Cutoff for Ising on the lattice

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Upper bound via sparse sets

- We showed:
  \[ \forall s \geq \frac{10d}{\alpha_s} \log \log n \ \forall W_s : \]
  \[ \mathbb{P}(\Delta_{W_s} \in S) \geq 1 - O(n^{-10d}) \]
  \[ \mathbb{P}(u \in \Delta_{W_s}) \leq \log^{-5d} n \ \forall u \]

- \[ \forall s \leq \log^{43} n \ \forall t \ \forall \sigma_0 : \]
  \[ \left\| \mathbb{P}_{\sigma_0} (X_{t+s} \in \cdot) - \mu \right\|_{TV} \leq \int \left\| \mathbb{P}_{\sigma_0} (X_t(\Delta_{W_s}) \in \cdot) - \mu|_{\Delta_{W_s}} \right\|_{TV} d\mathbb{P}(W_s) + O(n^{-10d}) \]

- **Corollary:**

Let \( t > 0 \) and \( \frac{10d}{\hat{\alpha}_s} \log \log n \leq s \leq \log^{4/3} n \). Then \( \exists \) measure \( \nu \) on the sparse sets \( S \) s.t. \( \nu(\{\Delta : u \in \Delta\}) < \log^{-5d} n \ \forall u \) and

\[ \left\| \mathbb{P}_{\sigma_0} (X_{t+s} \in \cdot) - \mu \right\|_{TV} \leq \int_S \left\| \mathbb{P}_{\sigma_0} (X_t(\Delta) \in \cdot) - \mu|_{\Delta} \right\|_{TV} d\nu(\Delta) + O(n^{-10d}) \]
The projection onto a sparse set

**Lemma:**

Let $\Delta \in S$ be a sparse set and $A_1, \ldots, A_{N_\Delta}$ be its component partition. Then for $\forall \sigma_0$ and $t \leq t_0$,

$$\left\| \mathbb{P}_{\sigma_0} (X_t(\Delta) \in \cdot) - \mu_{\Delta} \right\|_{TV} \leq \left\| \mathbb{P}_{\sigma_0} (\bar{X}_t^*(\cup B_i) \in \cdot) - \mu_{\cup B_i}^* \right\|_{TV} + O(n^{-10d})$$

where $(\bar{X}_t^*)$ is the product chain on $N_\Delta$ i.i.d. cubes $B_i^+$

**Proof:**

- Couple $X_t(\Delta)$ to $\bar{X}_t^*(\Delta)$ via $A_i^+ = B_{A_i}(\log^{3/2} n)$ to agree throughout $t \in [0, \log^{4/3} n]$.
- Inspect $\bar{X}_t^*(\Delta)$ started from equilibrium at time $t_0 = \log^{4/3} n$ to couple stationary measures.
- Decrease projection from $\Delta$ to $\cup B_i$ to conclude proof.
Concluding the upper bound

So far we showed:

Let \( t \leq \log^{4/3} n \) and \( \frac{10d}{\alpha_s} \log \log n \leq s \leq \log^{4/3} n \). Then

\[ \exists \text{ measure } \nu \text{ on } S \text{ s.t. } \nu(\{\Delta: u \in \Delta\}) < \log^{-5d} n \ \forall u \ \text{ and} \]

\[ \left\| \mathbb{P}_{\sigma_0}(X_{t+s} \in \cdot) - \mu \right\|_{TV} \leq \int \left\| \mathbb{P}_{\sigma_0}(\bar{X}_t^* (UB_i) \in \cdot) - \mu^*|_{UB_i} \right\|_{TV} d\nu(\Delta) + O(n^{-10d}) \]

For \( \Delta \in S \) with \( N_\Delta \) comp. apply Product Proposition:

\[ \max_{\sigma_0} \left\| \mathbb{P}_{\sigma_0}(\bar{X}_t^* (UB_i) \in \cdot) - \mu^*|_{UB_i} \right\|_{TV} \leq \sqrt{M_t} \]

where \( M_t = N_\Delta m_t \) and \( m_t = \left\| \mathbb{P}_{\sigma_0}(X_t^* (B) \in \cdot) - \mu^*|_B \right\|_{L^2(\mu^*|_B)}^2 \)

Integrate to get:

\[ \max_{\sigma_0} \left\| \mathbb{P}_{\sigma_0}(X_{t+s} \in \cdot) - \mu \right\|_{TV} \leq \left( \frac{n}{\log^5 n} m_t \right)^{1/2} + O(n^{-10d}) \]
Existence of cutoff

- Framework:
  - \((X_t)\) : Glauber dynamics for \(\mathbb{Z}_n^d\)
  - \((X_t^*)\) : Glauber dynamics on \(\mathbb{Z}_r^d\) for \(r = 3 \log^3 n\).
  - \(B\) : smaller cube within \(\mathbb{Z}_r^d\) of side-length \(2 \log^3 n\).

