Hybrid finite difference/finite element version of the immersed boundary method

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SUMMARY

The immersed boundary (IB) method is a framework for modeling systems in which an elastic structure is immersed in a viscous incompressible fluid. The IB formulation of such problems describes the elasticity of the structure in Lagrangian form and describes the momentum, viscosity, and incompressibility of the fluid-structure system in Eulerian form. Interactions between Lagrangian and Eulerian variables are mediated by integral transforms with delta function kernels. When discretized, the Lagrangian equations are approximated on a curvilinear mesh, the Eulerian equations are approximated on a Cartesian grid, and a regularized version of the delta function is used in approximations to the Lagrangian-Eulerian interaction equations. Here, we employ a version of the IB method that allows us to discretize the structure via standard Lagrangian finite element (FE) methods. Unlike most other extensions of the IB method that use FE structural discretizations, however, our approach retains a finite difference discretization of the incompressible Navier-Stokes equations. A key feature of our numerical scheme is that it enables the use of Lagrangian meshes with mesh spacings that are independent of the grid spacing of the background Eulerian grid. Results from computational experiments are included that demonstrate the accuracy and efficiency of our methodology. Copyright © 0000 John Wiley & Sons, Ltd.

1. INTRODUCTION

Since its introduction [1, 2], the immersed boundary (IB) method has been widely used to simulate biological fluid dynamics [3] and other problems in which a structure is immersed in a fluid flow [3–5]. In this paper, we consider the IB method for fluid-structure interaction problems involving an incompressible elastic body that is immersed in a viscous incompressible fluid [3]. The IB formulation of such problems uses a Lagrangian description of the immersed structure and an Eulerian description of the momentum, viscosity, and incompressibility of the fluid-structure system. Lagrangian and Eulerian variables are coupled by integral transforms with delta function kernels. When the continuous equations are discretized, the Lagrangian equations are approximated on a curvilinear mesh, the Eulerian equations are approximated on a Cartesian grid, and the Lagrangian-Eulerian interaction equations are approximated by replacing the singular delta function with a regularized delta function. One advantage of the IB formulation is that it enables the use of efficient Cartesian grid solvers, such as those based on fast Fourier transform (FFT) or multigrid

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Another strength of this approach is that it permits nonconforming discretizations of the fluid and structure. Specifically, the IB method does not require dynamically generated body-fitted meshes, a property that is especially useful for problems involving large structural deformations or contact between structures.

In many applications of the IB method, the elasticity of the immersed structure is described by systems of fibers that resist extension, compression, or bending. Such descriptions can be well suited for the highly anisotropic materials encountered in biological applications. Fiber models are also convenient to use in practice because they permit an especially simple definition and discretization. The fiber-based approach to elasticity modeling also presents certain challenges. For instance, it can be difficult to incorporate realistic shear properties or experimentally based constitutive laws into fiber models. Nonetheless, the fiber-based modeling approach typically employed with the IB method has facilitated significant work in biofluid dynamics [6–11], including three-dimensional simulations of cardiac fluid dynamics [12–21].

Over the past decade, several distinct research efforts have sought to use more general material models with the IB method. For instance, recent work has led to the development of an energy functional-based version of the conventional IB method that obtains a nodal approximation to the elastic forces generated by an immersed hyperelastic material via a finite element (FE) type approximation to the deformation of the material [22, 23]. Other work has also led to the development of the immersed structural potential method, which uses a meshless method to describe the mechanics of an incompressible hyperelastic structure immersed in fluid [24]. The formulation used by each of these numerical schemes relies on the availability of an elastic energy functional, i.e., these methods are specialized to hyperelastic material models. A separate line of research led to the development of the immersed finite element (IFE) method [25–27], which can be viewed as a generalization of the IB method in which FE methods are used for both the fluid and the structure. Like the IB method, the IFE method couples Lagrangian and Eulerian variables by discretized integral transforms with regularized delta function kernels, although different families of smoothed kernel functions are generally used by the two methods. Other work has led to the development of a fully variational IB method that avoids regularized delta functions altogether [28, 29].

In this paper, we introduce an alternative approach to using FE mechanics models with the IB method that combines a Cartesian grid finite difference method for incompressible fluid dynamics with a nodal FE method for nonlinear elasticity. To develop our numerical scheme, we use the continuous IB formulation introduced by Boffi et al. [28]. The mathematical formulation and numerical method require only the specification of a Lagrangian stress tensor to describe the material response of the immersed structure. Our numerical method also readily accommodates arbitrary choices of nodal finite elements, and it can be easily extended to treat noninterpolatory FE bases (e.g., NURBS).

We consider two weak formulations of the equations of motion suitable for use with standard $C^0$ FE methods for structural mechanics. One of these formulations, referred to herein as the unified weak form, is similar to those used by earlier IB-like methods [25–29]. The other, which does not appear to have been used previously to construct a numerical scheme, partitions the elastic force density into an internal force that is supported throughout the immersed elastic structure and a transmission force that is supported only on the surface of the immersed structure. We provide numerical examples that demonstrate that improvements in accuracy can be obtained by employing the partitioned formulation instead of the unified formulation, especially for cases in which the Lagrangian mesh is relatively coarse compared to the Eulerian grid.

Conventionally, regularized delta functions are used by the IB method both to spread the forces generated by the structure directly from the nodes of the Lagrangian mesh to the Eulerian grid and also to interpolate velocities from the Eulerian grid directly to the nodes of the Lagrangian mesh. This discrete Lagrangian-Eulerian coupling approach is also adopted by IB-like methods such as the immersed structural potential method [24] and the immersed finite element method [25–27]. A significant limitation of this approach is that if the physical spacing of the Lagrangian nodes is too large in comparison to the background Eulerian grid, severe “leaks” will develop at fluid-structure interfaces. An empirical rule that generally prevents such leaks is to require the Lagrangian mesh
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Figure 1. Lagrangian and Eulerian coordinate systems. The Lagrangian material coordinate domain is $U$, and the Eulerian physical coordinate domain is $\Omega$. The physical position of material point $s$ at time $t$ is $\chi(s,t)$, the physical region occupied by the structure is $\chi(U,t)$, and the physical region occupied by the fluid is $\Omega \setminus \chi(U,t)$.

to be approximately twice as fine as the Eulerian grid [3]. Because high Eulerian resolution is required to capture the thin boundary layers characteristic of moderate-to-high Reynolds number flows, however, following this empirical rule can require using very dense Lagrangian meshes that may generate significant numerical stiffness in the discretized equations. Moreover, for problems involving large deformations, a structural mesh that is initially “watertight” may deform in a manner that ultimately yields leaks as the simulation progresses.

A key contribution of this paper is that it introduces a new approach to discretizing the equations of Lagrangian-Eulerian interaction that overcomes this longstanding limitation of the IB method. Specifically, rather than spreading forces from and interpolating velocities to the nodes of the Lagrangian mesh, we instead spread forces from and interpolate velocities to quadrature points within the interiors of the Lagrangian finite elements. This approach takes advantage of the additional geometrical information provided by the FE description of the structure, which yields approximations not only to the current positions of the nodes of the Lagrangian mesh, but also to the positions of any material points within the structure. Such geometrical information is not readily available in conventional fiber-based IB methods, and this information does not appear to have been fully exploited in most earlier extensions of the IB method that incorporate FE-type structural discretizations. An important feature of our discretization approach is that obtaining a watertight structure simply requires a quadrature scheme with sufficiently many quadrature points to prevent leaks. Moreover, because it is straightforward to locally adapt the quadrature scheme to account for structural deformations, this approach can ensure that the structure remains watertight even in the presence of very large structural deformations. We present numerical tests that demonstrate that our scheme can yield accurate results even for Lagrangian meshes that are significantly coarser than the background Eulerian grid, and we demonstrate the potential gains in the computational efficiency offered by this approach.

2. CONTINUOUS FORMULATIONS

In the IB formulation of problems of fluid-structure interaction, the momentum, velocity, and incompressibility of the coupled fluid-structure system are described in Eulerian form, whereas the elasticity of the immersed structure is described in Lagrangian form. Let $x = (x_1, x_2, \ldots) \in \Omega \subset \mathbb{R}^d$, $d = 2$ or $3$, denote Cartesian physical coordinates, with $\Omega$ denoting the physical region that is occupied by the coupled fluid-structure system; let $s = (s_1, s_2, \ldots) \in U \subset \mathbb{R}^d$ denote Lagrangian material coordinates that are attached to the structure, with $U$ denoting the Lagrangian coordinate domain; and let $\chi(s,t) \in \Omega$ denote the physical position of material point $s$ at time $t$. The physical region occupied by the structure at time $t$ is $\chi(U,t) \subseteq \Omega$, and the physical region occupied by the fluid at time $t$ is $\Omega \setminus \chi(U,t)$. See Fig. 1. We do not assume that the Lagrangian coordinates are the initial coordinates of the elastic structure (i.e., we do not assume that $\chi(s,0) = s$), nor, more generally, do we require that $U \subseteq \Omega$.

