

A FETI-DP PRECONDITIONER FOR MORTAR METHODS IN THREE DIMENSIONS*

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Abstract. A FETI-DP method is developed for three-dimensional elliptic problems with mortar discretization. The mortar matching conditions are considered as the continuity constraints in the FETI-DP formulation. Among them, face average constraints are selected as primal constraints in our FETI-DP formulation to achieve an algorithm as scalable as two dimensional problems. A Neumann-Dirichlet preconditioner is used in the FETI-DP formulation and it gives the condition number bound

$$C \max_{i=1, \dots, N} \left\{ (1 + \log(H_i/h_i))^2 \right\},$$

where H_i and h_i are sizes of domain and mesh for each subdomain, respectively, and the constant C is independent of H_i , h_i , and coefficients of the elliptic problems. The proposed algorithm can be applied to two-dimensional elliptic problems with edge average constraints only as primal constraints. Numerical results are included.

Key words. FETI-DP, non-matching grids, mortar methods, preconditioner

AMS subject classifications. 65N30, 65N55

1. Introduction. FETI-DP methods were introduced by Farhat *et al.* [6] and applied to solving elliptic problems with conforming discretizations both in two and three dimensions [7]. In three dimensions, subdomains intersect with neighboring subdomains on faces, edges, or at corners, while they intersect on edges or at corners in two dimensions; the continuity of solution is imposed on faces and edges with dual variables and at corners with primal variables in the dual-primal FETI (FETI-DP) methods. However, numerical results in [6, 7] show that we need additional primal constraints for three dimensional problems to attain the same efficiency as two dimensional problems. For these constraints, additional Lagrange multipliers are introduced and they are treated as primal variables in the FETI-DP formulation. FETI-DP methods with various redundant constraints have been studied and their condition number bound was analyzed by Klawonn *et al.* [14, 15] for elliptic problems with heterogeneous coefficients. Numerical results were further provided in [12].

FETI-DP methods have been also applied to mortar finite elements methods [4, 5, 9, 17]. In [4, 5], the condition number bound of FETI-DP operator was analyzed for various types of preconditioners but it depends on ratios of mesh sizes between neighboring subdomains. In [9], a Neumann-Dirichlet preconditioner was proposed and analyzed for elliptic problems with heterogeneous coefficients. In this case, the condition number bound does not depend on the mesh sizes and the coefficients. Moreover, numerical results show that the Neumann-Dirichlet preconditioner works much more efficiently than other FETI-DP preconditioners

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for elliptic problems with highly discontinuous coefficients. For three dimensional problems, FETI methods with mortar discretizations were developed and their numerical results were provided in [16].

The primary contribution of our work is the extension of the FETI-DP method in [9] to three dimensional problems and to the second generation of mortar methods. In [9], vertex continuity constraints are introduced as primal constraints. However, for the three-dimensional case we need primal constraints other than the vertex constraints to get a method as scalable as the two-dimensional case in [9]. We select constraints that averages of the solution across subdomain interfaces are the same, which is so called face constraints in [15]. Similarly to the previous work in [9], we propose a Neumann-Dirichlet preconditioner for the FETI-DP formulation and show that the condition number bound

$$C \max_{i=1, \dots, N} \left\{ (1 + \log(H_i/h_i))^2 \right\}$$

for elliptic problems with discontinuous constant coefficients. Here, H_i and h_i are sizes of domain and mesh for each subdomain, respectively, and the constant C is independent of H_i , h_i , and the coefficients of elliptic problems. In our FETI-DP formulation, we follow a change of basis formulation introduced in [13]. The change of basis makes the analysis of FETI-DP algorithms easier when primal constraints other than the vertex continuity constraints are used. Moreover it gives an efficient and robust implementation of FETI-DP algorithms [10, 11].

We note that edge average constraints can be considered as primal constraints for two-dimensional problems. The continuity constraints at vertices can not be selected as primal constraints for the second generation of mortar methods [1]. We are able to extend the result in [9] to the second generation of mortar methods by introducing edge average constraints. Furthermore condition number bound estimate of this case can be carried out similarly to three dimensional case presented in this paper.

This paper is organized as follows. In Section 2, we introduce finite element spaces and norms and in Section 3, we derive the FETI-DP formulation with the mortar matching constraints, the primal constraints, and the Neumann-Dirichlet preconditioner. Section 4 is devoted to analyzing the condition number bound of the FETI-DP algorithm. Numerical results are provided in Section 5.

Throughout the paper, C or c ($\leq C$) denotes a generic positive constant that does not depend on any mesh parameters and the coefficients of elliptic problems.

2. Finite element spaces and norms.

2.1. A model problem and Sobolev spaces. Let Ω be a bounded polyhedral domain in \mathbb{R}^3 and $L^2(\Omega)$ be the space of square integrable functions defined in Ω equipped with the

norm

$$\|v\|_{L^2(\Omega)}^2 := \int_{\Omega} v^2 dx.$$

The space $H^1(\Omega)$ is the set of functions, which are square integrable up to the first weak derivatives, and the norm is given by

$$\|v\|_{H^1(\Omega)} := \left(\int_{\Omega} \nabla v \cdot \nabla v dx + \frac{1}{d_{\Omega}^2} \int_{\Omega} v^2 dx \right)^{1/2},$$

where d_{Ω} denotes the diameter of Ω .

We consider the following model elliptic problem:

For $f \in L^2(\Omega)$, find $u \in H^1(\Omega)$ such that

$$\begin{aligned} -\nabla \cdot (\rho(x) \nabla u(x)) &= f(x) \quad \text{in } \Omega, \\ u(x) &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Here, $\rho(x) > 0$ for all $x \in \Omega$ and $\rho(x) \in L^{\infty}(\Omega)$.

