A BDDC ALGORITHM FOR MORTAR DISCRETIZATION OF ELASTICITY PROBLEMS *

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Abstract. A BDDC (balancing domain decomposition by constraints) algorithm is developed for elasticity problems in three dimensions with mortar discretization on geometrically nonconforming subdomain partitions. Coarse basis functions in the BDDC algorithm are constructed from primal constraints on faces. These constrains are similar to the average matching condition and the moment matching condition on common faces or edges considered in [10, 7]. A condition number bound is proved to be $C(1+\log(H/h))^3$ for geometrically non-conforming partitions as well as to be $C(1+\log(H/h))^2$ for geometrically conforming partitions.

Key words. BDDC algorithm, mortar discretization, preconditioner, elasticity problems

AMS subject classifications. 65N30, 65N55

1. Introduction.

This study is intended for development of an efficient BDDC algorithm for solving mortar discretization of elasticity problems. The mortar discretization is applied to geometrically non-conforming subdomain partitions.

BDDC algorithms have been studied extensively in [16, 17, 13, 14, 15, 8] since its introduction by Dohrmann [4]. Coarse component of BDDC preconditioners is built from weighted sum of functions that minimize discrete local energies with certain primal constraints on subdomain interfaces. The BDDC algorithms solve linear systems of primal unknowns in contrast to FETI-DP algorithms, that have been applied successfully to solving partial differential equations during the last decade [5, 11, 12, 10, 7]. These features make the BDDC algorithms more robust and more flexible than the FETI-DP algorithms and BDD (Balancing Domain Decomposition) algorithms; see [17, 15]. On the other hand, the BDDC and FETI-DP algorithms have much in common. They share the same spectra except the eigenvalue 1 when the same set of primal constraints is chosen; see [16, 13]

Both of these methods have been applied to solving problems arising from mortar discretizations. In a recent study by the author jointly with Dryja and Widlund [8], a BDDC algorithm is developed for elliptic problems. This algorithm was shown to share the same spectra with the FETI-DP algorithm considered in [9] by the author and Lee. In addition, it is generalized to the geometrically non-conforming partitions with a slightly weaker condition number bound, $C(1+\log(H_i/h_i))^3$.

The FETI-DP algorithm in [9] has been extended to solving elasticity problems by the author [7]. However, the work is limited to geometrically conforming partitions. This study

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aims at developing a BDDC algorithm for the elasticity problems that allows the geometrically non-conforming partitions as well as shares the same spectra with the FETI-DP algorithm in [7].

In elasticity problems, a special difficulty arises in building certain functionals dual to rigid body motions. These functionals are important in obtaining a condition number bound independent of the number of subdomains. The most important ingredient of this study is a new set of primal constraints that makes it possible to construct such functionals. These constraints are different from those considered in [8] for elliptic problems on geometrically nonconforming partitions. In a geometrically non-conforming partition, any nonmortar interface $F \subset \partial\Omega_i$ may have a partition $\{F_{ij}\}_j$ by its mortar neighbors Ω_j such that $F_{ij} = \partial\Omega_i \cap \partial\Omega_j$. In [8], the primal constraints are applied to each F_{ij}. In this paper, however, we apply the new primal constraints to each nonmortar face F_l.

Using these new primal constraints we could generalize the Poincaré inequality proved for functions in the space; see Brenner [3],

$$v = (v_1, \cdots, v_N) \in \prod_{i=1}^{N} H(\Omega_i), \ \int_{F_{ij}} (v_i - v_j)\, ds = 0,$$

to functions in a more general space,

$$v = (v_1, \cdots, v_N) \in \prod_{i=1}^{N} H(\Omega_i), \ \int_{F} (v_i - \phi)\, ds = 0,$$

where $F \subset \partial\Omega_i$ is any nonmortar interface and $\{F_{ij}\}_j$ is the partition of F by its mortar neighbors Ω_j and $\phi = v_j$ on F_{ij}, for all j. In order to satisfy

$$\int_{F_{ij}} (v_i - v_j)\, ds = 0,$$

we need certain assumptions on meshes and Lagrange multiplier spaces for the geometrically nonconforming case.

This paper is organized as follows. In Section 2, we introduce a model compressible elasticity problem. In Section 3, a BDDC algorithm is developed for mortar discretization of the elasticity problem employing a coarse component that comes from the primal constraints across subdomain interfaces. Section 4 is devoted to analyzing the condition number of the BDDC algorithm.

Throughout this paper, c and C denote generic positive constants independent of mesh parameters, the number of subdomains, and coefficients of the elasticity problems. We will use h_i and H_i to denote the mesh size and the subdomain size of Ω_i, respectively.

2. A model problem. Let Ω be a polyhedral domain in $\mathbf{R}^3$. The Sobolev space $H^1(\Omega)$ is the set of functions in $L^2(\Omega)$ that are square integrable up to first weak derivatives and it is equipped with the standard Sobolev norm;

$$\|v\|_{1,\Omega}^2 := |v|_{1,\Omega}^2 + \|v\|_{0,\Omega}^2,$$

where $|v|_{1,\Omega}^2 = \int_\Omega \nabla v \cdot \nabla v \, dx$ and $\|v\|_{0,\Omega}^2 = \int_\Omega v^2 \, dx$.

We assume that $\partial\Omega$ is divided into two parts $\partial\Omega_D$ and $\partial\Omega_N$ on which a Dirichlet boundary condition and a natural boundary condition are specified, respectively. The subspace $H_D^1(\Omega) \subset H^1(\Omega)$ is a set of functions having zero trace on $\partial\Omega_D$. We introduce the vector valued Sobolev spaces, $[H_D^1(\Omega)]^3$ and $[H^1(\Omega)]^3$, equipped with the usual product norm.

We then consider the elasticity problem:

find $\mathbf{u} \in [H_D^1(\Omega)]^3$ such that

$$\int_\Omega G(\mathbf{x})\varepsilon(\mathbf{u}) : \varepsilon(\mathbf{v}) \, d\mathbf{x} + \int_\Omega G(\mathbf{x})\beta(\mathbf{x})\nabla \cdot \mathbf{u} \nabla \cdot \mathbf{v} \, d\mathbf{x} = \langle \mathbf{F}, \mathbf{v} \rangle \quad \forall \mathbf{v} \in [H_D^1(\Omega)]^3, \tag{2.1}$$

where $G = E/(1+\nu)$ and $\beta = \nu/(1-2\nu)$ are material parameters depending on the Young's modulus $E > 0$ and the Poisson ratio $\nu \in (0, 1/2]$. We assume that ν is bounded away from $1/2$ so that we exclude the case of incompressible elasticity problems. The linearized strain tensor is defined by

$$\varepsilon(\mathbf{u})_{ij} := \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \quad i, j = 1, 2, 3,$$

and the tensor product and the force term are given by

$$\varepsilon(\mathbf{u}) : \varepsilon(\mathbf{v}) = \sum_{i,j=1}^{3} \varepsilon_{ij}(\mathbf{u})\varepsilon_{ij}(\mathbf{v}), \quad \langle \mathbf{F}, \mathbf{v} \rangle = \int_\Omega \mathbf{f} \cdot \mathbf{v} \, d\mathbf{x} + \int_{\partial\Omega_N} \mathbf{g} \cdot \mathbf{v} d\sigma.$$

Here $\mathbf{f}$ is the body force and $\mathbf{g}$ is the surface force on the natural boundary part $\partial\Omega_N$.

The space $\mathbf{ker}(\varepsilon)$ has the following six rigid body motions as its bases, which are three translations

$$\mathbf{r}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \ \mathbf{r}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \ \mathbf{r}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \tag{2.2}$$

and three rotations

$$\mathbf{r}_4 = \frac{1}{H}\begin{pmatrix} x_2 - \widehat{x}_2 \\ -x_1 + \widehat{x}_1 \\ 0 \end{pmatrix}, \ \mathbf{r}_5 = \frac{1}{H}\begin{pmatrix} -x_3 + \widehat{x}_3 \\ 0 \\ x_1 - \widehat{x}_1 \end{pmatrix}, \ \mathbf{r}_6 = \frac{1}{H}\begin{pmatrix} 0 \\ x_3 - \widehat{x}_3 \\ -x_2 + \widehat{x}_2 \end{pmatrix}. \tag{2.3}$$

Here $\widehat{\mathbf{x}} = (\widehat{x}_1, \widehat{x}_2, \widehat{x}_3) \in \Omega$ and H is the diameter of Ω. This shift and the scaling make the L_2-norm of the six vectors scale in the same way with H.