To use an Eulerian description of the fluid and a Lagrangian description of the elasticity of the immersed structure, it is necessary to describe the stress of the fluid-structure system in both
Eulerian and Lagrangian forms. Let $\sigma = \sigma(x, t)$ denote the Cauchy stress tensor of the coupled fluid-structure system. Then

$$\sigma(x, t) = \begin{cases} \sigma^f(x, t) + \sigma^e(x, t) & \text{for } x \in \chi(U, t), \\ \sigma^f(x, t) & \text{otherwise,} \end{cases}$$

in which $\sigma^f(x, t)$ is the stress tensor of a viscous incompressible fluid, and $\sigma^e(x, t)$ is the stress tensor that describes the elasticity of the immersed structure. The fluid stress tensor is the usual one for a viscous incompressible fluid,

$$\sigma^f = -pI + \mu \left[ \nabla u + (\nabla u)^T \right],$$

in which $p = p(x, t)$ is the pressure, $\mu$ is the dynamic viscosity of the fluid, and $u = u(x, t)$ is the Eulerian velocity field.

To describe the elasticity of the structure with respect to the Lagrangian material coordinate system, it is convenient to use the first Piola-Kirchhoff elastic stress tensor $P^e(s, t)$, which is defined so that

$$\int_{\partial V} P^e(s, t) N \, dA(s) = \int_{\partial \chi(V, t)} \sigma^e(x, t) \, n \, d\sigma(x)$$

for any smooth region $V \subset U$, in which $N = N(s)$ is the outward unit normal along $\partial V$, and $n = n(x, t)$ is the outward unit normal along $\partial \chi(V, t)$. For simplicity, we shall restrict our attention to hyperelastic constitutive models, which may be characterized by a strain-energy functional $W^e = W^e(F)$, in which $\mathcal{F} = \mathcal{F}(s, t) = \nabla_s \chi(s, t) = \frac{\partial \chi}{\partial s}(s, t)$ is the deformation gradient associated with the mapping $\chi : (U, t) \rightarrow \Omega$. For such constitutive laws, $P^e(s, t) = \frac{\partial W^e}{\partial \mathcal{F}}(s, t)$. We remark, however, that the present formulation can accommodate incompressible materials that are described only in terms of a Lagrangian stress tensor, and that this formulation is therefore not restricted to hyperelastic material models.

### 2.1. Strong formulation

The strong form of the equations of motion for the fluid-structure system is:

$$\rho \left( \frac{\partial \mathbf{u}(x, t)}{\partial t} + \mathbf{u}(x, t) \cdot \nabla \mathbf{u}(x, t) \right) = \nabla \cdot \mathbf{\sigma}(x, t)$$

$$\nabla \cdot \mathbf{u}(x, t) = 0,$$

$$\mathbf{f}^e(x, t) = \int_{U} \mathbf{N} \cdot P^e(s, t) \, \delta(x - \chi(s, t)) \, ds$$

$$- \int_{\partial U} P^e(s, t) \mathbf{N} \, \delta(x - \chi(s, t)) \, dA(s),$$

$$\frac{\partial \chi}{\partial t}(s, t) = \int_{\Omega} \mathbf{u}(x, t) \, \delta(x - \chi(s, t)) \, dx,$$

in which $\rho$ is the mass density of the coupled fluid-structure system, $\mathbf{f}^e(x, t)$ is the Eulerian elastic force density, and $\delta(x) = \prod_{i=1}^{d} \delta(x_i)$ is the $d$-dimensional delta function. We omit the derivation of this formulation of the equations of motion, which was introduced by Boffi et al. [28].

In the equations of motion, Eqs. (4) and (5) are the standard Eulerian incompressible Navier-Stokes equations, except that the right-hand side of the momentum equation, Eq. (4), is augmented by an Eulerian elastic force density, $\mathbf{f}^e(x, t) = \nabla \cdot \mathbf{\sigma}^e(x, t)$, that is determined from the Lagrangian configuration of the immersed structure via Eq. (6). The conversion from the Lagrangian to the Eulerian description of the forces generated by the elasticity of the immersed structure is mediated by two integral transforms with delta function kernels. These integral transforms appear in Eq. (6).
and convert $\nabla_s \cdot \mathbf{P}^e$, the Lagrangian internal force density, and $-\mathbf{P}^e$, the Lagrangian transmission force density, into equivalent Eulerian densities. Notice that these two Lagrangian force densities have totally different characters: The internal force density is supported throughout $U$ and has units of force per unit volume, whereas the transmission force density is supported only along $\partial U$ and has units of force per unit area. Assuming that $\mathbf{P}^e(s, t)$ is sufficiently smooth, the internal force acts as a regular (i.e., nonsingular) body force on the fluid and may be treated with higher-order accuracy by the IB method \cite{28, 30}. The transmission force acts as a singular force layer on the fluid, and although this force will induce jumps in the pressure and shear stress along $\partial U$, these stress layers are supported only along $\partial U$. The transmission force density is supported only along $\partial U$ and has units of force per unit volume, whereas the transmission force density is supported only along $\partial U$.

### 2.2. Weak formulations

To obtain versions of the equations of motion that allow us to use standard $C^0$ FE methods for nonlinear elasticity, we consider two different formulations that each employ a weak form of the Lagrangian equations. These two formulations are equivalent in the continuous setting; however, when discretized, they yield numerical schemes that are generally different. Because we use a finite difference scheme to approximate the incompressible Navier-Stokes equations, we do not employ a weak formulation for the Eulerian equations.

We refer to our first weak form of the problem as the unified weak formulation because it includes only a single, unified body forcing term that accounts for both the regular internal elastic force density and the singular transmission elastic force density of the strong form of the equations. This formulation is:

$$
\rho \left( \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p + \mu \nabla^2 \mathbf{u} + \mathbf{f},
$$

$$
\nabla \cdot \mathbf{u} = 0,
$$

$$
\mathbf{f}(x, t) = \int_U \mathbf{F}(s, t) \delta(x - \chi(s, t)) \, ds,
$$

$$
\int_U \mathbf{F}(s, t) \cdot \mathbf{V}(s) \, ds = -\int_U \mathbf{P}^e(s, t) : \nabla_s \mathbf{V}(s) \, ds, \quad \forall \mathbf{V}(s),
$$

$$
\frac{\partial \chi}{\partial t}(s, t) = \int_\Omega \mathbf{u}(x, t) \delta(x - \chi(s, t)) \, dx,
$$

in which $\mathbf{f}(x, t)$ and $\mathbf{U}(s, t)$ are the Eulerian and Lagrangian total elastic force densities, and $\mathbf{V}(s)$ is an arbitrary Lagrangian test function that is not assumed to vanish on $\partial U$. This weak form of the equations of motion is similar to the formulation employed in the IFE method \cite{25–27} and the fully variational IB method \cite{28, 29}. Notice that $\mathbf{F}(s, t)$ accounts for both the internal and transmission force densities of the strong form of the equations in the sense that

$$
\int_U \mathbf{F}(s, t) \cdot \mathbf{V}(s) \, ds = -\int_U \mathbf{P}^e(s, t) : \nabla_s \mathbf{V}(s) \, ds
$$

\begin{align}
&= \int_U (\nabla \cdot \mathbf{P}^e(s, t)) \cdot \mathbf{V}(s) \, ds - \int_{\partial U} (\mathbf{P}^e(s, t) \mathbf{N}(s)) \cdot \mathbf{V}(s) \, dA(s),
\end{align}

where $\mathbf{N}(s)$ is the outward normal to $\partial U$, and $\mathbf{N}(s) \cdot \mathbf{V}(s) \, dA(s)$ is the normal component of force.
for any \( \mathbf{V}(s) \).