Let Ω be partitioned into non-overlapping polyhedral subdomains $\{\Omega_i\}_{i=1}^N$. We assume that the partition is geometrically conforming, which means that each subdomain intersects its neighboring subdomains on a full face, a full edge or at a vertex. Each subdomain Ω_i is equipped with a quasi uniform triangulation Ω_i^h , which consists of tetrahedrons. These triangulations need not be aligned across subdomain interfaces.

Each subdomain Ω_i is equipped with a finite element space

$$X_i := \{v \in H_D^1(\Omega_i) : v|_{\tau} \in P_1(\tau), \tau \in \Omega_i^h\},$$

where $H_D^1(\Omega_i) := \{v \in H^1(\Omega_i) : v = 0 \text{ on } \partial\Omega \cap \partial\Omega_i\}$ and $P_1(\tau)$ is a set of polynomials of degree ≤ 1 in τ . We assume that

$$\rho(x) = \rho_i, \quad \forall x \in \Omega_i,$$

where ρ_i is a positive constant. A bilinear form $a_i(\cdot, \cdot) : X_i \times X_i \rightarrow \mathbb{R}$ is defined as

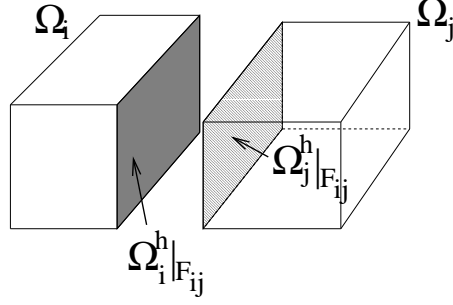
$$(2.1) \quad a_i(u_i, v_i) := \rho_i \int_{\Omega_i} \nabla u_i \cdot \nabla v_i dx.$$

We now introduce Sobolev spaces defined on the boundaries of subdomains. The space $H^{1/2}(\partial\Omega_i)$ is the trace space of $H^1(\Omega_i)$ equipped with the norm

$$\|w_i\|_{H^{1/2}(\partial\Omega_i)}^2 := |w_i|_{H^{1/2}(\partial\Omega_i)}^2 + \frac{1}{d_{\Omega_i}} \|w_i\|_{L^2(\partial\Omega_i)}^2,$$

where

$$|w_i|_{H^{1/2}(\partial\Omega_i)}^2 := \int_{\partial\Omega_i} \int_{\partial\Omega_i} \frac{|w_i(x) - w_i(y)|^2}{|x - y|^3} ds(x) ds(y).$$

FIG. 1. Mortar and nonmortar sides of F_{ij}

For any face $F_{ij} \in \partial\Omega_i$, $H_{00}^{1/2}(F_{ij})$ is the set of functions in $L^2(F_{ij})$ whose zero extension into $\partial\Omega_i$ is contained in $H^{1/2}(\partial\Omega_i)$ and is equipped with the norm

$$\|v\|_{H_{00}^{1/2}(F_{ij})}^2 := |v|_{H^{1/2}(F_{ij})}^2 + \int_{F_{ij}} \frac{v^2(x)}{\text{dist}(x, \partial F_{ij})} ds.$$

From Section 4.1 in [22], we have the following relation for $v \in H_{00}^{1/2}(F_{ij})$:

$$(2.2) \quad c\|\tilde{v}\|_{H^{1/2}(\partial\Omega_i)} \leq \|v\|_{H_{00}^{1/2}(F_{ij})} \leq C\|\tilde{v}\|_{H^{1/2}(\partial\Omega_i)},$$

where \tilde{v} is the zero extension of v to $\partial\Omega_i$, i.e., $\tilde{v} = v$ on F_{ij} and $\tilde{v} = 0$ on $\partial\Omega_i \setminus F_{ij}$.

2.2. Mortar matching conditions. Let us define

$$X := \prod_{i=1}^N X_i$$

and

$$(2.3) \quad W := \prod_{i=1}^N W_i,$$

where W_i is the trace space of X_i , i.e., $W_i = X_i|_{\partial\Omega_i}$. We will approximate the solution of the problem (2.1) in X . We note that the space X is not contained in $H^1(\Omega)$ because the triangles are non-matching across subdomain interfaces. In order to approximate the solution of the problem (2.1) in the nonconforming finite element space X , we impose the mortar matching condition on X , for which jumps of a function in X across a common face (interface) are orthogonal to a Lagrange multiplier space, i.e., $v = (v_1, \dots, v_N) \in X$ satisfies

$$(2.4) \quad \int_{F_{ij}} (v_i - v_j) \lambda_{ij} ds = 0 \quad \forall \lambda_{ij} \in M_{ij}, \forall F_{ij},$$

where M_{ij} is a Lagrange multiplier space given on the common interface $F_{ij} := \partial\Omega_i \cap \partial\Omega_j$.

On F_{ij} , we distinguish $\Omega_i^h|_{F_{ij}}$ and $\Omega_j^h|_{F_{ij}}$ as in Figure 1 and choose one as a mortar side and the other as a nonmortar side. On each nonmortar side, we define a finite element space

$$(2.5) \quad W_{ij} := \{v|_{F_{ij}} \in H_0^1(F_{ij}) : v \in X_{n(ij)}\},$$

where $n(ij)$ is the nonmortar side (nonmortar subdomain) of F_{ij} .

To get the optimal order approximation, we need the following abstract conditions on the space M_{ij} ;

(A.1) The basis $\{\xi_k^{ij}\}_{k=1}^{N_{ij}}$ are locally supported, that is, the number of elements in $\Omega_i^h|_{F_{ij}}$, which have nonempty intersections with the simply connected support of ξ_k^{ij} , is bounded independently of mesh sizes and F_{ij} .

