

$$\log \varphi_{\min}^{-1} \leq n^{o(\alpha_s/\text{gap})}$$

then $X_t$ exhibits cutoff at $\frac{1}{2} \text{gap}^{-1} \log n$ with window of order $O(\alpha_s^{-1} \log_+ \log \varphi_{\min}^{-1})$. 
Intuition: cutoff on the lattice

- Break up $\mathbb{Z}^d_n$ to cubes of side-length $\log^3 n$. Dynamics on such a cube:
  - $\alpha_s^{-1} = O(1)$
  - $\log \varphi_{\min}^{-1}(\sigma) = O(\log^{3d} n) = n^{o(1)}$
- Take non-adjacent cubes $Q_1, \ldots, Q_m$ ($m = (n/\log^3 n)^d$) and suppose as if the projection on those would predict mixing for the entire system:
  - Distance between cubes turn them $\approx$ independent.
  - Expect cutoff at $\frac{1}{2\text{gap}} \log m = \frac{1}{2\text{gap}} \log n + O(\log \log n)$ with window $O(\log \log n)$. 
Making this rigorous: sparse sets

**Definition:**

The set $\Lambda \subseteq V$ is **sparse** iff it can be partitioned into (not necessarily connected) components $\{A_i\}$ so that

1. $\text{diam}(A_i) = O(\log^3 n)$
2. $\text{dist}(A_i, A_j) \geq \log^2 n$

Let $\mathcal{S} = \{\Lambda \subseteq V : \Lambda \text{ is sparse}\}$.

**Motivation:**

- Small diameter $\Rightarrow$ can embed each component in a small box.
- Super logarithmic distances between components $\Rightarrow$ essentially independent.
Upper bound via sparse sets

**Theorem:**

Let $t > 0$ and $\frac{10d}{\hat{a}_s} \log \log n \leq s \leq \log^{4/3} n$. Then there exists a measure $\nu$ on the sparse sets $S$ such that $\nu(\{\Delta: u \in \Delta\}) < \log^{-5d} n \ \forall u$ and

$$\left\| \mathbb{P}_{\sigma_0}(X_{t+s} \in \cdot) - \mu \right\|_{TV} \leq \int_S \left\| \mathbb{P}_{\sigma_0}(X_t(\Delta) \in \cdot) - \mu|_\Delta \right\|_{TV} \ dv(\Delta) + O(n^{-10d})$$

**Assuming theorem, from here we can:**

- Box each component $A_i$ (extended a bit) inside $B_i$ then extend to a larger box.
- Couple dynamics to a product chain agreeing on the projections on $\cup B_i$
**L^1-L^2 reduction for Ising**

- **Framework:**
  - \((X_t)\): Glauber dynamics for \(\mathbb{Z}^d_n\)
  - \((X_t^*)\): Glauber dynamics on \(\mathbb{Z}^d_r\) for \(r = 3 \log^3 n\).
  - \(B\): smaller cube within \(\mathbb{Z}^d_r\) of side-length \(2\log^3 n\).

- **Define:**
  \[
  m_t \triangleq \max_{x_0} \left\| \mathbb{P}_{x_0} (X_t^*(B) \in \cdot) - \mu^*_B \right\|_{L^2(\mu^*_B)}^2
  \]

  (measure \(L^2\) convergence of the projection \((X_t^*) \hookrightarrow B\).)

- There are \(m = (n / \log^3 n)^d\) such disjoint cubes in \(\mathbb{Z}^d_n\), so as a lower bound take the proposition with

  \[
  M_t \triangleq (n / \log^3 n)^d m_t
  \]
$L^1 - L^2$ reduction for Ising (ctd.)

Recall:
\[
\mathfrak{m}_t \triangleq \max_{x_0} \left\| \mathbb{P}_{x_0} (X_t^*(B) \in \cdot) - \mu^* \right\|_{L^2(\mu^*_B)}^2
\]

Theorem:

Suppose
\[
\begin{align*}
10d \hat{\alpha}_s^{-1} \log \log n &\leq s < \log^{4/3} n \\
20d \hat{\alpha}_s^{-1} \log \log n &\leq t < \log^{4/3} n
\end{align*}
\]

where $\hat{\alpha}_s$ is the infimum over log-Sobolev constants. Then
\[
\begin{align*}
(n/\log^5 n)^d \mathfrak{m}_t &\to 0 \quad \Rightarrow \quad \limsup_{n \to \infty} \max_{x_0} \left\| \mathbb{P}_{x_0} (X_{t+s} \in \cdot) - \mu \right\|_{TV} = 0 \\
(n/\log^3 n)^d \mathfrak{m}_t &\to \infty \quad \Rightarrow \quad \liminf_{n \to \infty} \max_{x_0} \left\| \mathbb{P}_{x_0} (X_t \in \cdot) - \mu \right\|_{TV} = 1
\end{align*}
\]
Existence of cutoff

- Recall: \( m_t \triangleq \max_{x_0} \left\| P_{x_0}^t (X_t^* (B) \in \cdot) - \mu^*_B \right\|^2_{L^2 (\mu^*_B)} \)

\[
\begin{aligned}
&t^* \triangleq \inf \left\{ t : m_t \leq n^{-d} \log^{3d+1} n \right\}, \\
&s \triangleq 10d \hat{x}^{-1} \log \log n.
\end{aligned}
\]