When Ω is partitioned into a set of subdomains, the elasticity problem given on a floating subdomain has purely natural boundary condition. The Korn inequalities provided in Section

2 of [10] concern this case. In the following, Σ is any open subset of $\partial\Omega$ with positive measure and its diameter is comparable to that of Ω. Let $(\mathbf{u},\mathbf{r})_\Sigma$ be a L_2-inner product given by

$$(\mathbf{u},\mathbf{r})_\Sigma = \int_\Sigma \mathbf{u}\cdot\mathbf{r}\, ds.$$

We introduce two semi-norms provided for the space $[H^{1/2}(\Sigma)]^3$, which is a trace space of $[H^1(\Omega)]^3$,

$$|\mathbf{u}|_{1/2,\Sigma} := \inf_{\substack{\mathbf{v}\,\in\,\mathbf{H}^1(\Omega)\\ \mathbf{v}|_\Sigma\,=\,\mathbf{u}}} |\mathbf{v}|_{1,\Omega}, \quad |\mathbf{u}|_{E(\Sigma)} := \inf_{\substack{\mathbf{v}\,\in\,\mathbf{H}^1(\Omega)\\ \mathbf{v}|_\Sigma\,=\,\mathbf{u}}} \|\varepsilon(\mathbf{v})\|_{0,\Omega}.$$

The following Korn inequality is provided in [10, Lemma 6]:

LEMMA 2.1. *Let Ω be a Lipschitz domain and Σ be a subset of $\partial\Omega$ with positive measure. Then there exists a constant $c>0$ such that*

$$c|\mathbf{u}|_{1/2,\Sigma} \le |\mathbf{u}|_{E(\Sigma)} \le |\mathbf{u}|_{1/2,\Sigma},$$

for $\mathbf{u}\in[H^{1/2}(\Sigma)]^3$ satisfying $(\mathbf{u},\mathbf{r})_\Sigma = 0 \quad \forall \mathbf{r}\in \textbf{ker}(\varepsilon)$.

Another important inequality can be found in [10, Lemma 7]:

LEMMA 2.2. *Let Ω be a Lipschitz domain of the diameter H and $\Sigma\subset\partial\Omega$ be an open subset with positive measure. Then there exists a constant $C>0$ such that*

$$\inf_{\mathbf{r}\in\textbf{ker}(\varepsilon)} \|\mathbf{u}-\mathbf{r}\|^2_{0,\Sigma} \le CH|\mathbf{u}|^2_{E(\Sigma)} \quad \forall \mathbf{u}\in[H^{1/2}(\Sigma)]^3.$$

We note that the infimum above occurs when $\mathbf{u}-\mathbf{r}$ satisfies

$$(\mathbf{u}-\mathbf{r},\mathbf{s})_\Sigma = 0, \quad \forall \mathbf{s}\in\textbf{ker}(\varepsilon).$$

3. BDDC formulation for elasticity.

3.1. Mortar discretization.

We divide the domain Ω into a geometrically non-conforming partition $\{\Omega_i\}_{i=1}^N$, that is shape regular. We consider a compressible elasticity problem with coefficients $G(\mathbf{x})$ and $\beta(\mathbf{x})$ positive constants in each subdomain

$$G(\mathbf{x})|_{\Omega_i} = G_i, \quad \beta(\mathbf{x})|_{\Omega_i} = \beta_i.$$

The conforming P_1-finite element space $\mathbf{X}_i$ is associated to a quasi-uniform triangulation T_i of each subdomain Ω_i. In addition, functions in the space $\mathbf{X}_i$ satisfy the Dirichlet boundary condition on $\partial\Omega_i\cap\partial\Omega_D$. The triangulations $\{T_i\}_{i=1}^N$ may not match across the subdomain interfaces. We denote by $\mathbf{W}_i$ the trace space of $\mathbf{X}_i$ on $\partial\Omega_i$ and define the spaces

$$\mathbf{X} = \prod_{i=1}^N \mathbf{X}_i, \quad \mathbf{W} = \prod_{i=1}^N \mathbf{W}_i,$$

that have functions discontinuous across subdomain interfaces.

In three dimensions, a pair of subdomains can have a face, an edge, or a vertex in common. We will consider only the common faces as the interfaces of subdomains. In a geometrically non-conforming partition, a common face can be only a part of a subdomain face. The union of entire interfaces is denoted by

$$\Gamma = \left(\bigcup_{ij} \partial\Omega_i \cap \partial\Omega_j \right) \setminus \partial\Omega.$$

Among the interfaces, we select nonmortar faces F_l for which

$$\bigcup_l \overline{F}_l = \overline{\Gamma}, \quad F_l \cap F_k = \emptyset,\ l \neq k.$$

Since the subdomain partition can be geometrically non-conforming, a single nonmortar face $F_l \subset \partial\Omega_i$ may intersect several subdomain boundaries $\partial\Omega_j$. This provides F_l with a partition

$$\overline{F}_l = \bigcup_j \overline{F}_{ij}, \quad F_{ij} = \partial\Omega_i \cap \partial\Omega_j.$$

A dual or a standard Lagrange multiplier space $\mathbf{M}_l$ is given for each nonmortar face F_l. The space $\mathbf{M}_l$ is required to have the same dimension as the space

$$\overset{\circ}{\mathbf{W}}(F_l) := \mathbf{W}_i|_{F_l} \cap [H_0^1(F_l)]^3, \tag{3.1}$$

and to contain constant functions. Constructions of such Lagrange multiplier spaces have been studied in [1, 2, 18, 6]. On a nonmortar face $F_l \subset \partial\Omega_i$, a function $\boldsymbol{\phi}$, that is provided from the mortar neighbors of F_l, is given by

$$\boldsymbol{\phi} = \mathbf{v}_j \text{ on } F_{ij}, \quad \forall j,$$

where $\mathbf{v} = (\mathbf{v}_1, \cdots, \mathbf{v}_N) \in \mathbf{X}$. The mortar matching condition on $\mathbf{v} \in \mathbf{X}$ is

$$\int_{F_l} (\mathbf{v}_i - \boldsymbol{\phi}) \cdot \boldsymbol{\lambda}\, ds = 0, \quad \forall \boldsymbol{\lambda} \in \mathbf{M}_l,\ \forall F_l. \tag{3.2}$$

The mortar discretization of the problem (2.1) is to approximate the solution by Galerkin's method in the mortar finite element space

$$\widehat{\mathbf{X}} := \{ \mathbf{v} \in \mathbf{X} \ :\ \mathbf{v} \text{ satisfies the mortar matching condition (3.2)} \}.$$

3.2. Primal constraints and change of variables. We will first select certain primal constraints on functions in $\mathbf{X}$ (or $\mathbf{W}$) from the mortar matching condition (3.2) to construct coarse basis functions. These basis functions will be used to build a preconditioner in our BDDC algorithm. In addition, we introduce a change of variables (bases) for the unknowns (functions) in the space $\mathbf{W}$ based on the primal constraints.

Selection of the primal constraints is based on the study [10] by Klawonn and Widlund, or the study [7] by the author. On a nonmortar face F_l, we consider the rigid body motions $\{\mathbf{r}_k\}_{k=1}^6$ as in (2.2) and (2.3), where H is the diameter of the face F_l and $\widehat{\mathbf{x}}$ is a point in F_l. We define a projection $\mathbf{P}_l : [L^2(F_l)]^3 \to \mathbf{M}_l$ such that

$$\int_{F_l} (\mathbf{w} - \mathbf{P}_l(\mathbf{w})) \cdot \boldsymbol{\psi} \, ds = 0, \ \forall \ \boldsymbol{\psi} \in \mathbf{M}_l. \tag{3.3}$$

By using the projected rigid body motions $\{\mathbf{P}_l(\mathbf{r}_k)\}_{k=1}^6$, on each nonmortar face $F_l \subset \partial\Omega_i$ we select the following six primal constraints from (3.2):

$$\int_{F_l} (\mathbf{w}_i - \boldsymbol{\phi}) \cdot \mathbf{P}_l(\mathbf{r}_k) \, ds = 0 \quad \forall k = 1, \cdots, 6, \quad \forall F_l, \tag{3.4}$$

where $\boldsymbol{\phi} = \mathbf{w}_j$ on $F_{ij} \ (= \partial\Omega_i \cap \partial\Omega_j) \subset F_l$. Here we considered $\mathbf{w} = (\mathbf{w}_1, \cdots, \mathbf{w}_N) \in \mathbf{W}$ instead of $\mathbf{v} \in \mathbf{X}$.

In geometrically conforming partitions, i.e., F_l is a full face of two subdomains, the above constraints with $\{\mathbf{P}_l(\mathbf{r}_k)\}_{k=1}^3$ are face average matching condition because $\mathbf{P}_l(\mathbf{r}_k) = \mathbf{r}_k$, $k = 1, 2, 3$. The remaining constraints with $\{\mathbf{P}_l(\mathbf{r}_k)\}_{k=4}^6$, are similar to the moment matching constraints which were introduced for fully primal edges in [10]. We call the constraints with $\{\mathbf{P}_l(\mathbf{r}_k)\}_{k=1}^3$ the average constraints and call the constraints with $\{\mathbf{P}_l(\mathbf{r}_k)\}_{k=4}^6$ the moment constraints.

We now introduce a change of variables based on the primal constraints; see Li and Widlund [13] or Kim, Dryja and Widlund [8]. The change of variables leads to much simpler analysis of the BDDC algorithms.