(A.2) W_{ij} and M_{ij} have the same dimension.

(A.3) There is a constant C such that

$$\|\phi\|_{L^2(F_{ij})} \leq C \sup_{\psi \in M_{ij}} \frac{\int_{F_{ij}} \phi \psi \, ds}{\|\psi\|_{L^2(F_{ij})}} \quad \forall \phi \in W_{ij}.$$

(A.4) For $\mu \in H^{k-1/2}(F_{ij})$, there exists $\mu_h \in M_{ij}$ such that

$$\|\mu - \mu_h\|_{L^2(F_{ij})}^2 \leq Ch_i^{2k-1} |\mu|_{H^{k-1/2}(F_{ij})}^2,$$

where k is the order of finite elements in X_i .

The condition (A.4) implies that $1 \in M_{ij}$. In the following, we assume that the Lagrange multiplier space M_{ij} satisfies the above conditions; the standard Lagrange multiplier space in [2] and the Lagrange multipliers with dual basis in [8] are those examples.

In our FETI-DP formulation, we will use the mortar matching condition (2.4) as continuity constraints. These continuity constraints can further be written into

$$(2.6) \quad \sum_{i=1}^N B_i w_i = 0,$$

where $w_i = v_i|_{\partial\Omega_i}$. We note that the matrices B_i are not boolean matrices as in the original FETI (or FETI-DP) methods.

In the following, we will use the same notation for finite element functions and the corresponding vectors of nodal values. For example, w_i is used to denote a finite element function or the vector of nodal values of that function. The same applies to the notations for function spaces such as W_i, X, W , etc.

3. FETI-DP formulation.

3.1. FETI-DP operator. In this section, we formulate the FETI-DP operator for the problem (2.1) with the mortar matching condition as continuity constraints. For 3D elliptic problems, it was shown that using the primal variables at vertices is not enough to get the same condition number bound as 2D problems; see numerical results in [6, 7]. Hence, additional primal constraints are introduced to accelerate the convergence of the FETI-DP method.

For the 3D elliptic problems with conforming discretizations, Klawonn *et al.* [14] developed FETI-DP methods with various redundant constraints. They introduced edge average or face average constraints as primal constraints to achieve the same condition number bound as

2D elliptic problems. The continuity constraints on edges are that the averages of functions across a common edge are the same. The same is applied to faces also. In [15], they extended the results to a case with face constraints only.

In mortar discretizations, we can select the face constraints

$$\int_{F_{ij}} v_i ds = \int_{F_{ij}} v_j ds \quad \forall F_{ij}$$

or vertex constraints as primal constraints. We note that $1 \in M_{ij}$ and the vertex constraints can be considered for the first generation of the mortar methods [3] while the vertex constraints can not be used for the second generation of mortar methods [1].

We may impose the face average constraints by introducing additional Lagrange multipliers and then treat them as primal variables in the FETI-DP formulation; see [6, 7, 14]. In our FETI-DP formulation, we follow the change of basis (change of variables) formulation introduced in [13] that leads to much easier analysis and more robust implementation; see [10, 11].

On each interface F_{ij} , for $w_{ij} = w_i|_{F_{ij}}$ (or $w_{ij} = w_i|_{F_{ij}}$) we consider a change of variables so that

$$w_{ij} = T_{F_{ij}} \begin{pmatrix} w_{\Delta}^{(ij)} \\ w_{\Pi}^{(ij)} \end{pmatrix},$$

where $T_{F_{ij}}$ retains unknowns at the boundary of F_{ij} , $w_{\Pi}^{(ij)}$ is the average of w_{ij} on F_{ij} , i.e.,

$$w_{\Pi}^{(ij)} = \frac{\int_{F_{ij}} w_{ij} ds}{\int_{F_{ij}} 1 ds},$$

and the function $w_{\Delta}^{(ij)}$ has the average value zero on F_{ij} , i.e.,

$$\int_{F_{ij}} w_{\Delta}^{(ij)} ds = 0.$$

We note that $w_{\Delta}^{(ij)}$ is a function in the above equation and it can be represented using the change of base given by the transform $T_{F_{ij}}$

After the transforms, we express the unknowns w_i into

$$w_i = \begin{pmatrix} w_{\Delta}^{(i)} \\ w_{\Pi}^{(i)} \end{pmatrix},$$

where Π stands for the unknowns of primal variables, i.e., the averages on faces or unknowns at vertices, and Δ stands for the remaining unknowns. These notational conventions will be used throughout this paper.

We now consider a subspace \widetilde{W} of W that satisfies the primal constraints

$$\widetilde{W} := \left\{ w \in W : \int_{F_{ij}} (w_i - w_j) ds = 0 \right.$$

and w is continuous at subdomain vertices $\left. \right\}$.

Let W_Δ be a space of vectors

$$w_\Delta = \begin{pmatrix} w_\Delta^{(1)} \\ \vdots \\ w_\Delta^{(N)} \end{pmatrix},$$

and let W_Π be the space of primal variables w_Π . We then decompose the space \widetilde{W}

$$\widetilde{W} = W_\Delta \oplus W_\Pi.$$

We define

$$R_\Pi^{(i)} : W_\Pi \rightarrow W_\Pi|_{\Omega_i}$$

that restricts the primal variables to the primal variables in each subdomain.

