Homework Set 7

1. Show that if a sequence (x_n) in $\mathbb{R}$ converges to x, then $|x_n|$ converges to $|x|$. Is the converse true?

2. (Squeeze Theorem) Suppose that the real valued sequences (x_n), (y_n) and (z_n) satisfy $x_n \leq y_n \leq z_n$ for all n and that (x_n) and (z_n) converge to a common limit $p \in \mathbb{R}$. Show that (y_n) also converges to p.

3. Let us be given a metric space (X, d) and a sequence $(x_n)_{n\in\mathbb{N}}$ in X. A point $c \in X$ is said to be a *cluster point* of the sequence (x_n) if for all $\epsilon > 0$ there are infinitely many $n \in \mathbb{N}$ such that $d(x_n, c) < \epsilon$.

   (a) Show that c is a cluster point of (x_n) if and only if there is a subsequence (x_{n_k}) that converges to c.

   (b) Show that every limit point of the set $\{x_n : n \in \mathbb{N}\}$ (i.e., the range of (x_n)) is a cluster point of (x_n).

   (c) Show by an example that a cluster point of (x_n) may not necessarily be a limit point of the range of (x_n).
   (Hint: think of a case when a cluster point exists but there are no limit points of the range. We did something like this in class.)

   (d) Show that every bounded sequence in $\mathbb{R}$ has at least one cluster point. (This is the Bolzano-Weierstrass property for sequences.)

   (e) Show that a bounded sequence in $\mathbb{R}$ is convergent if and only if it has precisely one cluster point.

4. Find all the cluster points of the following real sequences. Justify your answers.

   (a) $1, 1, 2, 1, 3, 1, 4, 1, 5, 1, 6, \ldots$. Compare with 3(e): What was essential in that statement?

   (b) $1, 1, 2, 1, 2, 3, 1, 2, 3, 4, 1, 2, 3, 4, 5, \ldots$.

   (c) $x_n = (-1)^{n^2} + \cos(\frac{\pi}{2}n) + \frac{1}{n}$.

   (d) $0, 1, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \ldots$.

   (e) $x_n = 3n \pmod 5$.