Our second weak form of the problem, which we refer to as the partitioned weak formulation, is similar to the unified formulation, except that it treats the internal and transmission elastic force densities separately, as follows:

\[
\rho \left( \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p + \mu \nabla^2 \mathbf{u} + \mathbf{g} + \mathbf{t},
\]

\[
\nabla \cdot \mathbf{u} = 0,
\]

\[
g(x, t) = \int_{U} G(s, t) \delta(x - \chi(s, t)) \, ds,
\]

\[
\int_{U} G(s, t) \cdot \mathbf{V}(s) \, ds = -\int_{U} \mathbb{P}^e(s, t) : \nabla_s \mathbf{V}(s) \, ds
\]

\[
+ \int_{\partial U} \mathbb{P}^e(s, t) \mathbf{N}(s) \cdot \mathbf{V}(s) \, dA(s), \quad \forall \mathbf{V}(s),
\]

\[
t(x, t) = \int_{\partial U} T(s, t) \delta(x - \chi(s, t)) \, dA(s),
\]

\[
T = -\mathbb{P}^e \mathbf{N},
\]

\[
\frac{\partial \chi}{\partial t}(s, t) = \int_{\Omega} \mathbf{u}(x, t) \delta(x - \chi(s, t)) \, dx,
\]

in which \( g(x, t) \) and \( G(s, t) \) are the Eulerian and Lagrangian internal elastic force densities, \( t(x, t) \) and \( T(s, t) \) are the Eulerian and Lagrangian transmission elastic force densities, and \( \mathbf{V}(s) \) is an arbitrary Lagrangian test function. Notice that here, \( T(s, t) \) is equal to the transmission force density of the strong formulation, and that \( G(s, t) \) is weakly equivalent to the internal force density of the strong form of the equations, as can be shown by integrating the right-hand-side of Eq. (18) by parts.

3. SPATIAL DISCRETIZATION

We consider only the two-dimensional case for the remainder of the paper. The extension of the numerical scheme to the case \( d = 3 \) is straightforward, and implementations for \( d = 2 \) and \( 3 \) are provided by the open-source IBAMR software [31].

3.1. Eulerian discretization

To discretize the incompressible Navier-Stokes equations in space, we employ a staggered-grid finite difference scheme, an approach that yields superior accuracy when used with the IB method as compared to collocated discretizations (i.e., purely cell- or node-centered schemes) [32]. To simplify the exposition, we assume that \( \Omega \) is the unit square and is discretized on a regular \( N \times N \) Cartesian grid with grid spacings \( \Delta x_1 = \Delta x_2 = h = \frac{1}{N} \). Let \((i, j)\) label the individual Cartesian grid cells for integer values of \( i \) and \( j \), \( 0 \leq i, j < N \). The components of the Eulerian velocity field \( \mathbf{u} = (u_1, u_2) \) are approximated at the centers of the \( x_1 \)- and \( x_2 \)-edges of the Cartesian grid cells, i.e., at positions \( x_{i-\frac{1}{2}, j} = (ih, \left( j + \frac{1}{2} \right) h) \) and \( x_{i, j-\frac{1}{2}} = \left( \left( i + \frac{1}{2} \right) h, jh \right) \), respectively. A staggered scheme is also used for the Eulerian body force \( \mathbf{f} = (f_1, f_2) \). We use the notation \((u_1)_{i-\frac{1}{2}, j}, (u_2)_{i, j-\frac{1}{2}}, (f_1)_{i-\frac{1}{2}, j}, \) and \((f_2)_{i, j-\frac{1}{2}}\) to denote the discrete values of \( u \) and \( f \). The pressure \( p \) is approximated at the centers of the Cartesian grid cells, and its discrete values are denoted \( p_{i,j} \).

Let \( \nabla_h \cdot \mathbf{u} \approx \nabla \cdot \mathbf{u}, \nabla_h p \approx \nabla p, \) and \( \nabla_h^2 \mathbf{u} \approx \nabla^2 \mathbf{u} \) respectively denote second-order accurate finite difference approximations to the divergence, gradient, and Laplace operators [33]. In this approach, \( \nabla_h \cdot \mathbf{u} \) is defined at cell centers, whereas both \( \nabla_h p \) and \( \nabla_h^2 \mathbf{u} \) are defined at cell edges. We use a staggered-grid version [32, 33] of the xsPPM7 variant [34] of the piecewise parabolic method (PPM) [35] to discretize the nonlinear advection terms. Where needed, physical boundary conditions are discretized and imposed along \( \partial \Omega \) as described by Griffith [33].
If \( \mathbf{u} \) and \( \mathbf{v} \) are discrete staggered-grid vector fields, we denote by \([\mathbf{u}]\) and \([\mathbf{v}]\) the corresponding vectors of grid values. If \( \Omega \) has periodic boundaries, we define the discrete \( L^2 \) inner product on \( \Omega \) by

\[
(\mathbf{u}, \mathbf{v})_\chi = [\mathbf{u}]^T [\mathbf{v}] h^2.
\]

Minor adjustments to this definition are required when nonperiodic physical boundary conditions are used [33].

### 3.2. Lagrangian discretization

Let \( T_h = \cup_e U_e \) be a triangulation of \( U \) composed of elements \( U_e \). We denote by \( \{s_i\}_{l=1}^M \) the nodes of the mesh, and by \( \{\phi_l(s)\}_{l=1}^M \) interpolatory Lagrangian basis functions. Our formulation is independent of the choice of the nodal basis functions and may be used with any nodal finite elements. In particular, although our numerical examples consider only \( Q^1 \) (bilinear) elements, it is straightforward to employ higher-order elements. A generalization to noninterpolatory basis functions (e.g., NURBS) would be straightforward but has not yet been attempted.

We denote the time-dependent physical positions of the nodes of the Lagrangian mesh by \( \{\chi_l(t)\}_{l=1}^M \). Using the Lagrangian basis functions, we define an approximation to \( \chi(s, t) \) by

\[
\chi_h(s, t) = \sum_{l=1}^M \chi_l(t) \phi_l(s).
\]

Because we use interpolatory FE basis functions, \( \chi_h(s_i, t) = \chi_l(t) \). An approximation to the deformation gradient is given by

\[
\mathbf{F}_h(s, t) = \frac{\partial}{\partial s} \chi_h(s, t) = \sum_{l=1}^M \chi_l(t) \frac{\partial}{\partial s} \phi_l(s).
\]

Using \( \mathbf{F}_h(s, t) \), we compute directly \( \mathbf{P}_h^s(s, t) \) and \( \mathbf{T}_h(s, t) \), approximations to the first Piola-Kirchhoff stress tensor and the Lagrangian transmission force density, respectively. For \( C^0 \) Lagrangian basis functions, \( \chi_h(s, t) \) is a continuous function of \( s \), but \( \mathbf{F}_h(s, t) \) is generally discontinuous at internal element boundaries. Hence, \( \mathbf{P}_h^s \) and \( \mathbf{T}_h \) are also generally only piecewise continuous.

We approximate the Lagrangian force densities \( \mathbf{F}(s, t) \) and \( \mathbf{G}(s, t) \) by

\[
\mathbf{F}_h(s, t) = \sum_{l=1}^M \mathbf{F}_l(t) \phi_l(s), \quad \text{and} \quad \mathbf{G}_h(s, t) = \sum_{l=1}^M \mathbf{G}_l(t) \phi_l(s).
\]

The nodal values \( \{\mathbf{F}_l\}_{l=1}^M \) and \( \{\mathbf{G}_l\}_{l=1}^M \) must be determined from \( \mathbf{P}_h^s(s, t) \). By restricting the test functions to be linear combinations of the Lagrangian basis functions, we have that Eq. (11) becomes, after rearranging terms,

\[
\sum_{l=1}^M \left( \int_U \phi_l(s) \phi_m(s) \, ds \right) \mathbf{F}_l(t) = - \int_U \mathbf{P}_h^s(s, t) \nabla_s \phi_m(s) \, ds,
\]

for each \( m = 1, \ldots, M \). Similarly, Eq. (18) becomes

\[
\sum_{l=1}^M \left( \int_U \phi_l(s) \phi_m(s) \, ds \right) \mathbf{G}_l(t) = - \int_U \mathbf{P}_h^s(s, t) \nabla_s \phi_m(s) \, ds
\]

\[
\quad + \int_{\partial U} \mathbf{P}_h^s(s, t) \mathbf{N}(s) \phi_m(s) \, dA(s),
\]

where \( \mathbf{N}(s) \) is the outward unit normal at the boundary. This can be written in a more compact form as

\[
\sum_{l=1}^M \left( \int_U \phi_l(s) \phi_m(s) \, ds \right) \mathbf{G}_l(t) = - \int_U \mathbf{P}_h^s(s, t) \nabla_s \phi_m(s) \, ds
\]

\[
\quad + \int_{\partial U} \mathbf{P}_h^s(s, t) \mathbf{N}(s) \phi_m(s) \, dA(s).
\]

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for each \( m = 1, \ldots, M \).

Letting \([F]\) denote the vector of nodal coefficients of \( F_h \), we write Eq. (27) as
\[
[M] \, [F] = [B],
\]
in which \([M]\) is the mass matrix that has entries of the form \( \int_U \phi_i(s) \, \phi_m(s) \, ds \). Eq. (28) may be rewritten similarly. The mass matrix \([M]\) can also be used to evaluate the \( L^2 \) inner product of Lagrangian functions on \( U \). In particular, for any \( U_h(s, t) = \sum_l U_l(t) \phi_l(s) \) and \( V_h(s, t) = \sum_l V_l(t) \phi_l(s) \),
\[
(U_h, V_h)_s = [U]^T [M] [V].
\]

Different choices of mass matrices (e.g., lumped mass matrices) induce different discrete inner products on \( U \).