Let $S^{(i)}$ be the Schur complement matrix obtained from the bilinear form $a_i(\cdot, \cdot)$ in (2.1) and let $g^{(i)}$ be the Schur complement forcing vector obtained from $\int_{\Omega_i} f v_i dx$. After the change of variables, the matrix $S^{(i)}$ and vector $g^{(i)}$ are written into

$$S^{(i)} = \begin{pmatrix} S_{\Delta\Delta}^{(i)} & S_{\Delta\Pi}^{(i)} \\ S_{\Pi\Delta}^{(i)} & S_{\Pi\Pi}^{(i)} \end{pmatrix}, \quad g^{(i)} = \begin{pmatrix} g_\Delta^{(i)} \\ g_\Pi^{(i)} \end{pmatrix}.$$

We recall the mortar matching condition

$$\sum_{i=1}^N B_i w_i = 0.$$

Since $w \in \widetilde{W}$ satisfies the face average constraints and $1 \in M_{ij}$, the above continuity constraints are redundant for $w \in \widetilde{W}$. We consider a subspace \overline{M}_{ij} of M_{ij} that has one less basis than M_{ij} . We impose the mortar matching condition (2.4) with the Lagrange multiplier space \overline{M}_{ij} instead of M_{ij} and obtain its matrix representation

$$\sum_{i=1}^N \overline{B}_i w_i = 0.$$

The above constraints are then non-redundant constraints for $w \in \widetilde{W}$. We rewrite it as

$$(3.1) \quad \sum_{i=1}^N (\overline{B}_\Delta^{(i)} w_\Delta^{(i)} + \overline{B}_\Pi^{(i)} w_\Pi^{(i)}) = 0.$$

We note that the matrices $\overline{B}_\Delta^{(i)}$ are square and invertible.

Let

$$M_\Delta = \prod_{ij} \overline{M}_{ij}.$$

We then obtain the following mixed formulation of the problem (2.1) with the constraints (3.1):

Find $(w_\Delta, w_\Pi, \lambda) \in W_\Delta \times W_\Pi \times M_\Delta$ satisfying

$$(3.2) \quad \begin{aligned} S_{\Delta\Delta}w_\Delta + S_{\Delta\Pi}w_\Pi + B_\Delta^t\lambda &= g_\Delta, \\ S_{\Pi\Delta}w_\Delta + S_{\Pi\Pi}w_\Pi + B_\Pi^t\lambda &= g_\Pi, \\ B_\Delta w_\Delta + B_\Pi w_\Pi &= 0, \end{aligned}$$

where

$$\begin{aligned} S_{\Delta\Delta} &= \text{diag}_{i=1, \dots, N} \left(S_{\Delta\Delta}^{(i)} \right), \\ S_{\Delta\Pi} &= \begin{pmatrix} S_{\Delta\Pi}^{(1)} R_\Pi^{(1)} \\ \vdots \\ S_{\Delta\Pi}^{(N)} R_\Pi^{(N)} \end{pmatrix}, \\ S_{\Pi\Delta} &= S_{\Delta\Pi}^t, \\ S_{\Pi\Pi} &= \sum_{i=1}^N (R_\Pi^{(i)})^t S_{\Pi\Pi}^{(i)} R_\Pi^{(i)}, \\ B_\Delta &= \left(\bar{B}_\Delta^{(1)}, \dots, \bar{B}_\Delta^{(N)} \right), B_\Pi = \sum_{i=1}^N \bar{B}_\Pi^{(i)} R_\Pi^{(i)}, \\ g_\Delta &= \begin{pmatrix} g_\Delta^{(1)} \\ \vdots \\ g_\Delta^{(N)} \end{pmatrix}, g_\Pi = \sum_{i=1}^N (R_\Pi^{(i)})^t g_\Pi^{(i)}, w_\Delta = \begin{pmatrix} w_\Delta^{(1)} \\ \vdots \\ w_\Delta^{(N)} \end{pmatrix}. \end{aligned}$$

After eliminating w_Δ and w_Π from (3.2), we obtain

$$F_{DP}\lambda = d.$$

We note that

$$\langle F_{DP}\lambda, \lambda \rangle = \max_{w \in \tilde{W}} \frac{\langle Bw, \lambda \rangle^2}{\langle \tilde{S}w, w \rangle},$$

where

$$\begin{aligned} B &= \begin{pmatrix} B_\Delta & B_\Pi \end{pmatrix}, \\ \tilde{S} &= \begin{pmatrix} S_{\Delta\Delta} & S_{\Delta\Pi} \\ S_{\Pi\Delta} & S_{\Pi\Pi} \end{pmatrix}, \quad w = \begin{pmatrix} w_\Delta \\ w_\Pi \end{pmatrix}. \end{aligned}$$

More precisely we compute

$$F_{DP} = B\tilde{S}^{-1}B^t = F_{\Delta\Delta} + F_{\Delta\Pi}F_{\Pi\Pi}^{-1}F_{\Pi\Delta},$$

where

$$\begin{aligned}
F_{\Delta\Delta} &= B_{\Delta} S_{\Delta\Delta}^{-1} B_{\Delta}^t = \sum_{i=1}^N \bar{B}_{\Delta}^{(i)} (S_{\Delta\Delta}^{(i)})^{-1} (\bar{B}_{\Delta}^{(i)})^t, \\
F_{\Pi\Delta} &= -(B_{\Pi} - B_{\Delta} S_{\Delta\Delta}^{-1} S_{\Delta\Pi}) = - \sum_{i=1}^N (\bar{B}_{\Pi}^{(i)} - \bar{B}_{\Delta}^{(i)} (S_{\Delta\Delta}^{(i)})^{-1} S_{\Delta\Pi}^{(i)}) R_{\Pi}^{(i)}, \\
F_{\Delta\Pi} &= F_{\Pi\Delta}^t, \\
F_{\Pi\Pi} &= \sum_{i=1}^N (R_{\Pi}^{(i)})^t (S_{\Pi\Pi}^{(i)} - S_{\Pi\Delta}^{(i)} (S_{\Delta\Delta}^{(i)})^{-1} S_{\Delta\Pi}^{(i)}) R_{\Pi}^{(i)}.
\end{aligned}$$

From the above formula, we can see that the computation $F_{DP}\lambda$ can be done by applying matrix-vector multiplications in each subdomain except the term $F_{\Pi\Pi}^{-1}$.