On a nonmortar face $F_l \subset \partial\Omega_i$, a transform T_l will be given for $\mathbf{w}_i \in \mathbf{W}_i$ so as to

$$\mathbf{w}_i|_{F_l} = T_l \begin{pmatrix} \widehat{\mathbf{w}}_{\Delta,i} \\ \widehat{\mathbf{w}}_{\Pi,i} \end{pmatrix}, \tag{3.5}$$

with the unknowns $\widehat{\mathbf{w}}_{\Pi,i}$ of six components

$$\frac{\int_{F_l} \mathbf{w}_i \cdot \mathbf{P}_l(\mathbf{r}_k) \, ds}{\int_{F_l} \mathbf{P}_l(\mathbf{r}_k) \cdot \mathbf{P}_l(\mathbf{r}_k) \, ds}, \ k = 1, \cdots, 6,$$

and $\widehat{\mathbf{w}}_{\Delta,i}$ giving the value of six components zero, i.e.,

$$\int_{F_l} \mathbf{w}_{\Delta,i} \cdot \mathbf{P}_l(\mathbf{r}_k) \, ds = 0, \ k = 1, \cdots, 6, \ \text{ with } \mathbf{w}_{\Delta,i} = T_l \begin{pmatrix} \widehat{\mathbf{w}}_{\Delta,i} \\ 0 \end{pmatrix}.$$

We call $\widehat{\mathbf{w}}_{\Pi,i}$ primal variables. We further make the transform change only unknowns of which nodal basis are supported in the face F_l so that it retains the remaining unknowns. Therefore the transform can be applied to each nonmortar face F_l independently; see [8, Section 2.2].

We now consider a change of variables on mortar neighbors of F_l. The nonmortar face F_l is partitioned into $\{F_{ij}\}_j$, where $F_{ij} = \partial\Omega_i \cap \partial\Omega_j$. On each interface F_{ij} we introduce primal variables $\widehat{\mathbf{w}}_{\Pi,ij}$ of six components

$$\frac{\int_{F_{ij}} \mathbf{w}_j \cdot \mathbf{P}_l(\mathbf{r}_k)\, ds}{\int_{F_{ij}} \mathbf{P}_l(\mathbf{r}_k) \cdot \mathbf{P}_l(\mathbf{r}_k)\, ds},\ k = 1, \cdots, 6.$$

Similarly as before, we take a transform T_{ij} such that

$$\mathbf{w}_j|_{F_{ij}} = T_{ij} \begin{pmatrix} \widehat{\mathbf{w}}_{\Delta,ij} \\ \widehat{\mathbf{w}}_{\Pi,ij} \end{pmatrix},$$

with $\widehat{\mathbf{w}}_{\Delta,ij}$ giving the following values zero

$$\int_{F_{ij}} \mathbf{w}_{\Delta,ij} \cdot \mathbf{P}_l(\mathbf{r}_k)\, ds = 0,\ k = 1, \cdots, 6,\ \text{with } \mathbf{w}_{\Delta,ij} = T_{ij} \begin{pmatrix} \widehat{\mathbf{w}}_{\Delta,ij} \\ 0 \end{pmatrix},$$

and the transform retains unknowns of which nodal basis is not supported in F_{ij}.

The primal constraints (3.4) on F_l are then written into

$$(\widehat{\mathbf{w}}_{\Pi,i})_k = \sum_j A_{jk} (\widehat{\mathbf{w}}_{\Pi,ij})_k, \quad k = 1, \cdots, 6, \tag{3.6}$$

where

$$A_{jk} = \frac{\int_{F_{ij}} \mathbf{P}_l(\mathbf{r}_k) \cdot \mathbf{P}_l(\mathbf{r}_k)\, ds}{\int_{F_l} \mathbf{P}_l(\mathbf{r}_k) \cdot \mathbf{P}_l(\mathbf{r}_k)\, ds},$$

and $(\mathbf{v})_k$ denotes the k-th component of the vector $\mathbf{v}$.

REMARK 3.1. *We note that the primal constraints* (3.6) *are new. In the previous work on elliptic problems [8], primal constraints are given in every piece F_{ij} of F_l*

$$\int_{F_{ij}} (w_i - w_j)\, \psi_{ij}\, ds,$$

where ψ_{ij} is the sum of bases of M_l supported in $\overline{F}_{ij}$. Note that $\psi_{ij} = 1$ for the geometrically conforming partitions.

REMARK 3.2. *Instead of the projection $\mathbf{P}_l$, we might be able to introduce the projection $\mathbf{P}_{ij} : [L^2(F_{ij})]^3 \rightarrow \mathbf{M}_{ij}$, where $\mathbf{M}_{ij}$ is the space with its bases $\boldsymbol{\psi} \in \mathbf{M}_l$ supported in $\overline{F}_{ij}$. Correspondingly, we might select primal constraints similar to those in [8] such that*

$$\int_{F_{ij}} (\mathbf{w}_i - \mathbf{w}_j) \cdot \mathbf{P}_{ij}(\mathbf{r}_k)\, ds = 0.$$

However, the above constraints use more primal variables on the nonmortar face F_l than those in (3.6)*. In addition, the primal constraints in* (3.4) *(or in* (3.6) *) make it possible to build dual functionals that we will use to control rigid body motions of elasticity problems in our condition number analysis.*

These transforms can be applied independently to each nonmortar face F_l and each mortar face F_{ij}. After applying the change of variables, we write the unknowns in each subdomain boundary as

$$\widehat{\mathbf{w}}_i = \begin{pmatrix} \widehat{\mathbf{w}}_\Delta^{(i)} \\ \widehat{\mathbf{w}}_\Pi^{(i)} \end{pmatrix},$$

where the subscript Π denotes the primal variables. To simplify notations, we will omit the hat notation for the transformed unknowns. Accordingly the space $\mathbf{W}_i$ can be decomposed into

$$\mathbf{W}_i = \mathbf{W}_\Pi^{(i)} \times \mathbf{W}_\Delta^{(i)}.$$

Here the space $\mathbf{W}_\Pi^{(i)}$ consists of primal variables in Ω_i and $\mathbf{W}_\Delta^{(i)}$ is the space of the remaining unknowns. Similarly, after changing variables (bases), the Schur complement matrix S_i and the Schur complement forcing vector $\mathbf{g}_i$ of the local elasticity problem can be written into

$$S_i = \begin{pmatrix} S_{\Delta\Delta}^{(i)} & S_{\Delta\Pi}^{(i)} \\ S_{\Pi\Delta}^{(i)} & S_{\Pi\Pi}^{(i)} \end{pmatrix}, \quad \mathbf{g}_i = \begin{pmatrix} \mathbf{g}_\Delta^{(i)} \\ \mathbf{g}_\Pi^{(i)} \end{pmatrix}.$$

We introduce the space $\widetilde{\mathbf{W}}$ of functions satisfying the primal constraints,

$$\widetilde{\mathbf{W}} = \{\mathbf{w} \in \mathbf{W} \ : \ \mathbf{w} \text{ satisfies the primal constraints (3.4)}\}.$$

In the following, we derive a matrix representation of the mortar matching condition (3.2) on functions in this space.

We note that the primal variables of the nonmortar face F_l can be obtained from the primal variables of its mortar faces F_{ij}; see (3.6). We denote by $\mathbf{W}_\Pi$ the space of primal variables of all mortar faces and by $\mathbf{W}_\Delta$ the space of unknowns other than the primal variables, i.e.,

$$\mathbf{W}_\Delta = \prod_{i=1}^{N} \mathbf{W}_\Delta^{(i)}.$$

From the relation (3.6), we can define the mapping

$$R_\Pi^{(i)} \ : \ \mathbf{W}_\Pi \rightarrow \mathbf{W}_\Pi^{(i)}. \tag{3.7}$$

The matrix $R_\Pi^{(i)}$ is not a boolean matrix for the geometrically non-conforming case while it is a boolean matrix for the geometrically conforming case. Therefore, we can express unknowns in $\widetilde{\mathbf{W}}$ by using unknowns in the spaces $\mathbf{W}_\Pi$ and $\mathbf{W}_\Delta$, and the mortar matching condition (3.2) on $\mathbf{w} \in \widetilde{\mathbf{W}}$ can be written into

$$B_\Delta \mathbf{w}_\Delta + B_\Pi \mathbf{w}_\Pi = 0. \tag{3.8}$$

Since $\mathbf{w} \in \widetilde{\mathbf{W}}$ satisfies the primal constraints (3.4) that are selected from (3.2), the above constraints (3.8) are redundant on $\mathbf{w} \in \widetilde{\mathbf{W}}$. We can make them non-redundant after deleting six equations from (3.8) corresponding to each nonmortar faces F_l. We will use the same notation (3.8) for the non-redundant mortar matching constraints.

We further decompose

$$\mathbf{W}_\Delta = \mathbf{W}_{\Delta,n} \times \mathbf{W}_{\Delta,m},$$

where n and m denote unknowns at nonmortar faces (interior) and the remaining unknowns, respectively. The condition (3.8) is written into

$$B_n \mathbf{w}_n + B_m \mathbf{w}_m + B_\Pi \mathbf{w}_\Pi = 0, \tag{3.9}$$

We note that B_n is square and invertible, since the above equations are non-redundant and the number of unknowns $\mathbf{w}_n$ is equal to the number of the equations in (3.9).