- For \(t \leq \log^{4/3} n\) and \(10d\hat{a}_s^{-1} \log \log n \leq s \leq \log^{4/3} n\):
  \[
  \max_{\sigma_0} \left\| \mathbb{P}_{\sigma_0} (X_{t+s} \in \cdot) - \mu \right\|_{TV} \leq \left( \frac{n}{\log^5 n} \right)^d m_t \right)^{1/2} + O(n^{-10d})
  \]

- Matching lower bound: take order \((n/\log^3 n)^d\) such cubes (well-spaced) to get that for \(t > 20d\hat{a}_s^{-1} \log \log n\):
  \[
  \max_{\sigma_0} \left\| \mathbb{P}_{\sigma_0} (X_t \in \cdot) - \mu \right\|_{TV} \geq f \left( (n/\log^3 n)^d m_t \right) \quad \text{where} \quad \lim_{x \to \infty} f(x) = 1
  \]
  \[-O(n^{-10d})\]
Existence of cutoff

- Recall: \( m_t \triangleq \max_{x_0} \left\| \mathbb{P}_{x_0} (X_t^* (B) \in \cdot) - \mu_t^* \right\|_{L^2 (\mu_t^* | B)}^2 \)

and choose:
\[
\begin{align*}
t^* & \triangleq \inf \left\{ t : m_t \leq n^{-d} \log^{3d+1} n \right\}, \\
\sigma & \triangleq 10d \hat{\alpha}_s^{-1} \log \log n.
\end{align*}
\]
(by log-Sobolev inequalities \( t^* \asymp \log n \)).

- By def.:
\[
\begin{align*}
\frac{(n/ \log^3 n)^d}{m_t^*} = \log n & \to \infty \\
\frac{(n/ \log^5 n)^d}{m_t^*} = \log^{1-2d} n & \to 0
\end{align*}
\]

- Conclude:
\[
\begin{align*}
\max_{\sigma_0} \left\| \mathbb{P}_{\sigma_0} (X_{t^*+s} \in \cdot) - \mu \right\|_{TV} & \leq \log^{1-d} n + O(n^{-10d}) \to 0 \\
\max_{\sigma_0} \left\| \mathbb{P}_{\sigma_0} (X_{t^*} \in \cdot) - \mu \right\|_{TV} & \geq f(\log n) - O(n^{-10d}) \to 1
\end{align*}
\]

- Entire mixing occurs at interval \((t^*, t^* + s)\), i.e. cutoff at time \(t^*\) with window \(s = O(\log \log n)\).
Establishing cutoff location

Recall: \[ m_t \triangleq \max_{x_0} \left\| \mathbb{P}_{x_0} (X^*_t(B) \in \cdot) - \mu^*_B \right\|_{L^2(\mu^*_B)}^2 \]
\[ t^* \triangleq \inf \left\{ t : m_t \leq n^{-d} \log^{3d+1} n \right\}. \]

**Lemma:**

Set \( c_0 = \frac{12d}{\lambda_s} \), \( \frac{10d}{\lambda_s} \log \log n \leq t \leq \log^{4/3} n \) and \( r = 3 \log^3 n \).

Then: \[ e^{-\lambda(r)t - c_0 \log \log n} - O(n^{-10d}) \leq m_t \leq e^{-\lambda(r)t + c_0 \log \log n} \]

**Corollary:** Cutoff in terms of local spectral gap:

\[ t^* = \frac{d}{2\lambda(r)} \log n \quad \text{and window} \quad \leq \frac{40d}{\lambda_s \lambda} \log \log n \]
With some extra work

- The limit as $n \to \infty$ of $\lambda(n)$, spectral gap of Ising on $\mathbb{Z}^d_n$ exists and equals $\lambda_\infty$, the gap of infinite-vol. dynamics.
  - Convergence rate obtain from proof:
    $$|\lambda(n) - \lambda_\infty| < n^{-1/2+o(1)}$$

- Other boundary conditions e.g. all-plus:
  - Analyze locations of cubes in the system how many are adjacent to the outer boundary
  - Obtain an expression in terms of mixed eigenvalues, e.g. for $\mathbb{Z}^2_n$ with $+$ b.c. $\exists$ cutoff at
    $$t^* = (\lambda_\infty \wedge 2\lambda_{\text{H}})^{-1} \log n$$
General (believed) picture for the Glauber dynamics