3.2. Preconditioner. We derive a preconditioner from the similar idea to [9], in which a Neumann-Dirichlet preconditioner is derived from a dual norm on the Lagrange multiplier space by using a duality pairing between the Lagrange multiplier space and the finite element space on nonmortar sides. In the following, the idea is provided in more detail.

We further decompose the space \widetilde{W} into

$$\widetilde{W} = W_{\Delta} \oplus W_{\Pi} = W_{\Delta,n} \oplus W_{\Delta,m} \oplus W_{\Pi},$$

where the subscript n stands for the space of vectors for the unknowns at the interior of nonmortar faces and the subscript m stands for the remaining unknowns. In other words, we split a vector $w_{\Delta} \in W_{\Delta}$ into

$$w_{\Delta} = \begin{pmatrix} w_{\Delta,n} \\ w_{\Delta,m} \end{pmatrix},$$

where $w_{\Delta,n}$ are unknowns at the interior of nonmortar faces and $w_{\Delta,m}$ are the remaining unknowns. We recall the mortar matching condition

$$B_{\Delta} w_{\Delta} + B_{\Pi} w_{\Pi} = 0.$$

It is then written into

$$(3.3) \quad B_{\Delta,n} w_{\Delta,n} + B_{\Delta,m} w_{\Delta,m} + B_{\Pi} w_{\Pi} = 0.$$

Here, the matrix $B_{\Delta,n}$ is square and invertible.

We will propose the Neumann-Dirichlet preconditioner of the form

$$(3.4) \quad \widehat{F}_{DP}^{-1} = B D \widetilde{S} D B^t,$$

where B and D are given by

$$B = \begin{pmatrix} B_{\Delta,n} & B_{\Delta,m} & B_{\Pi} \end{pmatrix}, \quad D = \begin{pmatrix} D_{nn} & & \\ & D_{mm} & \\ & & D_{\Pi\Pi} \end{pmatrix}.$$

The Neumann-Dirichlet preconditioner provides the weights

$$(3.5) \quad D_{nn} = (B_{\Delta,n}^t B_{\Delta,n})^{-1}, \quad D_{mm} = 0, \quad D_{\text{III}} = 0.$$

This preconditioner is originated from a dual norm on the Lagrange multiplier space M_Δ ; see [9]. We recall the space $W_{\Delta,n}$ and \widetilde{W} . For $w \in \widetilde{W}$, we define a norm

$$\|w\|_{\widetilde{W}}^2 = \langle \widetilde{S}w, w \rangle.$$

Since a function $w_{\Delta,n} \in W_{\Delta,n}$ has the zero average on each face F_{ij} and has zero values at subdomain vertices, its zero extension $\widetilde{w}_{\Delta,n}$ to W satisfies the primal constraints, i.e., $\widetilde{w}_{\Delta,n} \in \widetilde{W}$. We may write

$$\widetilde{w}_{\Delta,n} = \begin{pmatrix} w_{\Delta,n} \\ 0 \\ 0 \end{pmatrix} \in \widetilde{W}.$$

We then define a norm for $w_{\Delta,n}$ by

$$\|w_{\Delta,n}\|_{W_{\Delta,n}}^2 = \langle \widetilde{S}\widetilde{w}_{\Delta,n}, \widetilde{w}_{\Delta,n} \rangle,$$

and a dual norm on the space M_Δ by

$$(3.6) \quad \|\lambda\|_{W'_{\Delta,n}} = \max_{w_{\Delta,n} \in W_{\Delta,n}} \frac{\langle B_{\Delta,n} w_{\Delta,n}, \lambda \rangle}{\|w_{\Delta,n}\|_{W_{\Delta,n}}}.$$

The Neumann-Dirichlet preconditioner \widehat{F}_{DP}^{-1} is given by

$$(3.7) \quad \langle \widehat{F}_{DP} \lambda, \lambda \rangle = \|\lambda\|_{W'_{\Delta,n}}^2.$$

Similarly, the matrix F_{DP} can be obtained from a dual norm

$$\langle F_{DP} \lambda, \lambda \rangle = \|\lambda\|_{\widetilde{W}'}^2,$$

where the dual norm is given by

$$\|\lambda\|_{\widetilde{W}'}^2 = \max_{w \in \widetilde{W}} \frac{\langle Bw, \lambda \rangle^2}{\|w\|_{\widetilde{W}}^2} = \max_{w \in \widetilde{W}} \frac{\langle Bw, \lambda \rangle^2}{\langle \widetilde{S}w, w \rangle}.$$

The preconditioner is originated from the idea that these two dual norms will be sufficiently close so as to get \widehat{F}_{DP}^{-1} as a good preconditioner for F_{DP} . The lower bound estimate can be done from

$$(3.8) \quad \|\lambda\|_{W'_{\Delta,n}}^2 = \max_{w_{\Delta,n} \in W_{\Delta,n}} \frac{\langle B\widetilde{w}_{\Delta,n}, \lambda \rangle^2}{\langle \widetilde{S}\widetilde{w}_{\Delta,n}, \widetilde{w}_{\Delta,n} \rangle} \leq \max_{w \in \widetilde{W}} \frac{\langle Bw, \lambda \rangle^2}{\|w\|_{\widetilde{W}}^2} = \|\lambda\|_{\widetilde{W}'}^2,$$

because $\widetilde{w}_{\Delta,n}$ is contained in \widetilde{W} . In the following section, we will provide an upper bound for the Neumann-Dirichlet preconditioner \widehat{F}_{DP}^{-1} .