3.3. A BDDC algorithm. We now formulate a BDDC operator for the mortar discretization of the elasticity problem (2.1). Since the matrix B_n in (3.9) is invertible,

$$\mathbf{w}_n = -B_n^{-1}(B_m \mathbf{w}_m + B_\Pi \mathbf{w}_\Pi).$$

We then define the matrix

$$R_\Gamma = \begin{pmatrix} -B_n^{-1}B_m & -B_n^{-1}B_\Pi \\ I & 0 \\ 0 & I \end{pmatrix}, \tag{3.10}$$

which maps the unknowns $(\mathbf{w}_m^t, \mathbf{w}_\Pi^t)^t$ to the unknowns $(\mathbf{w}_n^t, \mathbf{w}_m^t, \mathbf{w}_\Pi^t)^t$ satisfying the mortar matching condition (3.9). The mortar finite element space $\widehat{\mathbf{W}}$ can be described as

$$\widehat{\mathbf{W}} = \{\mathbf{w} = (\mathbf{w}_n, \mathbf{w}_m, \mathbf{w}_\Pi) \ : \ \mathbf{w} \text{ satisfies (3.9)}\}.$$

Here $\mathbf{w}_n$, $\mathbf{w}_m$, and $\mathbf{w}_\Pi$ are unknowns in the spaces $\mathbf{W}_{\Delta,n}$, $\mathbf{W}_{\Delta,m}$, and $\mathbf{W}_\Pi$, respectively. In addition, we express unknowns $(\mathbf{w}_n, \mathbf{w}_m, \mathbf{w}_\Pi)$ in the space $\widehat{\mathbf{W}}$ by using the matrix R_Γ and the unknowns $(\mathbf{w}_m, \mathbf{w}_\Pi)$,

$$\begin{pmatrix} \mathbf{w}_n \\ \mathbf{w}_m \\ \mathbf{w}_\Pi \end{pmatrix} = R_\Gamma \begin{pmatrix} \mathbf{w}_m \\ \mathbf{w}_\Pi \end{pmatrix}.$$

The mortar discretization of the elasticity problem is then given by

$$R_\Gamma^t \widetilde{S} R_\Gamma \begin{pmatrix} \mathbf{w}_m \\ \mathbf{w}_\Pi \end{pmatrix} = R_\Gamma^t \begin{pmatrix} \mathbf{g}_m \\ \mathbf{g}_\Pi \end{pmatrix}, \tag{3.11}$$

where

$$\widetilde{S} = \begin{pmatrix} S_{\Delta\Delta} & S_{\Delta\Pi} \\ S_{\Pi\Delta} & S_{\Pi\Pi} \end{pmatrix}, \quad \mathbf{g}_m = \begin{pmatrix} \mathbf{g}_m^{(1)} \\ \vdots \\ \mathbf{g}_m^{(N)} \end{pmatrix}, \quad \mathbf{g}_\Pi = \sum_i (R_\Pi^{(i)})^t \mathbf{g}_\Pi^{(i)}, \tag{3.12}$$

with $\mathbf{g}_m^{(i)}$, the part of $\mathbf{g}_\Delta^{(i)}$ to the unknowns other than the nonmortar interior, and

$$\begin{aligned} S_{\Delta\Delta} &= \operatorname{diag}_i \left(S_{\Delta\Delta}^{(i)} \right), \\ S_{\Pi\Delta} &= \sum_i (R_\Pi^{(i)})^t S_{\Pi\Delta}^{(i)}, \quad S_{\Delta\Pi} = S_{\Pi\Delta}^t, \\ S_{\Pi\Pi} &= \sum_i (R_\Pi^{(i)})^t S_{\Pi\Pi}^{(i)} R_\Pi^{(i)}. \end{aligned} \tag{3.13}$$

We note that $R_\Pi^{(i)}$ is the mapping defined in (3.7).

Our BDDC algorithm aims at solving (3.11) efficiently by PCGM (preconditioned conjugate gradient method) with an appropriate preconditioner. The preconditioner will use local problems in Ω_i and a coarse problem in Ω as its building blocks.

The coarse problem is obtained from the coarse finite element space, that will be constructed based on the primal constraints; see (3.4) or (3.6). In each subdomain, we solve the following problem

$$\begin{pmatrix} S_{\Delta\Delta}^{(i)} & S_{\Delta\Pi}^{(i)} \\ S_{\Pi\Delta}^{(i)} & S_{\Pi\Pi}^{(i)} \end{pmatrix} \begin{pmatrix} \Psi_\Delta^{(i)} \\ I_\Pi^{(i)} \end{pmatrix} = \begin{pmatrix} 0 \\ F_{\Pi\Pi}^{(i)} I_\Pi^{(i)} \end{pmatrix}, \tag{3.14}$$

where the matrix $I_\Pi^{(i)}$ is the identity matrix of its size equal to the number of primal variables in Ω_i. We then obtain

$$\Psi^{(i)} = \begin{pmatrix} \Psi_\Delta^{(i)} \\ I_\Pi^{(i)} \end{pmatrix} = \begin{pmatrix} -(S_{\Delta\Delta}^{(i)})^{-1} S_{\Delta\Pi}^{(i)} I_\Pi^{(i)} \\ I_\Pi^{(i)} \end{pmatrix} \tag{3.15}$$

and also

$$F_{\Pi\Pi}^{(i)} = S_{\Pi\Pi}^{(i)} - S_{\Pi\Delta}^{(i)} {S_{\Delta\Delta}^{(i)}}^{-1} S_{\Delta\Pi}^{(i)}. \tag{3.16}$$

The coarse finite element space is spanned by the columns of the matrix Ψ,

$$\Psi = \begin{pmatrix} \Psi^{(1)} R_\Pi^{(1)} \\ \vdots \\ \Psi^{(N)} R_\Pi^{(N)} \end{pmatrix}. \tag{3.17}$$

Each column $\boldsymbol{\psi}$ of the matrix Ψ is related to one primal unknown in $\mathbf{W}_\Pi$. Since $\boldsymbol{\psi} \in \mathbf{W}$ satisfies the primal constraints (3.6), we take $\overline{\boldsymbol{\psi}} = (\boldsymbol{\psi}_\Delta^t, \boldsymbol{\psi}_\Pi^t)^t$ from the vector $\boldsymbol{\psi}$ so that $\boldsymbol{\psi}_\Delta \in \mathbf{W}_\Delta$ and $\boldsymbol{\psi}_\Pi \in \mathbf{W}_\Pi$. The matrix $\overline{\Psi}$ with the columns $\overline{\boldsymbol{\psi}}$ is then expressed by

$$\overline{\Psi} = R_\Pi^t - \sum_{i=1}^N (R_\Delta^{(i)})^t (S_{\Delta\Delta}^{(i)})^{-1} S_{\Delta\Pi}^{(i)} R_\Pi^{(i)}, \tag{3.18}$$

where $R_{\Pi}^{t} : \mathbf{W}_{\Pi} \to \mathbf{W}_{\Delta} \times \mathbf{W}_{\Pi}$ and $(R_{\Delta}^{(i)})^{t} : \mathbf{W}_{\Delta}^{(i)} \to \mathbf{W}_{\Delta} \times \mathbf{W}_{\Pi}$ are zero extensions. Note that $\mathbf{W}_{\Delta} = \prod_{i=1}^{N} \mathbf{W}_{\Delta}^{(i)}$.

We introduce

$$R_{D,\Gamma} = \begin{pmatrix} D_{nn} & & \\ & D_{mm} & \\ & & D_{\Pi\Pi} \end{pmatrix} R_{\Gamma}, \tag{3.19}$$

where the matrices D_{nn}, D_{mm} and $D_{\Pi\Pi}$ will be specified later. We then propose the following preconditioner M^{-1} for the problem (3.11),

$$M^{-1} = R_{D,\Gamma}^{t} \left\{ \begin{pmatrix} S_{\Delta\Delta}^{-1} & 0 \\ 0 & 0 \end{pmatrix} + \overline{\Psi}(\Psi^{t} S \Psi)^{-1} \overline{\Psi}^{t} \right\} R_{D,\Gamma}, \tag{3.20}$$

where

$$S = \text{diag}_i \left(S_i \right), \quad S_i = \begin{pmatrix} S_{\Delta\Delta}^{(i)} & S_{\Delta\Pi}^{(i)} \\ S_{\Pi\Delta} & S_{\Pi\Pi} \end{pmatrix},$$

and $S_{\Delta\Delta}$ is given in (3.13).