To simplify notation, in the remainder of this paper we drop the subscript “\( h \)” from our numerical approximations to the Lagrangian variables.

### 3.3. Lagrangian-Eulerian interaction

As in the conventional IB method, we approximate the singular delta function kernel appearing in the Lagrangian-Eulerian interaction equations by a smoothed \( d \)-dimensional Dirac delta function \( \delta_h(x) \) that is of the tensor-product form \( \delta_h(x) = \prod_{i=1}^d \delta_h(x_i) \). In this work, we take the one-dimensional smoothed delta function \( \delta_h(x) \) to be the four-point delta function of Peskin [3].

To compute an approximation to \( f = (f_1, f_2) \) on the Cartesian grid, we construct for each element \( U^e \in T_h \) a Gaussian quadrature rule with \( N^e \) quadrature points \( s_Q^e \in U^e \) and weights \( \omega_Q^e \). For \( Q = 1, \ldots, N^e \). We then compute \( f_1 \) and \( f_2 \) on the edges of the Cartesian grid cells via
\[
(f_1)_{i-{1 \over 2}, j} = \sum_{U^e \in T_h} \sum_{Q=1}^{N^e} F_1(s_Q^e, t) \, \delta_h(x_{i-{1 \over 2}, j} - \chi(s_Q^e, t)) \, \omega_Q^e, \quad \text{and} \quad (31)
\]
\[
(f_2)_{i, j-{1 \over 2}} = \sum_{U^e \in T_h} \sum_{Q=1}^{N^e} F_2(s_Q^e, t) \, \delta_h(x_{i, j-{1 \over 2}} - \chi(s_Q^e, t)) \, \omega_Q^e, \quad \text{and} \quad (32)
\]
with \( F(s, t) = (F_1(s, t), F_2(s, t)) \). We use the shorthand
\[
f = S \, F,
\]

in which \( S = S(\chi) \) is the force-prolongation operator implicitly defined by Eqs. (31) and (32).

A corresponding velocity-restriction operator \( \mathcal{R} = \mathcal{R}(\chi) \) is used to determine the motion of the nodes of the Lagrangian mesh from the Cartesian grid velocity field via
\[
{d\chi \over dt} = \mathcal{R} \, u.
\]

There are many possible ways to construct \( \mathcal{R} \); however, we have found that an effective approach is to require \( \frac{d\chi}{dt} = \mathcal{R} \, u \) to satisfy the discrete power identity,
\[
(F, \frac{d\chi}{dt})_s = (f, u)_\chi, \quad \text{and} \quad (35)
\]

which implies that the semi-discrete unified formulation conserves energy during Lagrangian-Eulerian interaction [3]. This power identity can be rewritten as
\[
(F, \mathcal{R} \, u)_s = (S \, F, u)_\chi, \quad \text{and} \quad (36)
\]
i.e., \( \mathcal{R} = S^* \).
To construct $\mathcal{R}$ explicitly, it is convenient to use matrix notation. Identifying $[S]$ and $[R]$ with the matrix representations of $S$ and $\mathcal{R}$, we have that

$$[F] = [S][F], \quad \text{and}$$

$$\frac{d[x]}{dt} = [R][u].$$

Eq. (36) then becomes

$$[F]^T[M][R][u] = ([S][F])^T[u] h^2.$$  \hspace{1cm} (39)

If Eq. (39) is to hold for any $[F]$ and $[u]$, then $[R]$ must be defined via

$$[R] = [M]^{-1}[S]^T h^2.$$  \hspace{1cm} (40)

In our time-stepping scheme, which is stated below in Sec. 4, notice that we need only to apply $[R]$ to discrete velocity fields defined on the Cartesian grid. Specifically, we do not need to compute $[R]$ explicitly.

It is straightforward to show that this construction of $\mathcal{R}$ implies that $\frac{d[x]}{dt}(s, t)$ is an approximation to the $L^2$ projection of the Lagrangian vector field $U^{IB}(s, t) = (U_1^{IB}(s, t), U_2^{IB}(s, t))$, with

$$U_1^{IB}(s, t) = \sum_{i,j} (u_1)_{i,\frac{1}{2},j} \delta_h(x_{i,\frac{1}{2},j} - \chi(s, t)) h^2, \quad \text{and}$$

$$U_2^{IB}(s, t) = \sum_{i,j} (u_2)_{i,j,\frac{1}{2}} \delta_h(x_{i,j,\frac{1}{2}} - \chi(s, t)) h^2.$$  \hspace{1cm} (41)

Because the components of $U^{IB}(s, t)$ are not generally linear combinations of the Lagrangian basis functions, and therefore generally $\frac{d[x]}{dt} \neq U^{IB}$.

For the semi-discretized partitioned weak formulation, $\mathbf{g}$ is computed on the Cartesian grid via $\mathbf{g} = S \mathbf{G}$. The Eulerian transmission force density $\mathbf{t}$ is computed in a similar manner, but in this case, the numerical integration occurs only over those element boundaries that coincide with $\partial U$. We use the shorthand $\mathbf{t} = S^{out} \mathbf{T}$ to denote this operation. We use the same regularized delta function $\delta_h(\mathbf{x})$ to construct both $S$ and $S^{out}$. For simplicity, we use the same velocity-restriction operator for both formulations. This choice ensures that the two formulations coincide whenever $\mathbf{T} \equiv 0$.

The Lagrangian-Eulerian interaction operators introduced in this work are different from analogous operators generally used in standard IB methods. Standard IB methods and schemes such as the IFE method use regularized delta functions to apply nodal forces directly to the Cartesian grid and to interpolate Cartesian grid velocities directly to the Lagrangian nodes [3]. In such schemes,
\( \mathbf{f}(\mathbf{x}, t) \) would be approximated on the Eulerian grid by expressions similar to

\[
(f^B_{1,i-\frac{1}{2},j}) = \sum_{l=1}^{M} (F_1)_l(t) \delta_h(x_{i-\frac{1}{2},j} - x_l(t)) \omega^B_l, \quad \text{and} \\
(f^B_{2,i,j-\frac{1}{2}}) = \sum_{l=1}^{M} (F_2)_l(t) \delta_h(x_{i,j-\frac{1}{2}} - x_l(t)) \omega^B_l,
\]

in which \( f^B \) denotes the Eulerian force determined by the standard IB force-spreading operator and \( \omega^B_l \) denotes the volume associated with Lagrangian node \( l \). In this approach, each nodal force \( F_l \) is applied only to Cartesian grid cells within the support of \( \delta_h(x - x_l) \), and the Lagrangian mesh must therefore be finer than the Cartesian grid to avoid leaks \cite{3}. The corresponding approach to velocity restriction used by such methods would be to set \( \frac{dx}{dt} = \mathbf{U}^B(x_l, t) \).

Our force-prolongation operator can be seen as the composition of two operations: first, the nodal values of \( \mathbf{F} \) are interpolated from the mesh nodes to a “net” or “submesh” quadrature points within each element; then, \( \delta_h(x) \) is used to spread those interpolated force densities from the quadrature points to the Cartesian grid. See Fig. 2. Velocity restriction is similar. First, the Cartesian velocity field is interpolated to the quadrature points using \( \delta_h \); then, those samples of \( \mathbf{U}^B \) are then used to compute the right-hand-side of an \( L^2 \)-projection equation, and the solution of this projection equation determines \( \frac{dx}{dt} \). In our approach, the Lagrangian structure is watertight so long as the net of quadrature points is sufficiently dense. Denser nets of quadrature points can be obtained by increasing the order of the quadrature rule, and this may be done in an adaptive manner as the simulation progresses. In our numerical tests, we use adaptive Gaussian quadrature rules to construct \( \mathcal{S} \) and \( \mathcal{R} \) that provide, on average, at least a \( 3 \times 3 \) net of quadrature points per Cartesian grid cell.