From (3.6) and (3.7), we find the following form of the preconditioner

$$\widehat{F}_{DP}^{-1} = \sum_{i=1}^N \begin{pmatrix} (B_{\Delta,n}^{(i)})^{-1} & 0 \\ 0 & 0 \end{pmatrix} S_{\Delta\Delta}^{(i)} \begin{pmatrix} ((B_{\Delta,n}^{(i)})^t)^{-1} \\ 0 \end{pmatrix},$$

that provides the weights in (3.5). The computation $\widehat{F}_{DP}^{-1}\lambda$ can be done by solving a Neumann-Dirichlet problem in each subdomain, i.e.,

$$S_{\Delta\Delta}^{(i)} = (A_{\Delta\Delta}^{(i)} - A_{\Delta I}^{(i)}(A_{II}^{(i)})^{-1}A_{I\Delta}^{(i)}),$$

where

$$A^{(i)} = \begin{pmatrix} A_{II}^{(i)} & A_{I\Delta}^{(i)} & A_{I\Pi}^{(i)} \\ A_{\Delta I}^{(i)} & A_{\Delta\Delta}^{(i)} & A_{\Delta\Pi}^{(i)} \\ A_{\Pi I}^{(i)} & A_{\Pi\Delta}^{(i)} & A_{\Pi\Pi}^{(i)} \end{pmatrix}.$$

Here $A^{(i)}$ is the stiffness matrix of the bilinear form $a_i(u, v)$ for $u, v \in X_i$ and the subscripts I , Π , and Δ stand for the subdomain interior unknowns, the unknowns for the primal variables, and the remaining unknowns, respectively.

When we compute

$$S_{\Delta\Delta}^{(i)} \begin{pmatrix} ((B_{\Delta,n}^{(i)})^t)^{-1} \\ 0 \end{pmatrix} \lambda,$$

we solve the problem

$$A_{II}^{(i)} u_I^{(i)} = A_{I\Delta}^{(i)} \begin{pmatrix} ((B_{\Delta,n}^{(i)})^t)^{-1} \lambda \\ 0 \end{pmatrix},$$

where Neumann boundary condition, $((B_{\Delta,n}^{(i)})^t)^{-1}\lambda$, is given on the nonmortar faces and zero Dirichlet boundary condition is provided on the remaining part of the subdomain boundary.

4. Condition number bound estimation. On the interface F_{ij} , we assume that Ω_i is the nonmortar side and Ω_j is the mortar side. We denote the mesh sizes in each subdomains Ω_i and Ω_j by h_i and h_j , respectively. We recall the space W_{ij} in (2.5).

DEFINITION 4.1. We define a projection $\pi_{ij} : H_{00}^{1/2}(F_{ij}) \rightarrow W_{ij}$ for $v \in H_{00}^{1/2}(F_{ij})$ by

$$\int_{F_{ij}} (v - \pi_{ij}v) \lambda_{ij} ds = 0 \quad \forall \lambda_{ij} \in M_{ij}.$$

For the space M_{ij} satisfying the conditions (A.1)-(A.4) (see Section 2.2), we can show that the projection π_{ij} is continuous on the space $H_{00}^{1/2}(F_{ij})$ (see [8] or [21]);

$$(4.1) \quad \|\pi_{ij}v\|_{H_{00}^{1/2}(F_{ij})} \leq C \|v\|_{H_{00}^{1/2}(F_{ij})} \quad \forall v \in H_{00}^{1/2}(F_{ij}),$$

where C is a constant not depending on H_i and h_i . Moreover, the projection is continuous on the space $L^2(F_{ij})$.

LEMMA 4.2. For $w = (w_1, \dots, w_N) \in \widetilde{W}$, we have

$$\begin{aligned} \|\pi_{ij}(w_i - w_j)\|_{H_{00}^{1/2}(F_{ij})}^2 &\leq C \max_{l=i,j} \left\{ \left(1 + \log \frac{H_l}{h_l} \right)^2 \right\} \left(|w_i|_{H^{1/2}(\partial\Omega_i)}^2 \right. \\ &\quad \left. + \left(1 + \frac{h_j}{h_i} \right) |w_j|_{H^{1/2}(\partial\Omega_j)}^2 \right). \end{aligned}$$

Proof. Let \mathcal{N}_i^h denote the set of nodes in the finite element space W_i . We decompose w_i into

$$w_i = I_{F_{ij}}^{(i)}(w_i) + I_{\partial F_{ij}}^{(i)}(w_i) \quad \text{on } F_{ij},$$

where

$$I_{F_{ij}}^{(i)}(w_i) = \begin{cases} w_i(x) & \text{for nodes } x \in F_{ij} \cap \mathcal{N}_i^h, \\ 0 & \text{for nodes } x \in \partial F_{ij} \cap \mathcal{N}_i^h, \end{cases}$$

and

$$I_{\partial F_{ij}}^{(i)}(w_i) = \begin{cases} w_i(x) & \text{for nodes } x \in \partial F_{ij} \cap \mathcal{N}_i^h, \\ 0 & \text{for nodes } x \in F_{ij} \cap \mathcal{N}_i^h. \end{cases}$$

Similarly we have

$$w_j = I_{F_{ij}}^{(j)}(w_j) + I_{\partial F_{ij}}^{(j)}(w_j).$$

We consider

$$(4.2) \quad \|\pi_{ij}(w_i - w_j)\|_{H_{00}^{1/2}(F_{ij})}^2$$

$$(4.3) \quad \leq C \left(\|\pi_{ij}(I_{F_{ij}}^{(i)}(w_i))\|_{H_{00}^{1/2}(F_{ij})}^2 + \|\pi_{ij}(I_{\partial F_{ij}}^{(i)}(w_i))\|_{H_{00}^{1/2}(F_{ij})}^2 \right. \\ \left. + \|\pi_{ij}(I_{F_{ij}}^{(j)}(w_j))\|_{H_{00}^{1/2}(F_{ij})}^2 + \|\pi_{ij}(I_{\partial F_{ij}}^{(j)}(w_j))\|_{H_{00}^{1/2}(F_{ij})}^2 \right).$$