We will now express (3.20) in a simpler form. From the definition of Ψ in (3.17), we have

$$\Psi^{t} S \Psi = \sum_{i=1}^{N} (R_{\Pi}^{(i)})^{t} (\Psi^{(i)})^{t} S^{(i)} \Psi^{(i)} R_{\Pi}^{(i)}.$$

From (3.14)-(3.17), we obtain

$$\Psi^{t} S \Psi = \sum_{i=1}^{N} (R_{\Pi}^{(i)})^{t} F_{\Pi\Pi}^{(i)} R_{\Pi}^{(i)} = F_{\Pi\Pi}, \tag{3.21}$$

where

$$F_{\Pi\Pi} = S_{\Pi\Pi} - S_{\Pi\Delta} S_{\Delta\Delta}^{-1} S_{\Delta\Pi}.$$

We note that the definitions of the above matrices are given in (3.13). Using the block Cholesky decomposition of $\widetilde{S}$ as in Li and Widlund [13] and above, see also (3.12), we have

$$\begin{aligned} \widetilde{S}^{-1} = & \begin{pmatrix} S_{\Delta\Delta}^{-1} & 0 \\ 0 & 0 \end{pmatrix} \\ & + \left(R_{\Pi}^{t} - \sum_{i=1}^{N} (R_{\Delta}^{(i)})^{t} (S_{\Delta\Delta}^{(i)})^{-1} S_{\Delta\Pi}^{(i)} R_{\Pi}^{(i)} \right) F_{\Pi\Pi}^{-1} \\ & \left(R_{\Pi}^{t} - \sum_{i=1}^{N} (R_{\Delta}^{(i)})^{t} (S_{\Delta\Delta}^{(i)})^{-1} S_{\Delta\Pi}^{(i)} R_{\Pi}^{(i)} \right)^{t}. \end{aligned}$$

By combining the above equation with (3.18) and (3.21), we obtain

$$\widetilde{S}^{-1} = \begin{pmatrix} S_{\Delta\Delta}^{-1} & 0 \\ 0 & 0 \end{pmatrix} + \overline{\Psi}(\Psi^t S \Psi)^{-1}\overline{\Psi}^t.$$

Therefore, the BDDC operator, see (3.11), with the preconditioner M^{-1} in (3.20) can be simply written into

$$B_{DDC} = R_{D,\Gamma}^t \widetilde{S}^{-1} R_{D,\Gamma} R_\Gamma^t \widetilde{S} R_\Gamma. \tag{3.22}$$

4. Condition number estimate. In this section, we provide a condition number bound of the BDDC operator. We will construct functionals $\{f_m\}_{m=1}^6$, dual to the space **ker**(ε), which satisfy the following properties:

$$\begin{aligned} & f_m(\mathbf{r}_k) = \delta_{mk}, \quad m,k = 1,\cdots,6, \\ & |f_m(\mathbf{v})|^2 \leq C \frac{\|\mathbf{v}\|_{0,\partial\Omega_i}^2}{H_i^2} \quad \text{ for } \mathbf{v} \in [L^2(\partial\Omega_i)]^3. \end{aligned} \tag{4.1}$$

Here $\mathbf{r}_k$ are bases of **ker** (ε) scaled with respect to a face $F \subset \partial\Omega_i$; this means that we take $\widehat{\mathbf{x}} \in F$ and $H = \text{diam}(F)$ in (2.3). Such dual functionals were first introduced by Klawonn and Widlund [10]. An arbitrary rigid body motion $\mathbf{r}$ can then be represented by a linear combination of the elements of the bases $\{\mathbf{r}_k\}_{k=1}^6$,

$$\mathbf{r} = \sum_{k=1}^6 f_k(\mathbf{r})\mathbf{r}_k.$$

We now introduce six linearly independent functionals $\left\{g_k^l\right\}_{k=1}^6$ that are closely related to the primal constraints across a nonmortar face $F_l \subset \partial\Omega_i$,

$$g_k^l(\mathbf{v}) = \frac{\int_{F_l} \mathbf{v} \cdot \mathbf{P}_l(\mathbf{r}_k^l)\, ds}{\int_{F_l} \mathbf{P}_l(\mathbf{r}_k^l) \cdot \mathbf{P}_l(\mathbf{r}_k^l)\, ds}, \quad \text{ for } \mathbf{v} \in [L^2(F_l)]^3,\ k = 1,\cdots,6, \tag{4.2}$$

where $\mathbf{P}_l(\mathbf{r}_k^l)$ is the projection defined in (3.3) and $\left\{\mathbf{r}_k^l\right\}_{k=1}^6$ are six rigid body motions scaled with respect to the face F_l. Since these functionals are linearly independent, they provide bases of the dual space $\left(\textbf{ker}(\varepsilon)\right)'$. Thus there exists $\{\beta_{mk}\}_{m,k=1}^6$ such that

$$f_m^l = \sum_{k=1}^6 \beta_{mk} g_k^l, \quad m = 1,\cdots,6, \quad f_m^l(\mathbf{r}_k^l) = \delta_{mk}. \tag{4.3}$$

We will now show that $\{f_m^l\}_m$ satisfy (4.1). The projection $\mathbf{P}_l$ defined in (3.3) satisfies

$$\|\mathbf{P}_l(\mathbf{v}) - \mathbf{v}\|_{0,F_l}^2 \leq C h_i |\mathbf{v}|_{1/2,F_l}^2 \text{ for } \mathbf{v} \in [H^{1/2}(F_l)]^3$$

(see [18, Lemma 1.6]), so that the following estimate holds:

$$\|\mathbf{P}_l(\mathbf{r}_k^l)\|_{0,F_l}^2 \geq C H_i^2. \tag{4.4}$$

Here the constant C does not depend on any mesh parameters for sufficiently small h_i. From (4.2), (4.4), and Hölder's inequality, we obtain

$$|g_k^l(\mathbf{v})|^2 \leq C\frac{\|\mathbf{v}\|^2_{0,\partial\Omega_i}}{H_i^2} \text{ for } \mathbf{v} \in [L^2(\partial\Omega_i)]^3.$$

From this bound and (4.3), $\left\{f_m^l\right\}_{m=1}^6$ satisfy the bound in (4.1). We note that for $\mathbf{w} = (\mathbf{w}_1, \cdots, \mathbf{w}_N) \in \widetilde{\mathbf{W}}$, that satisfies the primal constraints, we have $g_k^l(\mathbf{w}_i) = g_k^l(\boldsymbol{\phi})$, see (4.2) and (3.4), so that

$$f_m^l(\mathbf{w}_i) = f_m^l(\boldsymbol{\phi}), \quad m = 1, \cdots, 6,$$

where $\boldsymbol{\phi} = \mathbf{w}_j$ on $F_{ij} \subset F_l$.

In the following, we will provide several lemmas which will be used to analyze the condition number bound of the BDDC operator. For a face $F \subset \partial\Omega_i$, the space $H_{00}^{1/2}(F)$ consists of the functions whose zero extension to the whole boundary $\partial\Omega_i$ belongs to the space $H^{1/2}(\partial\Omega_i)$ and is equipped with the norm

$$\|v\|_{H_{00}^{1/2}(F)} := \left(|v|^2_{H^{1/2}(F)} + \int_F \frac{v(x)^2}{\text{dist}(x, \partial F)}\, ds(x)\right)^{1/2},$$

where

$$|v|^2_{H^{1/2}(F)} = \int_F \int_F \frac{|v(x) - v(y)|^2}{|x - y|^3}\, ds(x)\, ds(y).$$

The norm can be extended to the product space $[H_{00}^{1/2}(F)]^3$ by using the usual product norm.

We now provide several inequalities for the mortar projection of functions. We recall that the space $\overset{\circ}{\mathbf{W}}(F_l)$, see (3.1), and the Lagrange multiplier space $\mathbf{M}_l$ given on the nonmortar face $F_l \subset \partial\Omega_i$.

DEFINITION 4.1. *The mortar projection* $\pi_l : [L^2(F_l)]^3 \to \overset{\circ}{\mathbf{W}}(F_l)$ *is defined by*

$$\int_{F_l} (\pi_l(\mathbf{v}) - \mathbf{v}) \cdot \boldsymbol{\psi}\, ds = 0 \quad \forall \boldsymbol{\psi} \in \mathbf{M}_l.$$

The mortar projection is continuous in both the L^2 and the $H_{00}^{1/2}$-norms when appropriate Lagrange multiplier spaces $\mathbf{M}_l$ are chosen; see [6, 18].

Let $F_l \subset \partial\Omega_i$ be a nonmortar face that is partitioned into $\left\{F_{ij}\right\}_j$ with $F_{ij} = \partial\Omega_i \cap \partial\Omega_j$ and let $\boldsymbol{\phi} = \mathbf{w}_j$ on F_{ij}. We then have the following result; see [8, Lemma 4.4 and Remark 4.5].

LEMMA 4.2.

$$\|\pi_l(\mathbf{w}_i)\|^2_{H_{00}^{1/2}(F_l)} \leq C\left(1 + log\frac{H_i}{h_i}\right)^2 \|\mathbf{w}_i\|^2_{1/2,\partial\Omega_i},$$

$$\|\pi_l(\boldsymbol{\phi})\|^2_{H_{00}^{1/2}(F_l)} \leq C\left(1 + log\frac{H_i}{h_i}\right)^3 \sum_j \|\mathbf{w}_j\|^2_{1/2,\partial\Omega_j}.$$

REMARK 4.3. *The additional factor* $(1 + log(H_i/h_i))$ *in the second estimate came from the geometrically non-conformity that causes* $\phi \in H^{1/2-\epsilon}(F_l)$, *with* $\epsilon > 0$. *When the partition is geometrically conforming, i.e.,* $F_l = F_{ij}$ *and* $\phi = \mathbf{w}_j$, *we obtain the usual estimate*

$$\|\pi_l(\mathbf{w}_j)\|^2_{H^{1/2}_{00}(F_l)} \leq C\left(1 + log\frac{H_i}{h_i}\right)^2 \|\mathbf{w}_j\|^2_{1/2,\partial\Omega_j}. \tag{4.5}$$

The proof of the following lemma is provided in [7, Lemma 4.6].

