# Practice Problems for the Final Exam

**Note:** The following is not a "sample final exam"; it is rather a collection of problems that are useful to study for the final.

1. Give at least one example of a convergent series and one example of a divergent series such that the root test cannot determine the outcome.

2. Show that if $\sum a_n$ converges absolutely, then $\sum \sin(a_n)$ also converges absolutely. Is the converse true?

3. Show that the series
$$\sum_{n=1}^{\infty} \frac{(-1)^n}{1+nx^2}$$
converges for every nonzero $x$.

4. Which of the following metric spaces are complete? Justify your answer.
   (a) The set of integers with the Euclidean metric.
   (b) The set of rational numbers with the Euclidean metric.
   (c) The set of rational numbers with the discrete metric.

5. Let $X$ be a metric space, $f : X \to \mathbb{R}$ be a continuous function and $a$ be a point in $X$. Show that if $f(a) > 0$, then there exist $\delta > 0$ such that $f(x) \geq \frac{1}{2}f(a)$ for all $x \in B_\delta(a)$.

6. Let $f : [a, b] \to \mathbb{R}$ be a continuous function such that $f(x) > 0$ for all $x \in [a, b]$. Show that there exists a number $c > 0$ such that $f(x) \geq c$ for all $x \in [a, b]$. Does the same conclusion hold if we replace $[a, b]$ by $(a, b)$?

7. For each of the following, state whether the given function is uniformly continuous or not. Justify your answer.
   (a) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = e^{-x^2}$.
   (b) $f : (0, 1) \to \mathbb{R}$, $f(x) = \sqrt{x}\sin(\frac{1}{x})$.
   (c) $f : (0, 1) \to \mathbb{R}$, $f(x) = \log(x)$.

8. Suppose a continuous function $f : [0, \infty) \to \mathbb{R}$ is known to be uniformly continuous on the set $(a, \infty)$ for some $a > 0$. Show that $f$ is uniformly continuous on the whole of $[0, \infty)$.

9. Let $X$ and $Y$ be two metric spaces and $f : X \to Y$ be a uniformly continuous function. Show that if $(x_n)$ is a Cauchy sequence in $X$, then $(f(x_n))$ is a Cauchy sequence in $Y$. Show by an example that this is not necessarily the case if we do not assume the continuity of $f$ to be "uniform".

10. Let $X$ be a metric space, $\phi : X \to \mathbb{R}$ be a continuous function, and $(a_n)$ be a sequence in $X$ that converges to the point $a$. Show that

$$\lim_{n\to\infty} \frac{\phi(a_1) + \phi(a_2) + \cdots + \phi(a_n)}{n} = \phi(a).$$

11. Show that every polynomial of odd degree with real coefficients must have at least one real root. (Hint: Intermediate value theorem.)

12. Show that for any integer $n \geq 3$, the polynomial $P_n(x) = x^n - 4x^2 + x + 1$ has at least 3 roots in $(-1, \infty)$. In particular, show that $P_4(x)$ has 4 real roots. (Hint: Apply the Intermediate Value Theorem using suitable end-points.)

**Connectedness:** (to be covered)

13. Show that a metric space $X$ is disconnected if and only if it has a nonempty proper subset $A$ which is both open and closed.

14. Show that a function $f$ from a connected metric space $X$ to a discrete metric space $D$ is continuous if and only if it is constant.

15. (a) Let $X$ be a metric space and $A$ be a subset of $X$. Show that if $A$ is connected, then the closure of $A$ is also connected.

    (b) Show that if we replace "closure" by "interior", the above statement is true for $X = \mathbb{R}$, but is false in general. (Hint: You can find a counter example in $\mathbb{R}^2$.)