- By def.: \[
\begin{aligned}
(n/ \log^3 n)^d m_{t^*} &= \log n \rightarrow \infty \\
(n/ \log^5 n)^d m_{t^*} &= \log^{1-2d} n \rightarrow 0
\end{aligned}
\]

- Remains to check range of \( t^* \):
  - Due to log-Sobolev inequalities \( t^* \asymp \log n \)

- By Theorem: entire mixing occurs at interval \((t^*, t^* + s)\)
  \(\Rightarrow\) cutoff at time \( t^* \) with window \( \leq s \).
Sparse sets upper bound

**Definition:**
The set $\Lambda \subset V$ is *sparse* (\(\Lambda \in S\)) if it can be partitioned into (not necessarily connected) components \(\{A_i\}\) so that

1. \(\text{diam}(A_i) \leq \frac{1}{2}\log^3 n\)
2. \(\text{dist}(A_i, A_j) \geq \log^2 n\)

**Theorem:**
Let \(t > 0\) and \(\frac{10d}{\bar{\alpha}_s} \log \log n \leq s \leq \log^{4/3} n\). Then there exists a measure \(\nu\) on the sparse sets \(S\) such that \(\nu(\{\Delta: u \in \Delta\}) < \log^{-5d} n\) \(\forall u\) and

\[
\left\| \mathbb{P}_{\sigma_0} (X_{t+s} \in \cdot) - \mu \right\|_{TV} \leq \int_S \left\| \mathbb{P}_{\sigma_0} (X_t(\Delta) \in \cdot) - \mu|_{\Delta} \right\|_{TV} d\nu(\Delta) + O(n^{-10d})
\]
Barrier dynamics

- Random map $G_s: \Omega \rightarrow \Omega$ (where $\Omega = \{\pm 1\}^V$) coupled to the Glauber dynamics.

**Definition**

For $s > 0$ define $G_s(X_0)$ as follows:

- Surround $\forall u \in V$ by $B_u(\log^{3/2} n)$, a ball of radius $\log^{3/2} n$ by graph metric.
- Impose periodic boundary ("barrier") on each ball.
- Run standard dynamics $(X_t)$ till time $s$ and use same site-choices and unit-variables for updates.
- Output: the spins at centers of $\{B_u(\log^{3/2} n) : u \in V\}$
The barrier dynamics map \( G_s \) can be coupled to the original Glauber dynamics \( X_t \) such that
\[
P\left( X_s = G_s(X_0) \quad \forall s \in [0, \log^{4/3} n] \right) \geq 1 - n^{-10d}.
\]

**Proof:**
- Use implicit coupling defining the barrier dynamics.
- Disagreement at \( u \Rightarrow \) sequence of updates at times \( t_1 < \cdots < t_\ell < \log^{4/3} n \) connects \( u \leftrightarrow \partial B_u(\log^{3/2} n) \):
  \[
P\left( \bigcup_{u,t} \left\{ X_t(u) \neq \tilde{X}_t(u) \right\} \right) \leq n^d \sum_{\ell \geq \log^{3/2} n} (2d)^\ell P(\text{Poisson}(\log^{4/3} n) \geq \ell)
  \leq C n^d e^{-c \log^{3/2} n} < n^{-10d}.
\]
Update support

- Update sequence for the barrier dynamics map $G_s$ in interval $[0, s]$:  
  - Seq. of triplets $(t_i, x_i, u_i)$
  - Given this: $G_s = g_{W_s}$ det. monotone.

**Definition:**

Let $W_s = \text{update seq. for barrier dynamics map } G_s$. The **support** of $W_s$ is the minimum subset $\Delta_{W_s} \subseteq V$ s.t. $g_{W_s}(\sigma_0)$ is determined by $\sigma_0(\Delta_{W_s})$ for $\forall \sigma_0$.

- Equiv.: $x \in \Delta_{W_s}$ if $\exists \sigma_0$ such that $g_{W_s}(\sigma_0) \neq g_{W_s}(\sigma_0^x)$. 

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Upper bound via update support

Lemma:
Let $W_s$ = random update seq. of the barrier dynamics map in the interval $(0, s)$ for some $s \leq \log^{4/3} n$. Then $\forall \sigma_0 \forall t > 0$

$$\left\| \mathbb{P}_{\sigma_0} (X_{t+s} \in \cdot) - \mu \right\|_{TV} \leq \int \left\| \mathbb{P}_{\sigma_0} (X_t (\Delta W_s) \in \cdot) - \mu|_{\Delta W_s} \right\|_{TV} d\mathbb{P}(W_s) + O(n^{-10d})$$

Proof:
- Couple dynamics to two instances of the barrier dynamics run for time $s$.
- Reduce to an integral over $L^1$ distances between the deterministic barrier-dynamics functions.
- Projection can only decrease $L^1$ distance.
Update support is sparse

- Most supports are sparse:
  - Volume decays exponentially
  - Components separated and small
- As time traverses, the effect of more and more sites becomes 0 (information flow stops at barriers of barrier dynamics).
Random support of update seq.
Lemma:

Let $W_s$ be the random update sequence of the barrier dynamics in the interval $(0, s)$ for some $s \geq \frac{10d}{\alpha_s} \log \log n$.