4. TEMPORAL DISCRETIZATION

4.1. Basic time-stepping scheme

Let \( \chi^n, \mathbf{u}^n, \) and \( p^{n-\frac{1}{2}} \) denote the approximations to the values of \( \chi \) and \( \mathbf{u} \) at time \( t^n \), and to the value of \( p \) at time \( t^{n-\frac{1}{2}} \), respectively. We advance the solution values forward in time by the time interval \( \Delta t \) as follows. First, we determine a preliminary approximation to the deformed structure configuration at time \( t^{n+\frac{1}{2}} \) via

\[
\frac{\chi^{n+1} - \chi^n}{\Delta t} = \mathcal{R}^n \mathbf{u}^n,
\]

with \( \mathcal{R}^n = \mathcal{R}(\chi^n) \), and we define an approximation to \( \chi \) at time \( t^{n+\frac{1}{2}} \) by

\[
\chi^{n+\frac{1}{2}} = \frac{\chi^{n+1} + \chi^n}{2}.
\]

Then, we solve

\[
\rho \left( \frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\Delta t} + \mathbf{A}^{n+\frac{1}{2}} \right) = -\nabla_h \rho^{n+\frac{1}{2}} + \mu \nabla_h^2 \frac{\mathbf{u}^{n+1} + \mathbf{u}^n}{2} + \mathbf{f}^{n+\frac{1}{2}},
\]

\[
\nabla_h \cdot \mathbf{u}^{n+1} = 0,
\]

\[
\mathbf{f}^{n+\frac{1}{2}} = \mathcal{S} \left( \chi^{n+\frac{1}{2}} \right) \mathbf{F} \left( \chi^{n+\frac{1}{2}} \right),
\]

\[
\frac{\chi^{n+1} - \chi^n}{\Delta t} = \mathcal{R} \left( \chi^{n+\frac{1}{2}} \right) \frac{\mathbf{u}^{n+1} + \mathbf{u}^n}{2},
\]

for \( \chi^{n+1}, \mathbf{u}^{n+1}, \) and \( p^{n+\frac{1}{2}} \), in which \( \mathbf{A}^{n+\frac{1}{2}} = \frac{3}{2} \mathbf{u}^n \cdot \nabla_h \mathbf{u}^n - \frac{1}{2} \mathbf{u}^{n-1} \cdot \nabla_h \mathbf{u}^{n-1} \) is computed via a PPM-type approximation to the nonlinear advection term \cite{32, 33}. The time-stepping scheme for the partitioned weak formulation is analogous. Notice that solving Eqs. (47)–(50) for \( \chi^{n+1} \),
requires the solution of a Crank-Nicolson-type discretization of the time-dependent incompressible Stokes equations. We solve this system of equations via the flexible GMRES (FGMRES) algorithm [36], using \( u^n \) and \( p^{n-\frac{1}{2}} \) as initial approximations to \( u^{n+1} \) and \( p^{n+\frac{1}{2}} \), and using a pressure-free projection method with multigrid subdomain solvers as a preconditioner [33].

### 4.2. Initial time step

Because time step-lagged values of \( u \) and \( p \) are used by the time-stepping scheme of Sec. 4.1, we cannot use that scheme for the initial time step. Instead, we use a two-step predictor-corrector method: First, we solve

\[
\rho \left( \frac{\tilde{u}^{n+1} - u^n}{\Delta t} + A^n \right) = -\nabla_h \tilde{p}^{n+\frac{1}{2}} + \mu \nabla_h^2 \frac{u^{n+1} + u^n}{2} + f^n, \tag{51}
\]

\[
\nabla_h \cdot \tilde{u}^{n+1} = 0, \tag{52}
\]

\[
\chi^{n+1} - \chi^n \Delta t = R(\chi^n) u^n, \tag{53}
\]

for \( \chi^{n+1}, \tilde{u}^{n+1}, \) and \( \tilde{p}^{n+\frac{1}{2}} \), with \( A^n = u^n \cdot \nabla_h u^n \). Because we do not have an initial value for the pressure, we use \( p = 0 \) as an initial guess for \( p^{n+\frac{1}{2}} \). Second, we set \( \chi^{n+\frac{1}{2}} = \frac{\chi^{n+1} + \chi^n}{2} \) and solve Eqs. (47)–(50) for \( \chi^{n+1}, u^{n+1}, \) and \( p^{n+\frac{1}{2}} \), except that we use \( A^{n+\frac{1}{2}} = u^{n+\frac{1}{2}} \cdot \nabla_h u^{n+\frac{1}{2}} \) with \( u^{n+\frac{1}{2}} = \frac{1}{2} (\tilde{u}^{n+1} + u^n) \).

### 5. IMPLEMENTATION

This version of the IB method is implemented in the open-source IBAMR software [31], which is a C++ framework for developing fluid-structure interaction models that use the IB method. IBAMR provides support for distributed-memory parallelism and adaptive mesh refinement (AMR). IBAMR relies upon the SAMRAI [37–39], PETSc [40–42], hypre [43, 44], and libMesh [45, 46] libraries for much of its functionality.

### 6. NUMERICAL RESULTS

#### 6.1. Thick elliptical shell

This set of numerical tests uses a thick elliptical shell like that employed in the context of the conventional IB method \([18, 30]\) and the fully variational IB method \([28]\) to demonstrate that the IB method can obtain higher-order convergence rates for certain problems. In these computations, the physical domain is \( \Omega = [0, 1] \times [0, 1] \) with periodic boundary conditions, and, following Boffi et al. \([28]\), the Lagrangian coordinate domain is \( U = [0, 2\pi R] \times [0, w] \), with \( R = 0.25 \) and \( w = 0.0625 \), and with periodic boundary conditions in the \( s_1 \) direction. The coordinate mapping \( \chi : (U, t) \rightarrow \Omega \) is given at time \( t = 0 \) by

\[
\chi(s, 0) = (\cos(s_1/R)(R + s_2) + 0.5, \sin(s_1/R)(R + \gamma + s_2) + 0.5). \tag{55}
\]

For \( \gamma = 0 \), the initial configuration of the shell is a circular annulus with inner radius \( R \) and thickness \( w \). We use \( \gamma = 0 \) for static problems and \( \gamma = 0.1 \) for dynamic problems. In either case, we discretize \( \Omega \) using an \( N \)-by-\( N \) Cartesian grid, and we discretize \( U \) using a \( 28M \)-by-\( M \) mesh of bilinear \((Q^1)\) elements, with \( M = \frac{1}{16} N^2 \). The Lagrangian discretization is constructed so that the nodes of the Lagrangian mesh are physically separated by a distance of approximately \( M_{fac} \Delta x \). Representative numerical results are shown in Fig. 3.

Although this is a relatively simple benchmark problem, the static version \((\gamma = 0)\) is one of the only test problems available for the IB method that we know of that permits a simple analytic
Figure 3. Representative results from the dynamic ($\gamma = 0.1$) version of the orthotropic shell problem of Sec. 6.1.2 for $N = 128$ and the partitioned (split) weak formulation over the time interval $0 \leq t \leq 1.25$. The computed pressure and structure deformation for $M_{\text{fac}} = 4$ are shown in A and B. The computed deformations obtained with $M_{\text{fac}} = 1$ and $M_{\text{fac}} = 4$ are compared in C. Notice the high level of agreement between the computed deformations in C.

solution. Moreover, because certain choices of $P^e$ yield problems for which the IB method is able to attain higher-order convergence rates, this problem allows us to verify that our implementation attains its designed order of accuracy.

6.1.1. Anisotropic shell We first consider an idealized anisotropic shell that is defined in terms of the strain-energy functional [28],

$$W^e(F) = \frac{\mu^e}{2w} \left\| \frac{\partial \chi}{\partial s_1} (s, t) \right\|^2 = \frac{\mu^e}{2w} F_{\alpha 1} F_{\alpha 1},$$

with repeated indices implying summation. In this case,

$$P^e(s, t) = \frac{\partial W^e}{\partial u^e}(s, t) = \frac{\mu^e}{w} \left( \frac{\partial \chi}{\partial s_1} 0 \right) = \frac{\mu^e}{w} \left( \begin{array}{c} F_{11} \\ F_{21} \end{array} \right).$$

This energy functional and stress tensor correspond to an idealized anisotropic elastic material composed of a continuous family of extension-resistant fibers that wrap the thick shell. Because $U$ is periodic in the $s_1$ direction, $P^e \mathbf{N} \equiv 0$ along $\partial U$. This reflects the fact that none of the fibers terminate along the boundary of the structure. Because the transmission force vanishes in this case, the unified and partitioned weak formulations are identical.
convergence rates observed in the discrete order convergence rates being observed in the discrete order convergence is observed in the oscillation of the shell. Fig. 5 summarizes the error data at time $t = 512$ and $t = 50$. We consider the time interval $t = 256$, and $t = 512$.

Second-order convergence rates are observed for the pressure in the discrete $L^1$, $L^2$, and $L^\infty$ norms. Convergence rates for the deformation are somewhat less regular, with nearly second-order convergence rates being observed in the discrete $L^1$ norm and between first- and second-order convergence rates observed in the discrete $L^2$ and $L^\infty$ norms.

