Homework Set 9

1. (a) Let (x_n) be a sequence in $\mathbb{R}$. Define the sequence (s_n) of "partial arithmetic means" by

$$s_n = \frac{x_1 + x_2 + \cdots + x_n}{n}, \qquad n = 1, 2, \ldots$$

Show that if $x_n \to a$, then $s_n \to a$.

(b) Construct a bounded sequence (x_n) for which (s_n) does not converge. (Hint: let (x_n) oscillate between two values, but for longer and longer durations.)

(c) Construct an unbounded nonnegative sequence (x_n) for which $s_n \to 0$.

2. Evaluate the limit $\lim_{n\to\infty} \left(\frac{1}{n!}\right)^{1/n}$.
(Hint: Show and use the simple bound $n! > (n/2)^{n/2}$.)

3. Find the radius of convergence of the following power series:
(a) $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ (b) $\sum_{n=1}^{\infty} \frac{2^n}{n} z^n$ (c) $\sum_{n=1}^{\infty} n^{\sqrt{n}} z^n$

4. Let (a_n) be a sequence of nonnegative real numbers such that $\sum a_n$ diverges. Show that $\sum \sqrt{a_n}$ also diverges.

5. (a) (**Abel's test**) Let (x_n) be a bounded monotonic sequence of real numbers and $\sum y_n$ be a convergent series. Show that the series $\sum x_n y_n$ is convergent.
(Hint: Say x_n is increasing. Let $x = \lim x_n$ and write $x_n y_n = -(x - x_n)y_n + x y_n$. Apply Dirichlet's test (Theorem 3.42) to the first term.)

(b) Show that the series $\sum_{n=2}^{\infty} \frac{1}{\sqrt[n]{2}} \frac{(-1)^n}{\log n}$ is convergent.
(Hint: Set $x_n = \frac{1}{\sqrt[n]{2}}$ and $y_n = \frac{(-1)^n}{\log n}$ in Abel's test. Justify this selection.)

6. (a) Show that if $\limsup |a_n| > 0$, then $\limsup |a_n|^{1/n} \geq 1$.

(b) Interpret the series $\sum_{n=0}^{\infty} z^{n^2}$ as a power series in the variable z and find its radius of convergence.