- **Setting:** Ising model on the lattice \((\mathbb{Z}/n\mathbb{Z})^d\).
  - **Belief:** For some critical inverse-temperature \(\beta_c\):
    - **Low temperature:** \((\beta > \beta_c)\)
      - \(\text{gap}^{-1}\) and \(t_{\text{mix}}\) are exponential in the surface area.
    - **Critical temperature:** \((\beta = \beta_c)\)
      - \(\text{gap}^{-1}\) and \(t_{\text{mix}}\) are polynomial in the surface area.
        - Exponent of \(\text{gap}^{-1}\) is universal (the dynamical critical exponent \(z\)).
    - **High temperature:** \((\beta < \beta_c)\)
      - **Rapid mixing:** \(\text{gap}^{-1} = O(1)\) and \(t_{\text{mix}} \asymp \log n\)
      - Mixing occurs abruptly, i.e. there is cutoff.
Mixing on the square lattice

- High & low temperature regimes fully settled:
  \[ \beta < \beta_c \]  
  \[ \text{gap}^{-1} = O(1) \]  
  \[ t_{\text{mix}} = O(\log n) \]
  \[ \beta > \beta_c \]  
  \[ \text{gap}^{-1} = \exp(c(\beta) + o(1))n \]  
  \[ t_{\text{mix}} = \exp(c(\beta) + o(1))n \]

- Power law at the critical \[ \beta = \beta_c = \frac{1}{2} \log(1 + \sqrt{2}) \]
  - Numerical experiments: universal exponent of \( \sim 2.17 \)  
    [Ito ‘93], [Wang, Hatano, Suzuki ‘95], [Grassberger ‘95],  
    [Nightingale, Blöte ‘96], [Wang, Hu ‘97], ...
  - No previous sub-exponential upper bounds ...
  - Only geometries with analogous critical power-law:  
    Mean-field [Ding, L., Peres ‘09], Regular tree [Ding, L., Peres ‘10]
Recent result

**Theorem:** ([L.-Sly])

Consider Glauber dynamics for the critical Ising model on a finite $n \times n$ box $\Lambda \subset \mathbb{Z}^2$ with an arbitrary boundary condition $\tau$. There exists an absolute $C > 0$ (independent of $\Lambda, \tau$) so that the mixing of the dynamics is at most $n^C$.

**Corollary:**

Perfect simulation (zero error approximation) for the 2D critical Ising model with arbitrary boundary conditions.

Best previous samplers gave approximated sample for specific homogenous boundaries [JS’93], [RW’96].
Another recent result

- Mixing believed to be polynomial...
  - [Martinelli ’94]: mixing $\leq \exp(n^{1/2+o(1)})$ for $n \times n$ box in $\mathbb{Z}^2$ and large enough $\beta$.
  - Breakthrough by [Martinelli, Toninelli ’10]: $t_{\text{mix}} \leq \exp(n^\epsilon)$ for any $\epsilon > 0$ and large enough $\beta$.

- More recently: **Quasi-polynomial down to $\beta_c$**

**THEOREM** ([L., Martinelli, Toninelli, Sly]):

Consider Glauber dynamics for the Ising model on an $n \times n$ box $\Lambda \subset \mathbb{Z}^2$ with all-plus boundary. For any $\beta > \beta_c$ there exists $C(\beta) > 0$ so that the mixing time is at most $n^{C \log n}$. 
Glauber dynamics on $\mathbb{Z}^2$

- Present state-of-the-art bounds on mixing:

\[
\beta < \beta_c \\
\text{gap}^{-1} = O(1) \\
t_{\text{mix}} = \frac{1}{2} \lambda^{-1}_\infty \log n + O(\log \log n) \\
\text{Cutoff}
\]

\[
\beta_c \\
n^{7/4} \leq \text{gap}^{-1} \leq t_{\text{mix}} \leq n^c
\]

\[
\beta > \beta_c \\
\text{Free b.c.:} \\
gap^{-1} = \exp(c(\beta) + o(1))n \\
t_{\text{mix}} = \exp(c(\beta) + o(1))n
\]

\[
\text{Plus b.c.:} \\
gap^{-1} \leq t_{\text{mix}} \leq n^{O(\log n)}
\]
Open problems

- Ising model on the 2D lattice…
  - Calculate the precise *dynamical critical exponent*.
  - Show the Glauber dynamics is *polynomial* at low temperatures under all-plus b.c.

- Ising model on the 3D lattice…
  - Establish strong spatial mixing throughout the high temperature regime (zero external field).
  - Establish power-law behavior at criticality and sub-exponential low-temp mixing under all-plus b.c.
Thank you