When we set $\gamma = 0$, the initial configuration of the elastic shell is an equilibrium configuration of the continuous problem, so that $u(x, t) \equiv 0$. Requiring $\int_{\Omega} p(x, t) \, dx = 0$, it can be shown [28] that

$$p(x, t) = \begin{cases} p_0 + \frac{\mu e}{R} (R + w - r) & r \leq R, \\ p_0 + \frac{\mu e}{w} \frac{1}{\pi} (R + w - r) & R < r \leq R + w, \\ p_0 & R + w < r, \end{cases}$$

with $r = \|x - (0.5, 0.5)\|$ and $p_0 = \frac{\pi \mu e}{3\pi} \left( R^2 - \frac{(R+w)^3}{R} \right)$. We set $\rho = 1$, $\mu = 1$, and $\mu e = 1$, and we consider the time interval $0 \leq t \leq 3$. Fig. 4 summarizes the error data at time $t = 3$ for $N = 64, 128, 256$, and $512$ and $M_{\text{fac}} = 1$ and $4$, with $\Delta t = 0.25\Delta x$. Second-order convergence rates are observed in the discrete $L^1$, $L^2$, and $L^\infty$ norms for the velocity field. Second-order convergence rates are also observed for the pressure in the discrete $L^1$ norm; however, because the pressure field is $C^0$ but not $C^1$, only first-order convergence rates are observed for the pressure in the discrete $L^\infty$ norm, and intermediate convergence rates are observed in the $L^2$ norm.

We also consider the case in which $\gamma = 0.1$, so that the initial configuration of the shell is not in equilibrium. We set $\rho = 1$, $\mu = 0.01$, and $\mu e = 1$, yielding a Reynolds number of approximately 50. We consider the time interval $0 \leq t \leq 0.75$, which corresponds to approximately one damped oscillation of the shell. Fig. 5 summarizes the error data at time $t = 0.75$ for $N = 64, 128, 256$, and $512$ and $M_{\text{fac}} = 1$ and $4$, with $\Delta t = 0.25\Delta x$. Essentially second-order convergence rates are observed in the discrete $L^1$, $L^2$, and $L^\infty$ norms for the velocity field. Essentially second-order convergence rates are also observed for the pressure in the discrete $L^1$ norm, whereas only first-order convergence is observed in the $L^\infty$ norm, and intermediate convergence rates are observed in the $L^2$ norm. Convergence rates for the deformation are somewhat less regular, with nearly second-order convergence rates being observed in the discrete $L^1$ norm and between first- and second-order convergence rates observed in the discrete $L^2$ and $L^\infty$ norms.
Notice that in all cases, accuracy is largely similar for both choices of $M_{\text{fac}}$, suggesting that the scheme yields results that are largely independent of the relative coarseness of the Lagrangian mesh. In particular, because the convergence rates for $u$ are similar for both choices of $M_{\text{fac}}$, these results suggest that the scheme does not allow leaks at fluid-structure interfaces, even for Lagrangian meshes that are rather coarse compared to the Eulerian grid.

### 6.1.2. Orthotropic shell

The second elasticity model that we use for this problem is an orthotropic material model defined in terms of the strain-energy functional [28],

$$W^e(F) = \frac{\mu^e}{2w} \left( \left\| \frac{\partial \chi}{\partial s_1} (s, t) \right\|^2 + \left\| \frac{\partial \chi}{\partial s_2} (s, t) \right\|^2 \right) = \frac{\mu^e}{2w} F : \bar{F},$$

for which

$$\bar{F}^e(s, t) = \frac{\partial W^e}{\partial \bar{F}}(s, t) = \frac{\mu^e}{w} \bar{F}(s, t).$$

This energy functional and stress tensor correspond to an elastic material composed of two continuous families of fibers. The first family of fibers wraps the elliptical shell circumferentially, and the second family is composed of radial fibers that are orthogonal to the circumferential fibers. Because one family of fibers terminates along the fluid-structure interfaces, there are singular force layers along $\partial \chi(U, t)$ that must be balanced by discontinuities in the pressure and viscous stress. Therefore, in this case the discretized unified and partitioned formulations yield different results.
We first set $\gamma = 0$, which is an equilibrium configuration of the continuous problem, so that $u(x, t) \equiv 0$. Requiring $\int_\Omega p(x, t) \, dx = 0$, it can be shown [28] that

$$p(x, t) = \begin{cases} p_0 + \mu^e \left( \frac{1}{R} - \frac{1}{R+w} \right) & r \leq R, \\ p_0 + \frac{\mu^e}{w} \left( \frac{R}{R+w} - \frac{R}{R+w} \right) & R < r \leq R + w, \\ p_0 & R + w < r, \end{cases}$$

with $r = ||x - (0.5, 0.5)||$ and $p_0 = \pi \mu^e \left( 3wR + R^2 - \frac{(R+w)^2}{R} \right)$. We set $\rho = 1$, $\mu = 1$, and $\mu^e = 1$, and we consider the time interval $0 \leq t \leq 3$. Fig. 6 summarizes the error data at time $t = 3$ for $N = 64, 128, 256$, and 512 and $M_{fac} = 1$ and 4, with $\Delta t = 0.25 \Delta x$. First-order convergence rates are observed for $u$ in all norms. First-order convergence rates are also observed for $p$ in the $L^2$ norm. Because $p$ possesses discontinuities at fluid-structure interfaces for this problem, however, the present method yields convergence rates of 0.5 in the $L^1$ norm and does not converge in the $L^\infty$ norm.

We also consider the case in which $\gamma = 0.1$, so that the initial configuration of the shell is not in equilibrium. Because there is no exact analytic solution available for this problem, we estimate the convergence rates in a standard way by computing the norms of the differences of quantities computed on an $N$-by-$N$ Cartesian grid and corresponding Lagrangian mesh, and those computed on a $2N$-by-$2N$ Cartesian grid and corresponding Lagrangian mesh. We set $\rho = 1$, $\mu = 0.01$, and $\mu^e = 1$, yielding a Reynolds number of approximately 50. We consider the time interval $0 \leq t \leq 1.25$, which corresponds to approximately one damped oscillation of the shell. Fig. 7 summarizes the error data at time $t = 0.75$ for $N = 64, 128, 256$, and 512 and $M_{fac} = 1$ and 4.
with $\Delta t = 0.25\Delta x$. Essentially first-order convergence rates are observed for $u$ and $\chi$ in all norms, whereas $p$ exhibits first-order convergence in only the $L^1$ norm.

For this problem, we find that the unified and partitioned formulations yield similar accuracy in most cases for $u$, and that the unified formulation can offer modestly better accuracy for $\chi$. By contrast, the partitioned formulation offers significantly better accuracy for the pressure for relatively coarse Lagrangian meshes. This property appears also to result in improvements in volume conservation; see Sec. 6.2.

6.2. Soft elastic disc

These tests demonstrate the volume-conservation properties of our method by considering the problem of a soft elastic disc immersed in a lid-driven cavity flow. The physical domain is $\Omega = [0, 1]^2$, and the physical boundary conditions are $u \equiv 0$ along the left ($x_1 = 0$), right ($x_1 = 1$), and bottom ($x_2 = 0$) boundaries of $\Omega$, and $u \equiv (1, 0)$ along the top ($x_2 = 1$) boundary of $\Omega$. The immersed structure is a disc of radius 0.2 that is initially centered about the position $(0.6, 0.5)$. The flow induced by the driven lid brings the structure nearly into contact with the moving upper boundary of the domain. This near contact is handled automatically by the IB formulation of the problem. Close to $\partial \Omega$, we use a modified regularized delta function formulation [19] to ensure that force and torque are conserved and that velocity is interpolated accurately during Lagrangian-Eulerian interaction. No additional specialized methods are required by the present scheme to handle this case.
We use an isotropic neo-Hookean material model

\[ W^e(F) = \frac{\mu^e}{2} \|F\|, \]

for which

\[ P^e(s, t) = \mu^e F. \]

Because generally \( P^e N \neq 0 \), the solution will generally possess discontinuities in the pressure and viscous stress at fluid-structure interfaces, and we expect the IB method to yield no better than first-order convergence rates for this example. This flow also possesses well-known corner singularities that also act to reduce the convergence rate of the incompressible flow solver. Although it is possible to devise numerical schemes that accurately treat the corner singularities present in the classical lid-driven cavity flow [47], we do not employ such a method in this work.