LEMMA 4.4. *For the bases* $\{\mathbf{r}^l_m\}_{m=1}^6$ *of* $\mathbf{ker}(\varepsilon)$ *scaled with respect to the face* $F_l \subset \partial\Omega_i$, *we have*

$$\|\pi_l(\mathbf{r}^l_m)\|^2_{H^{1/2}_{00}(F_l)} \leq C\left(1 + log\frac{H_i}{h_i}\right) H_i \|\mathbf{r}^l_m\|^2_{\infty,F_l}.$$

DEFINITION 4.5. *A nonmortar face* $F_l \subset \partial\Omega_i$ *is called a simple face if it has only one mortar subdomain* Ω_j *such that* $F_l = \partial\Omega_i \cap \partial\Omega_j$.

ASSUMPTION 4.6. *On each simple face* $F_l\,(= \partial\Omega_i \cap \partial\Omega_j)$, *its nonmortar subdomain* Ω_i *and its mortar subdomain* Ω_j *have been chosen so as to satisfy* $G_i \leq G_j$.

LEMMA 4.7. *Let* $F_l \subset \Omega_i$ *be a simple face, i.e.,* $F_l = \partial\Omega_i \cap \partial\Omega_j$, *and satisfy Assumption 4.6. Then, for* $\mathbf{w} = (\mathbf{w}_1, \cdots, \mathbf{w}_N) \in \widetilde{\mathbf{W}}$, *we have*

$$G_i\|\pi_l(\mathbf{w}_i - \mathbf{w}_j)\|^2_{H^{1/2}_{00}(F_l)} \leq C\left(1 + log\frac{H_i}{h_i}\right)^2 \left(|\mathbf{w}_i|^2_{S_i} + |\mathbf{w}_j|^2_{S_j}\right),$$

where $|\mathbf{w}_l|^2_{S_l} = \langle S_l\mathbf{w}_l, \mathbf{w}_l\rangle$ *for* $l = i, j$.

Proof. Let $\{\mathbf{r}^l_m\}_{m=1}^6$ be bases of **ker**(ε) scaled with respect to the face F_l and $\{f^l_m\}_{m=1}^6$ be the corresponding dual functionals to the bases. Since $\mathbf{w} = (\mathbf{w}_1, \cdots, \mathbf{w}_N) \in \widetilde{\mathbf{W}}$ satisfies the primal constraints across the face F_l, the following identity holds

$$\sum_{m=1}^6 f^l_m(\mathbf{w}_i)\mathbf{r}^l_m = \sum_{m=1}^6 f^l_m(\mathbf{w}_j)\mathbf{r}^l_m.$$

We then have

$$\begin{aligned}\|\pi_l(\mathbf{w}_i - \mathbf{w}_j)\|^2_{H^{1/2}_{00}(F_l)} \leq{}& 2\left\|\pi_l\left(\mathbf{w}_i - \sum_{m=1}^6 f^l_m(\mathbf{w}_i)\mathbf{r}^l_m\right)\right\|^2_{H^{1/2}_{00}(F_l)} \\ &+ 2\left\|\pi_l\left(\mathbf{w}_j - \sum_{m=1}^6 f^l_m(\mathbf{w}_j)\mathbf{r}^l_m\right)\right\|^2_{H^{1/2}_{00}(F_l)}.\end{aligned}$$

We now estimate

$$\begin{aligned}
&\left\|\pi_l\left(\mathbf{w}_i-\sum_{m=1}^{6} f_m^l(\mathbf{w}_i)\mathbf{r}_m^l\right)\right\|^2_{H_{00}^{1/2}(F_l)}\\
&=\left\|\pi_l\left(\mathbf{w}_i-\mathbf{r}_i-\sum_{m=1}^{6} f_m^l(\mathbf{w}_i-\mathbf{r}_i)\mathbf{r}_m^l\right)\right\|^2_{H_{00}^{1/2}(F_l)}\\
&\le C\left(\|\pi_l(\mathbf{w}_i-\mathbf{r}_i)\|^2_{H_{00}^{1/2}(F_l)}+\sum_{m=1}^{6}|f_m^l(\mathbf{w}_i-\mathbf{r}_i)|^2\|\pi_l(\mathbf{r}_m^l)\|^2_{H_{00}^{1/2}(F_l)}\right),
\end{aligned}$$

where $\mathbf{r}_i \in$ **ker** (ε) is an arbitrary rigid body motion. We note that $\mathbf{r}=\sum_{m=1}^{6} f_m^l(\mathbf{r})\mathbf{r}_m^l$ for any $\mathbf{r}\in$ **ker** (ε).

From Lemmas 4.2, 2.1, and 2.2, the first term of the above expression is bounded by

$$\|\pi_l(\mathbf{w}_i-\mathbf{r}_i)\|^2_{H_{00}^{1/2}(F_l)}\le C\left(1+\log\frac{H_i}{h_i}\right)^2|\mathbf{w}_i|^2_{E(\partial\Omega_i)}\le C\frac{1}{G_i}\left(1+\log\frac{H_i}{h_i}\right)^2|\mathbf{w}_i|^2_{S_i}, \tag{4.6}$$

and from (4.1), Lemmas 4.4 and 2.2, the second term is bounded by

$$\begin{aligned}
&|f_m^l(\mathbf{w}_i-\mathbf{r}_i)|^2\ \|\pi_l(\mathbf{r}_m^l)\|^2_{H_{00}^{1/2}(F_l)}\\
&\le C\frac{\|\mathbf{w}_i-\mathbf{r}_i\|^2_{0,\partial\Omega_i}}{H_i^2}\left(1+\log\frac{H_i}{h_i}\right)H_i\|\mathbf{r}_m^l\|^2_{\infty,F_l}\\
&\le C\left(1+\log\frac{H_i}{h_i}\right)|\mathbf{w}_i|^2_{E(\partial\Omega_i)}\\
&\le C\frac{1}{G_i}\left(1+\log\frac{H_i}{h_i}\right)|\mathbf{w}_i|^2_{S_i}.
\end{aligned}\tag{4.7}$$

Similarly, we obtain

$$\|\pi_l(\mathbf{w}_j-\mathbf{r}_j)\|^2_{H_{00}^{1/2}(F_l)}\le C\frac{1}{G_j}\left(1+\log\frac{H_i}{h_i}\right)^2|\mathbf{w}_j|^2_{S_j},$$

and

$$|f_m^l(\mathbf{w}_j-\mathbf{r}_j)|^2\ \|\pi_l(\mathbf{r}_m^l)\|^2_{H_{00}^{1/2}(F_l)}\le C\frac{1}{G_j}\left(1+\log\frac{H_i}{h_i}\right)|\mathbf{w}_j|^2_{S_j}.$$

Using Assumption 4.6, i.e., $G_i\le G_j$, the required bound has been shown. □

DEFINITION 4.8. *A pair of subdomains Ω_j and Ω_k has a simple path if there exists a path $(\Omega_j,\Omega_{k_1},\cdots,\Omega_{k_p},\Omega_k)$ connecting Ω_j and Ω_k through simple nonmortar faces, i.e., all common faces appearing in the path are simple faces.*

DEFINITION 4.9. *We call that a nonmortar face $F_l\subset\partial\Omega_i$, that is not a simple face, has simple mortar neighbors if any pair of subdomains (Ω_j,Ω_k) in its mortar neighbors has a simple path of which length is less than TOL and the subdomains Ω_m in the path satisfy*

$$\frac{G_i}{G_m}\le TOL,$$

where G_i and G_m are the material coefficients given in the subdomains Ω_i and Ω_m, respectively.

ASSUMPTION 4.10. *We assume that any nonmortar face, that is not a simple face, has simple mortar neighbors.*

REMARK 4.11. *When most of the nonmortar faces are simple faces and when nonmortar subdomains are chosen to have smaller G_i, Assumption 4.10 can be satisfied for such partitions with a relatively small TOL.*

LEMMA 4.12. *Let a nonmortar face $F_l \subset \Omega_i$, that is not a simple face, satisfy Assumption 4.10. For $\mathbf{w} = (\mathbf{w}_1, \cdots, \mathbf{w}_N) \in \widetilde{\mathbf{W}}$, we then have*

$$G_i \|\pi_l(\mathbf{w}_i - \boldsymbol{\phi})\|^2_{H^{1/2}_{00}(F_l)} \le C(TOL) \left(1 + log\frac{H_i}{h_i}\right)^3 \left(|\mathbf{w}_i|^2_{S_i} + \sum_{m \in P(F_l)} |\mathbf{w}_m|^2_{S_m} \right),$$

where $P(F_l)$ is the set of subdomain indices appearing in the simple paths connecting any two mortar neighbors in $\{\Omega_j\}_j$ and the constant $C(TOL)$ depends only on TOL.

