The Cutoff Phenomenon

- Describes a sharp transition in the convergence of finite ergodic Markov chains to stationarity.

Steady convergence:
- It takes a while to reach distance $\frac{1}{2}$ from stationarity then a while longer to reach distance $\frac{1}{4}$, etc.

Abrupt convergence:
- Distance from equilibrium quickly drops from 1 to 0.
Cutoff: formal definition

A family of chains \((X^n_t)\) is said to have cutoff if:

\[
\lim_{n \to \infty} \frac{t_{\text{mix}}(\varepsilon)}{t_{\text{mix}}(1 - \varepsilon)} = 1 \quad \forall \ 0 < \varepsilon < 1.
\]

i.e., \(t_{\text{mix}}(\alpha) = (1+o(1))t_{\text{mix}}(\beta)\) for any \(0 < \alpha, \beta < 1\).

A sequence \((w_n)\) is called a cutoff window if

\[
\begin{align*}
w_n &= o(t_{\text{mix}}(\frac{1}{4})) , \\
(t_{\text{mix}}(\varepsilon) - t_{\text{mix}}(1 - \varepsilon)) &= O_{\varepsilon}(w_n) \quad \forall \ 0 < \varepsilon < 1. 
\end{align*}
\]
Basic examples

Lazy discrete-time simple random walk

On the hypercube $\{0,1\}^n$:
- Exhibits cutoff at $\frac{1}{2} n \log n + O(n)$ [Aldous ’83]

On the $n$-cycle:
- No cutoff.
The importance of cutoff

- Suppose we run Glauber dynamics for the Ising Model satisfying $t_{\text{mix}} \asymp f(n)$ for some $f(n)$.
- Cutoff $\iff \exists$ some $c_0 > 0$ so that:
  - Must run the chain for at least $\sim c_0 \cdot f(n)$ steps to even reach distance $(1 - \varepsilon)$ from $\mu$.
  - Running it any longer than that is essentially redundant.
- Proofs usually require (and thus provide) a deep understanding of the chain (its reasons for mixing).
- Many natural chains are believed to have cutoff, yet proving cutoff can be extremely challenging.
Cutoff History

- Random walks on graphs and groups:
  - Discovered:
    - Random transpositions on $S_n$ [Diaconis, Shahshahani ‘81]
    - RW on the hypercube, Riffle-shuffle [Aldous ‘83]
  - Named “Cutoff Phenomenon” in the top-in-at-random shuffle analysis [Diaconis, Aldous ‘86]
  - RWs on finite groups [Saloff-Coste ‘04]
  - RWs on random regular graphs [L., Sly ‘10]
- One-dimensional Markov chains:
  - Birth-and-Death chains [Diaconis, Saloff-Coste ‘06], [Ding, L., Peres ‘09]
- No proofs of cutoff except when stationary distribution is completely understood and has many symmetries [till recently]
Peres’ Product Criterion

- **Question** [Diaconis ’96]: How can we determine whether a given Markov chain exhibits cutoff?
- **Observation** [Peres ‘04]: if a reversible chain has cutoff then
  \[
  \text{gap} \cdot t_{\text{mix}}(\frac{1}{4}) \to \infty
  \]
or equivalently:
  \[
  t_{\text{rel}} = o(t_{\text{mix}}(\frac{1}{4})).
  \]
- **Proof:**
  - Key fact: every reversible Markov chain satisfies
    \[
    t_{\text{mix}}(\varepsilon) \geq (t_{\text{rel}} - 1) \log(\frac{1}{2\varepsilon}).
    \]
  - Assume that \( t_{\text{rel}} \geq 1 + \delta t_{\text{mix}}(\frac{1}{4}) \) for some \( \delta > 0 \).
  - It follows that \( t_{\text{mix}}(\varepsilon) \geq f(\varepsilon) \cdot t_{\text{mix}}(\frac{1}{4}) \) where \( f(\varepsilon) \to \infty \) as \( \varepsilon \to 0 \).
  \( \Rightarrow \) No (pre) cutoff.
Peres’ Product Criterion (ctd.)

- The condition $\not\times$ is necessary for cutoff. Is it also sufficient, giving a method to determine the existence of cutoff?

- [Aldous ’04]: unfortunately not: the product-condition $\not\times$ does not imply cutoff (explicit construction).

- Even so, Peres conjectured that for many natural families of chains, cutoff occurs iff $\not\times$. (e.g., holds for birth-and-death chains [Ding, L., Peres ’09]).

$$\not\times \quad \text{gap} \cdot t_{\text{mix}}(\frac{1}{4}) \rightarrow \infty \quad \text{cutoff}$$

- Notable conjectured examples:
  - Ising on lattices; Potts model on lattices; Gas Hard-core model on lattices; lattice Colorings; Anti–ferromagnetic Ising / Potts model, Spin–glass, Arbitrary boundary conditions / external field; ...
Recently: cutoff for Ising on lattices

**Theorem** [L., Sly]:

Let $\beta_c = \frac{1}{2}\log(1+\sqrt{2})$ be the critical inverse-temperature for the Ising model on $\mathbb{Z}^2$. Then the continuous-time Glauber dynamics for the Ising model on $(\mathbb{Z}/n\mathbb{Z})^2$ with periodic boundary conditions at $0 \leq \beta < \beta_c$ has cutoff at $(1/\lambda_\infty) \log n$ where $\lambda_\infty$ is the spectral gap of the dynamics on the infinite volume lattice.

Analogous result holds for *any* dimension $d \geq 1$:

- Cutoff at $(d/2\lambda_\infty) \log n$.
- E.g., cutoff at $[2(1 - \tanh(2\beta))]^{-1} \log n$ for $d = 1$.
Random walk on the hypercube

- Glauber dynamics for infinite temperature ($\beta=0$) Ising $\equiv$ lazy RW on the hypercube $\{-1,1\}^n$:
  - Stationary distribution is uniform.
  - Spins evolve independently.
- [Aldous ’83]: Cutoff at $\frac{1}{2}n \log n + O(n)$.
  - Twice faster than trivial upper bound.
  - Constant window in continuous time version.
Cutoff for RW on hypercube (ctd.)

- Symmetry ⇒ Start at the all-plus state.
- Symmetry ⇒ Mixing of magnetization \( S_t = \sum_{i=1}^{n} X_t(i) \) [a birth & death chain] determines entire mixing:
  \[
  \left\| \mathbb{P}_+ (X_t \in \cdot) - \pi \right\|_{TV} = \left\| \mathbb{P}_+ (S_t \in \cdot) - \pi_S \right\|_{TV}.
  \]
  - To bound the coupling-time of this 1d chain it thus suffices to couple it from its extreme ends +, −.
- Magnetizations contract to within \( \sqrt{n} \) from each other:
  \[
  \mathbb{E}_+ [S_t] = n(1 - \frac{1}{n})^t, \quad \mathbb{E}_- [S_t] = -n(1 - \frac{1}{n})^t.
  \]
  - At time \( t = \frac{1}{2} n \log n \) the expected distance between the chains is \( O(\sqrt{n}) \).
- Afterwards: distance is a biased RW drifting towards 0. Comparing to SRW ⇒ takes \( O(n) \) further steps to hit 0.