From the continuity of π_{ij} in $H_{00}^{1/2}(F_{ij})$ and a face lemma [18, Lemma 4.24], we estimate the first and the third terms of the above equation

$$(4.4) \quad \|\pi_{ij}(I_{F_{ij}}^{(i)}(w_i))\|_{H_{00}^{1/2}(F_{ij})}^2 \leq C \left(1 + \log \frac{H_i}{h_i} \right)^2 \|w_i\|_{H^{1/2}(\partial\Omega_i)}^2,$$

$$(4.5) \quad \|\pi_{ij}(I_{F_{ij}}^{(j)}(w_j))\|_{H_{00}^{1/2}(F_{ij})}^2 \leq C \left(1 + \log \frac{H_j}{h_j} \right)^2 \|w_j\|_{H^{1/2}(\partial\Omega_j)}^2.$$

We now estimate the fourth term in (4.3)

$$(4.6) \quad \begin{aligned} \|\pi_{ij}(I_{\partial F_{ij}}^{(j)}(w_j))\|_{H_{00}^{1/2}(F_{ij})}^2 &\leq Ch_i^{-1} \|\pi_{ij}(I_{\partial F_{ij}}^{(j)}(w_j))\|_{L^2(F_{ij})}^2, \\ &\leq Ch_i^{-1} \|I_{\partial F_{ij}}^{(j)}(w_j)\|_{L^2(F_{ij})}^2, \\ &\leq Ch_i^{-1} h_j \|I_{\partial F_{ij}}^{(j)}(w_j)\|_{L^2(\partial F_{ij})}^2, \\ &\leq Ch_i^{-1} h_j \left(1 + \log \frac{H_j}{h_j} \right) \|w_j\|_{H^{1/2}(\partial\Omega_j)}^2. \end{aligned}$$

Here we have used an inverse inequality, the continuity of π_{ij} in $L^2(F_{ij})$, and an edge lemma [18, Lemma 4.17].

Similarly, we obtain

$$(4.7) \quad \|\pi_{ij}(I_{\partial F_{ij}}^{(i)}(w_i))\|_{H_{00}^{1/2}(F_{ij})}^2 \leq C \left(1 + \log \frac{H_i}{h_i}\right) \|w_i\|_{H^{1/2}(\partial\Omega_i)}^2.$$

Since $w = (w_1, \dots, w_N) \in \widetilde{W}$, w_i and w_j have the same average over the common face F_{ij} , i.e.,

$$c_{ij} = \frac{\int_{F_{ij}} w_i ds}{\int_{F_{ij}} 1 ds} = \frac{\int_{F_{ij}} w_j ds}{\int_{F_{ij}} 1 ds}.$$

We can replace w_i and w_j in (4.2) by $w_i - c_{ij}$ and $w_j - c_{ij}$, respectively, and then replace the norm by the semi-norm in the above estimates (4.4)-(4.7) using a Poincaré inequality.

Combining (4.3) with (4.4)-(4.7), the desired estimate follows. \square

REMARK 4.3. *The face average constraints are important in applying a Poincaré inequality to the above analysis, while the continuity constraints at vertices are not necessary. We are able to consider face constraints only as the primal constraints. This makes it possible to extend our FETI-DP formulation to the second generation of mortar methods. We note that vertex constraints can not be used as primal constraints for the second generation of mortar discretization.*

To obtain a condition number bound that does not depend on mesh sizes and the coefficients, we need the following assumptions.

ASSUMPTION 4.4. *On a common interface F_{ij} , we choose the subdomain Ω_i with smaller ρ_i as the nonmortar side and the subdomain Ω_j with larger ρ_j as the mortar side.*

ASSUMPTION 4.5. *For each F_{ij} , we assume that*

$$(4.8) \quad \frac{h_j}{h_i} \simeq C \left(\frac{\rho_j}{\rho_i}\right)^\gamma, \text{ with } 0 \leq \gamma \leq 1,$$

where Ω_i is the nonmortar side and Ω_j is the mortar side, and the constant C does not depend on any mesh parameters h_i and the coefficients ρ_i .

The Assumption 4.4 is conventionally used in the analysis of the mortar methods; see [19, Remark 3.1]. The Assumptions 4.4 and 4.5 together imply that the subdomain Ω_i with smaller coefficient ρ_i will have finer discretization (smaller h_i) that is practically meaningful. We note that numerical results in [19, Section 1.5] show that the optimal ratio h_j/h_i tends to become $(\rho_j/\rho_i)^{1/4}$ as an adaptivity strategy is applied successively to elliptic problems in 2D.

We now estimate the upper bound of the FETI-DP algorithm.

LEMMA 4.6. *Under Assumptions 4.4 and 4.5, for $\lambda \in M_\Delta$, we have*

$$\langle F_{DP}\lambda, \lambda \rangle \leq C \max_{i=1, \dots, N} \left\{ \left(1 + \log \frac{H_i}{h_i}\right)^2 \right\} \langle \widehat{F}_{DP}\lambda, \lambda \rangle,$$

where the constant C does not depend on h_i , H_i , and ρ_i .