Proof. Let

$$\mathcal{F}_l(\mathbf{v}) = \sum_{m=1}^{6} f^l_m(\mathbf{v}) \mathbf{r}^l_m,$$

where $\{f^l_m\}_m$ are dual functionals provided for the bases $\{\mathbf{r}^l_m\}_m$, that are scaled with respect to the face F_l. We note that $\mathcal{F}_l(\mathbf{r}) = \mathbf{r}$ for any rigid body motion $\mathbf{r}$. On F_l, we define $\boldsymbol{\phi} = \mathbf{w}_j$ on $F_{ij} (:= \partial\Omega_i \cap \partial\Omega_j) \subset F_l$. Since $\mathbf{w}$ satisfies the primal constraints, we have $\mathcal{F}_l(\mathbf{w}_i) = \mathcal{F}_l(\boldsymbol{\phi})$ and obtain

$$\begin{aligned} &\|\pi_l(\mathbf{w}_i - \boldsymbol{\phi})\|^2_{H^{1/2}_{00}(F_l)} \\ &\le 2\|\pi_l\left(\mathbf{w}_i - \mathcal{F}_l(\mathbf{w}_i)\right)\|^2_{H^{1/2}_{00}(F_l)} + \|\pi_l\left(\boldsymbol{\phi} - \mathcal{F}_l(\boldsymbol{\phi})\right)\|^2_{H^{1/2}_{00}(F_l)}. \end{aligned} \tag{4.8}$$

We estimate the first term of (4.8) as in (4.6) and (4.7),

$$\|\pi_l\left(\mathbf{w}_i - \mathcal{F}_l(\mathbf{w}_i)\right)\|^2_{H^{1/2}_{00}(F_l)} \le C\frac{1}{G_i}\left(1 + \log\frac{H_i}{h_i}\right)^2 |\mathbf{w}_i|^2_{S_i}. \tag{4.9}$$

Applying Lemma 4.2 to the second term of (4.8) gives

$$\|\boldsymbol{\phi} - \mathcal{F}_l(\boldsymbol{\phi})\|^2_{H^{1/2}_{00}(F_l)} \le \left(1 + \log\frac{H_i}{h_i}\right)^3 \sum_j \|\mathbf{w}_j - \mathcal{F}_l(\boldsymbol{\phi})\|^2_{1/2,\partial\Omega_j}.$$

It suffices to prove that

$$\sum_j \|\mathbf{w}_j - \mathcal{F}_l(\boldsymbol{\phi})\|^2_{1/2,\partial\Omega_j} \le C \sum_{m \in P(F_l)} \frac{1}{G_m} |\mathbf{w}_m|^2_{S_m}, \tag{4.10}$$

where $P(F_l)$ is the set of subdomain indices appearing in the simple paths connecting any two subdomains in $\{\Omega_j\}_j$ that are mortar neighbors of Ω_i on F_l.

We now consider

$$\begin{aligned}
&\|\mathbf{w}_j - \mathcal{F}_l(\boldsymbol{\phi})\|^2_{1/2,\partial\Omega_j} \\
&\le 2\|\mathbf{w}_j - \mathbf{r}_j\|^2_{1/2,\partial\Omega_j} + 2\|\mathbf{r}_j - \mathcal{F}_l(\boldsymbol{\phi})\|^2_{1/2,\partial\Omega_j} \\
&\le C\frac{1}{G_j}|\mathbf{w}_j|^2_{S_j} + 2\|\mathcal{F}_l(\mathbf{r}_j - \boldsymbol{\phi})\|^2_{1/2,\partial\Omega_j}. \qquad (4.11)
\end{aligned}$$

The second term in (4.11) is estimated by

$$\begin{aligned}
&\|\mathcal{F}_l(\mathbf{r}_j - \boldsymbol{\phi})\|^2_{1/2,\partial\Omega_j} \\
&\le C\frac{\|\mathbf{r}_j - \boldsymbol{\phi}\|^2_{0,F_l}}{H_i^2}\max_m\left\{\|\mathbf{r}^l_m\|^2_{1/2,\partial\Omega_j}\right\} \\
&\le C\frac{1}{H_i^2}\frac{H_j^3}{H_i^2}\sum_{k,\, F_{ik}\subset F_l}\|\mathbf{r}_j - \mathbf{w}_k\|^2_{0,F_{ik}} \\
&\le C\frac{1}{H_i}\sum_{k,\, F_{ik}\subset F_l}\|\mathbf{r}_j - \mathbf{w}_k\|^2_{0,F_{ik}}. \qquad (4.12)
\end{aligned}$$

Here we have used the regularity of the partition, i.e., $H_i \simeq H_j$ for neighboring subdomains Ω_i and Ω_j, and

$$\|\mathbf{r}^l_m\|^2_{1/2,\partial\Omega_j} \le \|\mathbf{r}^l_m\|^2_{1,\Omega_j} \le CH_j^3/H_i^2.$$

We now consider the term $\|\mathbf{r}_j - \mathbf{w}_k\|^2_{0,F_{ik}}$. When $k = j$, from Lemmas 2.2 and 2.1 we obtain

$$\frac{1}{H_i}\|\mathbf{r}_j - \mathbf{w}_j\|^2_{0,F_{ij}} \le C\frac{1}{G_j}|\mathbf{w}_j|^2_{S_j}.$$

When $k \neq j$, we note that there exists a path $(\Omega_j, \Omega_{k_1}, \cdots, \Omega_{k_p}, \Omega_k)$, where each neighboring subdomains Ω_q and Ω_r in the path intersect on a simple face F_{qr}. Let

$$\mathcal{F}_{qr}(\mathbf{v}) = \sum_m f^{qr}_m(\mathbf{v})\mathbf{r}^{qr}_m,$$

where $\{f^{qr}_m\}_m$ are dual functionals to the bases $\{\mathbf{r}^{qr}_m\}_m$ of **ker**(ε) scaled with respect to the face F_{qr}. We note that $\mathcal{F}_{qr}(\mathbf{w}_q) = \mathcal{F}_{qr}(\mathbf{w}_r)$.

We then obtain

$$\begin{aligned}
&\|\mathbf{r}_j - \mathbf{w}_k\|^2_{0,F_{ik}} \\
&= \|\mathbf{r}_j - \mathcal{F}_{jk_1}(\mathbf{w}_j) + \mathcal{F}_{jk_1}(\mathbf{w}_{k_1}) - \mathcal{F}_{k_1k_2}(\mathbf{w}_{k_1}) + \cdots \\
&\cdots + \mathcal{F}_{k_{p-1}k_p}(\mathbf{w}_{k_p}) - \mathcal{F}_{k_pk}(\mathbf{w}_{k_p}) + \mathcal{F}_{k_pk}(\mathbf{w}_k) - \mathbf{w}_k\Big\|^2_{0,F_{ik}} \\
&\le C\left(\|\mathcal{F}_{jk_1}(\mathbf{r}_j - \mathbf{w}_j)\|^2_{0,F_{il}}\right. \\
&\quad + \|\mathcal{F}_{jk_1}(\mathbf{w}_{k_1} - \mathbf{r}_{k_1})\|^2_{0,F_{ik}} + \|\mathcal{F}_{k_1k_2}(\mathbf{w}_{k_1} - \mathbf{r}_{k_1})\|^2_{0,F_{ik}} + \cdots \\
&\quad + \|\mathcal{F}_{k_{p-1}k_p}(\mathbf{w}_{k_p} - \mathbf{r}_{k_p})\|^2_{0,F_{ik}} + \|\mathcal{F}_{k_pk}(\mathbf{w}_{k_p} - \mathbf{r}_{k_p})\|^2_{0,F_{ik}} \\
&\quad \left. + \|\mathcal{F}_{k_pk}(\mathbf{w}_k - \mathbf{r}_k)\|^2_{0,F_{ik}} + \|\mathbf{r}_k - \mathbf{w}_k\|^2_{0,F_{ik}}\right). \qquad (4.13)
\end{aligned}$$

We note that the rigid body motions $\{\mathbf{r}_m^{qr}\}_{m=1}^6$ scaled with respect to the face F_{qr} in the path satisfy

$$\|\mathbf{r}_m^{qr}\|_{\infty,F_{ik}} \leq C(TOL),$$

since the length of the path is less than TOL and the subdomain partition is regular. Using the above bound and Lemma 2.2, each terms in (4.13) can be bounded in a similar way to

$$\begin{aligned}
\|\mathcal{F}_{k_{p-1}k_p}(\mathbf{w}_{k_p}-\mathbf{r}_{k_p})\|_{0,F_{ik}}^2 &\leq C\frac{1}{H_{k_p}^2}\|\mathbf{w}_{k_p}-\mathbf{r}_{k_p}\|_{F_{k_{p-1}k_p}}^2 \max_m\left\{\|\mathbf{r}_m^{k_{p-1}k_p}\|_{\infty,F_{ik}}^2\right\} H_i^2 \\
&\leq C(TOL)\frac{H_i^2}{H_{k_p}}\frac{1}{H_{k_p}}\|\mathbf{w}_{k_p}-\mathbf{r}_{k_p}\|_{0,\partial\Omega_{k_p}}^2, \\
&\leq C(TOL)\frac{H_i^2}{H_{k_p}}|\mathbf{w}_{k_p}|_{E(\partial\Omega_{k_p})}^2.
\end{aligned}$$

Therefore we obtain

$$\begin{aligned}
\frac{1}{H_i}\|\mathbf{r}_j-\mathbf{w}_k\|_{0,F_{ik}}^2 &\leq C(TOL)\sum_{m\in P(j,k)}\frac{H_i}{H_m}|\mathbf{w}_m|_{E(\partial\Omega_m)}^2 \\
&\leq C(TOL)\sum_{m\in P(j,k)}\frac{1}{G_m}|\mathbf{w}_m|_{S_m}^2, \qquad (4.14)
\end{aligned}$$

where $P(j,k)$ is the set of subdomain indices appearing in the path from Ω_j to Ω_k. Since the subdomain partition is regular, the factors H_i and H_m cancel each other in the final estimate. Combining (4.14) with (4.12) and (4.11), we obtain the estimate (4.10). Using $G_i/G_m \leq TOL$ for $m \in P(j,k)$, the desired bound follows. □

REMARK 4.13. *Using the technic used above, we are able to extend the Poincaré inequality shown by Brenner [3] to a more general case, i.e.,*

$$v=(v_1,\cdots,v_N)\in\prod_{i=1}^N H(\Omega_i),\ \int_F (v_i-\phi)\,ds=0,$$

where $F \subset \partial\Omega_i$ is any nonmortar interface and $\{F_{ij}\}_j$ is a partition of F by its mortar neighbors Ω_j and $\phi = v_j$ on F_{ij}.