Then

\[ \mathbb{P}(\Delta_{W_s} \in S) \geq 1 - O(n^{-10d}) \]

and

\[ \mathbb{P}(u \in \Delta_{W_s}) \leq \log^{-5d} n \quad \forall u. \]

Proof:

- Estimate the probability that a full copy $B_u(\log^{3/2} n)$ of the barrier-dynamics is "trivial" (coupling).
- No long $(\varepsilon \log n)$ path of nontrivial balls by a first moment argument.
Upper bound via sparse sets

We showed:

\( \forall s \geq \frac{10d}{\alpha_s} \log \log n \ \forall W_s : \)

\[
P(\Delta_{W_s} \in S) \geq 1 - O(n^{-10d})
\]

\[
P(u \in \Delta_{W_s}) \leq \log^{-5d} n \ \forall u
\]

\( \forall s \leq \log^{43} n \ \forall t \ \forall \sigma_0 : \)

\[
\left\| P_{\sigma_0}(X_{t+s} \in \cdot) - \mu_{TV} \right\| \leq \int \left\| P_{\sigma_0}(X_t(\Delta_{W_s}) \in \cdot) - \mu_{\Delta_{W_s}} \right\|_{TV} \ dP(W_s) + O(n^{-10d})
\]

**Corollary:**

Let \( t > 0 \) and \( \frac{10d}{\tilde{\alpha}_s} \log \log n \leq s \leq \log^{4/3} n \). Then \( \exists \) measure \( \nu \) on the sparse sets \( S \) s.t. \( \nu(\{\Delta: u \in \Delta\}) < \log^{-5d} n \ \forall u \) and

\[
\left\| P_{\sigma_0}(X_{t+s} \in \cdot) - \mu \right\|_{TV} \leq \int \left\| P_{\sigma_0}(X_t(\Delta) \in \cdot) - \mu_{\Delta} \right\|_{TV} \ d\nu(\Delta) + O(n^{-10d})
\]
The projection onto a sparse set

**Lemma:**

Let \( \Delta \in S \) be a sparse set and \( A_1, \ldots, A_{N_\Delta} \) be its component partition. Then for \( \forall \sigma_0 \) and \( t \leq t_0 \),

\[
\left\| \mathbb{P}_{\sigma_0} (X_t(\Delta) \in \cdot) - \mu_{|\Delta} \right\|_{TV} \leq \left\| \mathbb{P}_{\sigma_0} (\bar{X}_t^* (\cup B_i) \in \cdot) - \mu^*_{|\cup B_i} \right\|_{TV} + O(n^{-10d})
\]

where \((\bar{X}_t^*)\) is the product chain on \( N_\Delta \) i.i.d. cubes \( B_i^+ \)

**Proof:**

- Couple \( X_t(\Delta) \) to \( \bar{X}_t^*(\Delta) \) via \( A_i^+ = B_{A_i}(\log^{3/2} n) \) to agree throughout \( t \in [0, \log^{4/3} n] \).
- Inspect \( \bar{X}_t^* (\Delta) \) started from equilibrium at time \( t_0 = \log^{4/3} n \) to couple stationary measures.
- Decrease projection from \( \Delta \) to \( UB_i \) to conclude proof.
Concluding the upper bound

- So far we showed:

Let \( t \leq \log^{4/3} n \) and \( \frac{10d}{\tilde{\alpha}_s} \log \log n \leq s \leq \log^{4/3} n \). Then there exists a measure \( \nu \) on \( S \) such that \( \nu(\{\Delta: u \in \Delta\}) < \log^{-5d} n \) \( \forall u \) and

\[
\nu \left( \{ \frac{P_{\sigma_0}(X_{t+s} \in \cdot) - \mu}{\left\| P_{\sigma_0}(X_{t+s} \in \cdot) - \mu\right\|_{TV}} \leq \int \nu(\Delta) \in \cdot \right) - \mu^*_{UB_i} \right\|_{TV} d\nu(\Delta) + O(n^{-10d})
\]

- For \( \Delta \in S \) with \( N_\Delta \) compact, apply Product Proposition:

\[
\max_{\sigma_0} \left\| P_{\sigma_0}(\bar{X}_t^{\ast}(UB_i) \in \cdot) - \mu^*_{UB_i} \right\|_{TV} \leq \sqrt{M_t}
\]

where \( M_t = N_\Delta m_t \) and \( m_t = \left\| P_{\sigma_0}(X_t^*(B) \in \cdot) - \mu^*_B \right\|_{L^2(\mu^*_B)}^2 \)

- Integrate to get:

\[
\max_{\sigma_0} \left\| P_{\sigma_0}(X_t \in \cdot) - \mu \right\|_{TV} \leq \left( \frac{n}{\log^5 n} d m_t \right)^{1/2} + O(n^{-10d})
\]