As in previous studies of this problem [48, 49], we set \( \mu = 0.01 \), \( \rho = 1 \), and \( \mu^e = 0.1 \). The initial velocity is \( u \equiv 0 \), and the initial deformation is \( \chi(s, 0) \equiv s \). The physical domain is discretized on an \( N \times N \) Cartesian grid, and the Lagrangian coordinate domain is discretized using a semistructured mesh of bilinear \((Q^1)\) elements that is constructed so that the coarsest element is approximately a factor of \( M_{fac} \) coarser than the background Eulerian grid. We consider the time interval \( 0 \leq t \leq 10 \), during which the disc is subjected to slightly more than one rotation within the cavity. The structure becomes entrained in the shearing flow along the cavity lid from \( t \approx 4 \) until \( t \approx 6 \), and during this time is subjected to extremely large deformations. Sample results are shown in Figs. 8 and 9. Fig. 10 shows the relative change in disc area for different values of \( N \) and \( M_{fac} \). Over this time interval,
the maximum area change yielded by the unified formulation is less than 0.5% for \( N = 64 \) and \( M_{\text{fac}} = 4 \), and is approximately 0.2% for \( N = 64 \) and \( M_{\text{fac}} = 1 \). For the partitioned formulation, the maximum area change is approximately 0.2% for \( N = 64 \) and both \( M_{\text{fac}} = 1 \) and \( M_{\text{fac}} = 4 \). With either formulation, the maximum volume change converges to zero at a first-order rate.

In general, the partitioned formulation appears to yield superior volume conservation for this problem, and its volume-conservation properties seem to be relatively insensitive to the relative coarseness of the Lagrangian mesh. Using either formulation, volume errors converge to zero at essentially a first-order rate. These results compare favorably to those obtained by the IFE method, which, even for relatively fine Lagrangian meshes, yields volume losses of up to 20% when applied to the same test problem without using a volume-conservation algorithm, and which exhibits volume losses of approximately 2.5% when using a volume-conservation algorithm [49].

6.3. Collapsible channel flow

These tests consider the problem of steady pressure-driven flow in a two-dimensional collapsible channel [50]. The fluid properties are \( \rho = 1 \) gm/cm\(^3\) and \( \mu = 0.01 \) ba s. The channel is of length \( L = 40 \) cm and width \( D = 1 \) cm and is centered in the region \( \Omega = [0, L] \times [0, H] \) with \( H = 3D \). In the reference configuration, the channel walls have thicknesses \( w = 0.01D \) and are straight. A portion of the upper wall of length \( L_u = 5 \) cm is flexible, and an upstream portion of the upper wall of length \( L_u = 5 \) cm and a downstream portion of length \( L_d = 30 \) cm are rigid. The entire length of the lower wall is rigid. The tangential velocity is clamped to zero along \( \partial \Omega \). At steady-state, the upstream pressure in the channel is \( p_u = 0.45 \) Pa, the downstream pressure is \( p_d = 0 \) Pa, and the external pressure is \( p_e = 1.755 \) Pa. See Fig. 11. Near \( \partial \Omega \), we again use a modified regularized delta function formulation [19] to ensure that force and torque are conserved and that velocity is interpolated accurately during Lagrangian-Eulerian interaction.
Figure 11. A partially-collapsible channel of length $L$ and width $D$. The channel is centered in the region $\Omega = [0, L] \times [0, H]$ with $H = 3D$. A portion of the upper wall of length $L_u$ is flexible, and an upstream portion of the upper wall of length $L_d$ and a downstream portion of length $L_d$ are both rigid. The entire length of the lower wall is rigid. We set the upstream pressure in the channel to $p_u$, the downstream pressure to $p_d$, and the external pressure to $p_e$.

Figure 12. Trajectory of a material point on the flexible section of the collapsible channel for two different grid resolutions. The $y$ position of material point $s = (7.5, 2.0)$, which is initially located in the center of the flexible portion of the upper wall, is plotted as a function of time for $N = 48$ and 96. Although there are differences in the dynamics, notice that the steady-state configurations are the same for both grid spacings.

Figure 13. Comparison between IB and ALE results for the collapsible channel benchmark. The steady-state position of the upper wall in the vicinity of the flexible region is shown as computed by the IB (grey) and ALE (black) methods. Despite differences between the two schemes, the deformations predicted by both methods are essentially identical.
In the flexible portion of the wall, we use a St. Venant-Kirchhoff elasticity model, for which
\[
W^e = \frac{\lambda^e}{2} \text{tr}(E)^2 + \mu^e \text{tr}(E^2),
\]
with \(\lambda = \frac{1}{2}(\mathbf{C} - \mathbb{I})\) and \(\mathbf{C} = F^T F\). The constitutive parameters are given by \(\lambda^e = \frac{1}{2} E^e/(1 + \nu^e)\) and \(\nu^e = (E^e \nu^e)/(1 + \nu^e)(1 - 2\nu^e)\), in which \(E^e = 71.8\) kPa is the Young’s modulus of the material and \(\nu^e = 0.45\) is the Poisson ratio. Rigidity constraints are enforced by a feedback-forcing approach. Specifically, an additional Lagrangian constraint force \(\mathbf{F}(s, t)\) is added to the elastic force density. The additional constraint force is determined via
\[
\mathbf{F}(s, t) = \kappa^e(s)(s - \chi(s, t)),
\]
in which \(\kappa^e(s) > 0\) is a penalty parameter that is set to zero within the flexible part of the channel wall. Notice that as \(\kappa^e \to \infty, \chi(s, t) \to s\).

We compare the results produced by the unified version of the present IB method to results that were obtained by an arbitrary Lagrangian-Eulerian (ALE) method [50, 51]. An important difference between the physical problems considered by the ALE method and the present IB method is that the physical system treated by ALE method neglects the inertia associated with any material exterior to the channel and instead treats the exterior as a constant-pressure reservoir. By contrast, the IB model considers the channel to be totally immersed in fluid. (In principle, neglecting the inertia of the exterior region is possible within the framework of the IB method, but doing so requires a variable-density incompressible Navier-Stokes solver. Our present implementation only provides a constant-density solver.) Consequently, the dynamics produced by the two physical models will generally be different. Despite these differences, however, both physical models admit identical steady-state equilibrium configurations. We thereby compare only the static configurations produced by the two numerical methods.

We discretize \(\Omega\) using an \(\frac{L}{H} N\)-by-\(N\) Cartesian grid with \(N = 48\) and \(96\), and we discretize each channel wall using a single layer of bilinear elements, each of length \(\Delta x\) and width \(w\) in the reference configuration. Notice that for either value of \(N, w \ll \Delta x\). We use \(\Delta t = 4.05e^{-3}\Delta x\), and in the rigid portions of the channel walls, we set \(\kappa^e = 1.0e3\Delta x/\Delta t^2\), which is approximately the largest value of \(\kappa^e\) that is stable for the physical and numerical parameters considered. We increase the driving and loading pressures from zero over a time interval of \(1\) s. The system undergoes large, oscillatory deformations during the initial part of the simulation, but by \(t = 100\) s the system has reached equilibrium. At steady state, \(Re \approx 80\). The IB method yields converged steady-state deformations for \(N = 48\); see Fig. 12. The final configurations generated by the IB and ALE methods are shown in Fig. 13. It can be seen that the deformations generated by the two methods are virtually identical. The maximum \(y\) displacements determined by the two methods also agree to within \(0.5\%\): in the IB model, the maximum \(y\) displacement of the centerline of the upper wall is \(0.3654\) cm, whereas in the ALE model, it is \(0.3641\) cm. Recall that the thickness of the beam is \(w = 0.01\) cm. Given the differences between the physical models and numerical schemes, the agreement between the IB and ALE methods is excellent.

6.4. Turek-Hron fluid-structure interaction benchmark

This section presents preliminary results obtained by our IB method for the Turek-Hron fluid-structure interaction benchmark problem [52], which considers channel flow past a structure composed of a rigid cylinder and a trailing elastic bar; see Fig. 14. We use the version of this problem at \(Re = 200\), for which the fluid properties are \(\rho = 1000\) kg/m\(^2\) and \(\mu = 1\) Pa·s, and for which the mass density of the elastic bar is the same as that of the fluid. The channel is \(\Omega = [0, L] \times [0, H]\) with \(L = 2.5\) m and \(H = 0.41\) m. No-slip and no-penetration boundary conditions (\(u \equiv 0\)) are used along the bottom (\(x_2 = 0\)) and top (\(x_2 = H\)) boundaries of \(\Omega\). A parabolic inflow profile of the form
\[
u_1(x_2, t) = 1.5U(t)(\frac{H/2 - x_2}{H/2})^2, \quad \text{with}
\]
\[
\tilde{U}(t) = (2.0 \text{ m/s}) \begin{cases} \frac{1}{2} (1 - \cos(\pi t/2)) & \text{if } t < 2, \\ 1 & \text{otherwise}, \end{cases}
\]
Figure 14. Sample results from the IB version of the Turek-HrOn fluid-structure interaction benchmark problem [52] at $Re = 200$. The fluid vorticity ranges from $-30 \text{ m/s}^2$ (blue) to $+30 \text{ m/s}^2$ (red). The structure is shown in black.