A condition number analysis of BDDC algorithms has been carried out using an appropriate average operator by Li and Widlund [14]. The same technic was used for a condition number estimate of the BDDC algorithm for mortar discretization by Kim, Dryja, and Widlund [8]. A bound for an average operator E_D in a certain norm is central in the analysis. We recall the definitions of R_Γ and $R_{D,\Gamma}$ in (3.10) and (3.19), respectively. We note that

$$R_\Gamma\ :\ \mathbf{W}_{\Delta,m}\times\mathbf{W}_\Pi \rightarrow \mathbf{W}_{\Delta,n}\times\mathbf{W}_{\Delta,m}\times\mathbf{W}_\Pi,$$

where $\mathbf{W}_\Pi$ is the space of primal variables, $\mathbf{W}_{\Delta,n}$ is the space of unknowns on nonmortar faces other than the primal variables, and $\mathbf{W}_{\Delta,m}$ is the space of the remaining unknowns on subdomain interfaces. We also note that

$$\mathbf{W}_\Delta = \mathbf{W}_{\Delta,n}\times\mathbf{W}_{\Delta,m}, \quad \widetilde{\mathbf{W}} = \mathbf{W}_\Delta\times\mathbf{W}_\Pi$$

.

We now define the operator E_D by

$$E_D = R_\Gamma R_{D,\Gamma}^t. \tag{4.15}$$

Here the weight matrix D in (3.19) will be chosen so that

$$\text{(P1)}\quad R_\Gamma^t R_{D,\Gamma} = R_{D,\Gamma}^t R_\Gamma = I$$

$$\text{(P2)}\quad |E_D \mathbf{w}|_{\widetilde{S}}^2 \le C \max_i \left\{ \left(1+\log\frac{H_i}{h_i}\right)^3 \right\} |\mathbf{w}|_{\widetilde{S}}^2,$$

where $|\mathbf{w}|_{\widetilde{S}}^2 = \langle \widetilde{S}\mathbf{w}, \mathbf{w}\rangle$; see (3.12). We then have

$$R_\Gamma^t R_{D,\Gamma} \begin{pmatrix} \mathbf{w}_m \\ \mathbf{w}_\Pi \end{pmatrix} = \begin{pmatrix} -B_m^t (B_n^t)^{-1} D_{nn} \mathbf{z}_n + D_{mm}\mathbf{w}_m \\ -B_\Pi^t (B_n^t)^{-1} D_{nn} \mathbf{z}_n + D_{\Pi\Pi}\mathbf{w}_\Pi \end{pmatrix},$$

where

$$\mathbf{z}_n = -B_n^{-1}(B_m \mathbf{w}_m + B_\Pi \mathbf{w}_\Pi).$$

In order to satisfy property (P1), the weight matrix D is chosen to be

$$D_{nn} = 0, \quad D_{mm} = I, \quad D_{\Pi\Pi} = I. \tag{4.16}$$

Using Lemma 4.12, we can establish property (P2) for the operator E_D with the weight matrix D just given.

LEMMA 4.14. *With Assumptions 4.6 and 4.10, the operator E_D satisfies*

$$|E_D \mathbf{w}|_{\widetilde{S}}^2 \le C(TOL) \max_i \left\{ \left(1+ log\frac{H_i}{h_i}\right)^3 \right\} |\mathbf{w}|_{\widetilde{S}}^2.$$

Proof. With the weight matrix D in (4.16), the operator E_D in (4.15) is given by

$$E_D \begin{pmatrix} \mathbf{w}_n \\ \mathbf{w}_m \\ \mathbf{w}_\Pi \end{pmatrix} = \begin{pmatrix} \mathbf{w}_n - B_n^{-1}(B_n \mathbf{w}_n + B_m \mathbf{w}_m + B_\Pi \mathbf{w}_\Pi) \\ \mathbf{w}_m \\ \mathbf{w}_\Pi \end{pmatrix}.$$

Let

$$\mathbf{z}_n = \mathbf{w}_n - B_n^{-1}(B_n \mathbf{w}_n + B_m \mathbf{w}_m + B_\Pi \mathbf{w}_\Pi).$$

We take $\mathbf{z}_i$ by restricting the unknowns $(\mathbf{z}_n, \mathbf{w}_m, \mathbf{w}_\Pi)$ to the subdomain Ω_i. Similarly, we take $\mathbf{w}_i$ from $(\mathbf{w}_n, \mathbf{w}_m, \mathbf{w}_\Pi)$. Let

$$\mathbf{z} = (\mathbf{z}_1, \cdots, \mathbf{z}_n), \quad \mathbf{w} = (\mathbf{w}_1, \cdots, \mathbf{w}_n).$$

We note that $\mathbf{w}$ satisfies the primal constraints and $\mathbf{z}$ satisfies the mortar matching condition. Each $\mathbf{z}_i$ is of the form

$$\mathbf{z}_i = \mathbf{w}_i - \sum_{F \subset \partial\Omega_i} E_F^{(i)} \pi_F(\mathbf{w}_i - \boldsymbol{\phi}),$$

where F is a nonmortar face in $\partial\Omega_i$, π_F is the mortar projection given on the face F, $E_F^{(i)}$ is the zero extension of functions defined on F to $\partial\Omega_i \backslash F$, and $\boldsymbol{\phi} = \mathbf{w}_j$ on $F_{ij} (:= \partial\Omega_j \cap \partial\Omega_i) \subset F$. We then obtain

$$\begin{aligned}
|E_D \mathbf{w}|_{\widetilde{S}}^2 &= \sum_{i=1}^N \langle S_i \mathbf{z}_i, \mathbf{z}_i \rangle \\
&\leq C \sum_{i=1}^N \left(\langle S_i \mathbf{w}_i, \mathbf{w}_i \rangle + \sum_{F \subset \partial\Omega_i} \langle S_i E_F^{(i)} \pi_F(\mathbf{w}_i - \boldsymbol{\phi}), E_F^{(i)} \pi_F(\mathbf{w}_i - \boldsymbol{\phi}) \rangle \right) \\
&\leq C \sum_{i=1}^N \langle S_i \mathbf{w}_i, \mathbf{w}_i \rangle + \sum_{i=1}^N \sum_{F \subset \partial\Omega_i} G_i \|\pi_F(\mathbf{w}_i - \boldsymbol{\phi})\|_{H_{00}^{1/2}(F)}^2 \\
&\leq C(TOL) \max_i \left\{ \left(1 + \log \frac{H_i}{h_i}\right)^3 \right\} \sum_{i=1}^N \langle S_i \mathbf{w}_i, \mathbf{w}_i \rangle \\
&\leq C(TOL) \max_i \left\{ \left(1 + \log \frac{H_i}{h_i}\right)^3 \right\} \langle \widetilde{S} \mathbf{w}, \mathbf{w} \rangle.
\end{aligned}$$

Here we have used that $\langle S_i \mathbf{w}_i, \mathbf{w}_i \rangle \leq C G_i |\mathbf{w}_i|_{H^{1/2}(\partial\Omega_i)}^2$, $|E_F^{(i)} \pi_F(\mathbf{w}_i - \boldsymbol{\phi})|_{H^{1/2}(\partial\Omega_i)} \simeq \|\pi_F(\mathbf{w}_i - \boldsymbol{\phi})\|_{H_{00}^{1/2}(F)}$, and Lemma 4.12. □

By using the properties (P1) and (P2), we can show the following condition number bound of the BDDC operator in (3.22); see [8, Theorem 4.7] for the proof.

THEOREM 4.15. *If E_D satisfies the properties (P1) and (P2) then the BDDC operator has the following condition number bound*

$$\kappa(B_{DDC}) \leq C(TOL) \max_i \left\{ \left(1 + log \frac{H_i}{h_i}\right)^3 \right\}.$$

REMARK 4.16. *The analysis above can be modified for the geometrically conforming case resulting in the condition number bound, see Lemma 4.7,*

$$\kappa(B_{DDC}) \leq C \max_i \left\{ \left(1 + log \frac{H_i}{h_i}\right)^2 \right\}.$$

We note that all common interfaces are simple faces in the geometrically conforming case so that the constant C does not depend on TOL.

REMARK 4.17. *The weights D make the BDDC algorithm well connected to the FETI-DP algorithm in [7]. These BDDC and FETI-DP algorithms share the same spectra except* *possibly the eigenvalue 1; this fact was also proved for elliptic problems in [8]. Here we emphasize that this study consider a more general case than the previous studies [7, 8]. It extends the BDDC algorithm to geometrically non-conforming partitions with the primal constraints imposed on nonmortar faces* F_l *not on the every pieces* F_{ij} *of* F_l.

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