**Convergence and closure**

**Theorem 1:** Let $(X, d)$ be a metric space, $E \subset X$ and $(x_n)$ be a sequence in $E$. If $x_n \to x$, then $x \in \overline{E}$.

*Proof*: If $x \in E$, then there is nothing to prove since $E \subset \overline{E}$. So let us assume that $x \notin E$. The convergence of $x_n$ to $x$ implies that for all $\epsilon > 0$, there exists $n \in \mathbb{N}$ such that $x_n \in B_\epsilon(x)$. Since $x \notin E$ and $x_n \in E$, we see that $x_n \in B_\epsilon(x) \cap E \backslash \{x\}$. In other words, for all $\epsilon > 0$, we have $B_\epsilon(x) \cap E \backslash \{x\} \neq \emptyset$. This says $x \in E'$, and hence $x \in \overline{E}$.

**Theorem 2:** For all $x \in \overline{E}$, there exists a sequence $(x_n)$ in $E$ that converges to $x$.

*Proof*: If $x \in E$, just pick $x_n = x$ for all $n$. If $x \in E'$, then for each $n = 1, 2, \ldots$, let $x_n$ be an arbitrary point in the nonempty set $B_{\frac{1}{n}}(x) \cap E \backslash \{x\}$. Since $d(x_n, x) \to 0$, we have $x_n \to x$.

**Theorem 1 + Theorem 2:** $x \in \overline{E}$ if and only if there exists a sequence in $E$ that converges to $x$.

**Theorem 3:** $E$ is closed if and only if the limit of every convergent sequence in $E$ belongs to $E$.

*Proof*: $(\Rightarrow)$ Let $E$ be closed, $(x_n)$ be a sequence in $E$, and $x_n \to x$. Theorem 1 implies that $x \in \overline{E}$. Since $E = \overline{E}$, $x \in E$.

$(\Leftarrow)$ Let $x$ be an arbitrary element in $\overline{E}$. By Theorem 2, there is a sequence $(x_n)$ in $E$ whose limit is $x$. By the assumption, the limit of $(x_n)$ belongs to $E$, therefore $x \in E$. Since $x$ is arbitrary, this implies $\overline{E} \subset E$, hence $E = \overline{E}$.

**Moral of the story:** Taking the closure of a set $E$ is the same as adding to $E$ all the points in $X$ that are limits of (convergent) sequences in $E$.