Figure 15. $x$ and $y$ displacements of the control point $A$ and lift and drag forces for the fluid-structure interaction benchmark problem for $N = 328$.

is imposed along the $x_1 = 0$ boundary, and zero-tangential-slip and zero-normal-traction boundary conditions are imposed at $x_1 = L$. 
The disc has center $C = (0.2 \text{ m}, 0.2 \text{ m})$ and radius $R = 0.05 \text{ m}$ and is held rigid via a penalty approach like that used in Sec. 6.3. The elastic bar is attached to the back of the disc and has length $l = 0.35 \text{ m}$ and width $h = 0.02 \text{ m}$. A St. Venant-Kirchhoff model is used to describe the elasticity of the bar with $\mu^e = 2.0\times10^6 \text{ MPa}$ and $\nu^e = 0.4$. A control point $A$ with initial position $(0.6 \text{ m}, 0.2 \text{ m})$ is attached to the center of the trailing edge of the bar.

There is a difference between the benchmark problem defined by Turek and Hron [52] and the present IB model: In the original problem specification, the elastic bar is compressible. Consequently, in the present work, the flexible bar is treated as an incompressible material. (Treating a compressible structure within the framework of the IB method appears to require employing a compressible Navier-Stokes solver and adopting a compressible formulation of both the fluid and the structure. Such an extension of the present formulation is beyond the scope of this work.) The extent to which this problem is sensitive to this difference has not yet been fully characterized; however, our results suggest that the differences between the compressible and incompressible versions of this benchmark problem may be relatively minor (see below).

We discretize $\Omega$ using an $\frac{L}{H}N$-by-$N$ Cartesian grid. The structure is the superposition of a circular disc and an elastic bar, which are both discretized using bilinear elements with edge lengths that are no larger than $\Delta x$. We set $\Delta t = 7.8125\times10^{-3} \Delta x$ and use $\kappa^e = 1.0\times10^3 \Delta x/\Delta t^2$, which is approximately the largest value of $\kappa^e$ that is stable for these physical and numerical parameters. Sample results obtained using the unified formulation with $N = 328$ are shown in Figs. 14 and 15. Over the time interval $[19.5 \text{ s}, 20 \text{ s}]$, the $x$ displacement of control point $A$ has mean $-2.18e-3 \text{ m}$ and amplitude $2.19e-3 \text{ m}$; the $y$ displacement has mean $1.34e-3 \text{ m}$ and amplitude $3.16e-2 \text{ m}$; the lift force has mean $2.80 \text{ N}$ and amplitude $1.84e+2 \text{ N}$; and the drag force has mean $4.47e+2 \text{ N}$ and amplitude $40.9 \text{ N}$. The frequency of the $x$ displacements of $A$ is approximately $11.0/\text{s}$, and the frequency of the $y$ displacements is approximately $5.4/\text{s}$. Although there does not yet appear to be a reference solution for this benchmark problem, these values are all within the range of comparable results obtained by various other numerical methods that have been applied to this problem [52, 53]. Notice that these earlier results all considered the elastic bar to be compressible.

### 6.5. Timing results

We conclude by presenting preliminary timing data obtained for the dynamic version of the problem of Sec. 6.1.2. We remark, however, that we have not yet fully optimized our implementation of this method. For these tests, we perform the first 64 time steps for various choices of $N$ and $M_{\text{fac}}$ using the partitioned version of the method. We also simulate the conventional IB method within our implementation by using the unified formulation, using a diagonally-lumped mass matrix, setting $M_{\text{fac}} = 0.5$, and using the trapezoidal rule to construct $S$ and $R$. We record the wall-clock time required using the C function `clock_gettime()` with the `CLOCK_REALTIME` timer. Timings were performed using a single core of a compute server equipped with four quad-core 2.3 GHz AMD Barcelona CPUs and 32 GB RAM. Results are summarized in Table I. Notice that larger values of $M_{\text{fac}}$, which correspond to `coarser` Lagrangian meshes, always yield smaller runtimes. Also notice that the Lagrangian-Eulerian interaction operators introduced in this work generally yield better performance than those of the conventional IB approach except for the finest Lagrangian meshes. These preliminary results suggest that the reductions in the number of Lagrangian degrees

<table>
<thead>
<tr>
<th>$M_{\text{fac}}$</th>
<th>0.5</th>
<th>1</th>
<th>2</th>
<th>4</th>
<th>conventional IB</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N$</td>
<td>64</td>
<td>0.368</td>
<td>0.156</td>
<td>0.100</td>
<td>0.089</td>
</tr>
<tr>
<td>128</td>
<td>1.630</td>
<td>0.649</td>
<td>0.383</td>
<td>0.318</td>
<td>1.525</td>
</tr>
<tr>
<td>256</td>
<td>6.441</td>
<td>2.953</td>
<td>1.762</td>
<td>1.416</td>
<td>6.065</td>
</tr>
</tbody>
</table>

Table I. Average wall-clock time (in seconds) per time step for the dynamic version of the problem of Sec. 6.1.2. We also provide timings for a version of this method similar to the standard IB method, in which we use a diagonally-lumped mass matrix, set $M_{\text{fac}} = 0.5$, and use the trapezoidal rule to construct $S$ and $R$. The largest value of $\kappa$ obtained using the unified formulation with $\Delta$ no larger than $\Delta$ was used.
of freedom permitted by the Lagrangian-Eulerian interaction operators introduced herein can yield significant improvements in performance. We expect that these differences would be even more pronounced in three-dimensional models.

7. CONCLUSIONS

In this paper, we have described a version of the IB method for problems of fluid-structure interaction that uses general incompressible elasticity models with unstructured FE discretizations while retaining a Cartesian grid finite difference scheme for the incompressible Navier-Stokes equations. A feature of the present method is that it uses standard methods for both the Lagrangian and Eulerian equations. In practice, it should be feasible to use this approach to couple existing structural analysis and incompressible flow codes by passing forces from the structural code to the fluid solver, and by passing velocities from the fluid solver back to the structural code. Although we only consider cases in which the elasticity of the structure is described by a hyperelastic constitutive law defined in terms of a strain-energy functional, our numerical algorithm does not rely on the availability of such an energy functional and requires only a routine to evaluate the material stress tensor.

Our method is based on a formulation of the continuous IB equations introduced by Boffi et al. [28]. We consider two different statements of these equations that each use a weak formulation of the Lagrangian equations of nonlinear elasticity. One of these formulations treats the internal and boundary stresses within the immersed elastic structure using a single, unified, volumetric elastic force density. The other partitions these stresses into an internal elastic force density supported on the interior of the structure, and a transmission elastic force density supported on fluid-structure interfaces. Both weak formulations are demonstrated to yield similar convergence rates, but the partitioned formulation is seen to yield higher accuracy for Lagrangian meshes that are relatively coarse in comparison to the background Eulerian grid. The partitioned formulation is also seen to yield superior volume conservation in comparison to the unified formulation. A limitation of the partitioned formulation is that it does not satisfy a discrete power identity that implies that energy is conserved during Lagrangian-Eulerian interaction. Such a power identity is satisfied by the unified formulation, and may be necessary to obtain an unconditionally stable implicit time-stepping scheme [54].

A key contribution of this paper is that it introduces a new approach to Lagrangian-Eulerian interaction that overcomes a longstanding limitation of the IB method, namely the requirement that the Lagrangian mesh be approximately twice as fine as the background Eulerian grid to avoid leaks at fluid-structure interfaces. Numerical examples demonstrate that our scheme permits the use of Lagrangian meshes that are at least four times as coarse as the background Eulerian grid without leak. We speculate that the Lagrangian mesh may be arbitrarily coarse with respect to the background Eulerian grid without leak as long as the numerical quadrature scheme used to discretize the Lagrangian-Eulerian interaction equations has a sufficiently dense net of quadrature points. In two spatial dimensions, our scheme has been demonstrated to reduce the number of Lagrangian degrees of freedom by a factor of at least $8^2 = 64$ in comparison to the conventional IB method for selected problems. In three spatial dimensions, we expect that we would be able to see savings on the order of $8^3 = 512$ or more in some cases. These reductions in the number of Lagrangian degrees of freedom are also shown to translate into improved wall-clock timings. In practice, whether it is possible to use a coarse structural mesh is clearly problem dependent. The advantage of the present approach is that it enables the analyst to employ spatial discretizations that are tailored to the requirements of the structural model rather than dictated by the background Eulerian discretization.

Finally, we speculate that the partitioned formulation could be useful in constructing higher-order versions of the IB method. In particular, the partitioned formulation is well suited for developing a hybrid approach in which the IB method is used to treat the volumetric internal force density, and another method is used to treat the singular transmission force density. For instance, because the transmission force density of the partitioned formulation is defined on a closed surface, it may be possible to treat it with higher-order accuracy by a version of the immersed interface method.
[55–60]. Such an extension of this method could yield a fully second-order accurate generalization of the IB method for problems, like those considered in this work, in which the immersed structure is of codimension zero with respect to the fluid.

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