Proof. We note that

$$(4.9) \quad \langle F_{DP}\lambda, \lambda \rangle = \|\lambda\|_{\widetilde{W}'}^2 = \max_{w \in \widetilde{W}} \frac{\langle Bw, \lambda \rangle^2}{\langle \widetilde{S}w, w \rangle},$$

$$(4.10) \quad \langle \widehat{F}_{DP}\lambda, \lambda \rangle = \|\lambda\|_{W'_{\Delta,n}}^2 = \max_{w_{\Delta,n} \in W_{\Delta,n}} \frac{\langle B_{\Delta,n}w_{\Delta,n}, \lambda \rangle^2}{\|w_{\Delta,n}\|_{W_{\Delta,n}}^2}.$$

From the definitions of B and π_{ij} , we have

$$\langle Bw, \lambda \rangle^2 = \left(\sum_{ij} \int_{F_{ij}} \pi_{ij}(w_i - w_j) \lambda_{ij} ds \right)^2.$$

Let

$$z_{ij} = \pi_{ij}(w_i - w_j).$$

From $1 \in M_{ij}$ and the definition of π_{ij} , we have

$$\int_{F_{ij}} z_{ij} ds = \int_{F_{ij}} (w_i - w_j) ds = 0,$$

Since z_{ij} has zero average on F_{ij} and has zero values on ∂F_{ij} , after the transform introduced in Section 3.1 we may write

$$(4.11) \quad \langle Bw, \lambda \rangle = \sum_{ij} \int_{F_{ij}} \pi_{ij}(w_i - w_j) \lambda_{ij} ds = \langle B_{\Delta,n}z_{\Delta,n}, \lambda \rangle^2,$$

where $z_{\Delta,n} = z_{ij}$ on F_{ij} and $z_{\Delta,n} \in W_{\Delta,n}$. We note that $z_{\Delta,n}$ can be a function or a vector of unknowns, i.e., it is a function in the equation $z_{\Delta,n} = z_{ij}$, by using the change of base, and it is a vector of unknowns in the term $B_{\Delta,n}z_{\Delta,n}$, by using the change of unknowns. From the above relation (4.11) and (4.10), we get

$$(4.12) \quad \langle Bw, \lambda \rangle^2 \leq \langle \widehat{F}_{DP}\lambda, \lambda \rangle \|z_{\Delta,n}\|_{W_{\Delta,n}}^2.$$

We will show that

$$(4.13) \quad \|z_{\Delta,n}\|_{W_{\Delta,n}}^2 \leq C \max_i \left\{ \left(1 + \log \frac{H_i}{h_i} \right)^2 \right\} \langle \widetilde{S}w, w \rangle.$$

The desired estimate then follows from (4.9) and (4.12).

Let $\widetilde{z}_{\Delta,n}$ be the zero extension of $z_{\Delta,n}$ to the all interfaces, i.e., into the space W . It is easy to see that $\widetilde{z}_{\Delta,n} \in \widetilde{W}$. Let $z_i = \widetilde{z}_{\Delta,n}|_{\partial\Omega_i}$, the restriction of $\widetilde{z}_{\Delta,n}$ to the subdomain Ω_i .

We then obtain

$$\begin{aligned}
\|z_{\Delta,n}\|_{W_{\Delta,n}}^2 &= \langle \tilde{S} \tilde{z}_{\Delta,n}, \tilde{z}_{\Delta,n} \rangle \\
&= \sum_i \langle S^{(i)} z_i, z_i \rangle \\
&\leq C \sum_i \rho_i |z_i|_{H^{1/2}(\partial\Omega_i)}^2 \\
&\leq C \sum_i \sum_j \rho_i \|z_{ij}\|_{H_{00}^{1/2}(F_{ij})}^2 \\
&= C \sum_i \sum_j \rho_i \|\pi_{ij}(w_i - w_j)\|_{H_{00}^{1/2}(F_{ij})}^2 \\
&\leq C \max_{l=1,\dots,N} \left\{ \left(1 + \log \frac{H_l}{h_l} \right)^2 \right\} \left(\sum_{ij} \left(\langle S^{(i)} w_i, w_i \rangle \right. \right. \\
&\quad \left. \left. + \left(1 + \frac{h_j}{h_i} \right) \frac{\rho_i}{\rho_j} \langle S^{(j)} w_j, w_j \rangle \right) \right).
\end{aligned}$$

Here we used the well-known inequality

$$\rho_i |z_i|_{H^{1/2}(\partial\Omega_i)}^2 \leq \langle S^{(i)} z_i, z_i \rangle \leq C \rho_i |z_i|_{H^{1/2}(\partial\Omega_i)}^2,$$

the relation in (2.2), and Lemma 4.2.

From Assumptions 4.4 and 4.5, we obtain

$$\left(1 + \frac{h_j}{h_i} \right) \frac{\rho_i}{\rho_j} \leq C \left(1 + \left(\frac{\rho_j}{\rho_i} \right)^\gamma \right) \frac{\rho_i}{\rho_j} \leq C \left\{ \frac{\rho_i}{\rho_j} + \left(\frac{\rho_i}{\rho_j} \right)^{1-\gamma} \right\} \leq C,$$

where Ω_i is the nonmortar side of F_{ij} . Therefore, we have shown the desired estimate (4.13) with the constant C independent of h_i , H_i , and ρ_i . \square

REMARK 4.7. *The above analysis can be applied to two dimensional problems with edge average constraints only as primal constraints.*

From the lower bound estimate in (3.8) and the upper bound estimate in Lemma 4.6, we then have the condition number bound of the FETI-DP algorithm with the Neumann-Dirichlet preconditioner.

THEOREM 4.8. *Under Assumptions 4.4 and 4.5,*

$$\kappa(\widehat{F}_{DP}^{-1} F_{DP}) \leq C \max_i \left\{ \left(1 + \log \frac{H_i}{h_i} \right)^2 \right\},$$

where the constant C does not depend on h_i , H_i , and ρ_i .

5. Numerical Results. In this section, we provide numerical tests for the FETI-DP formulation developed in this paper. We consider the following model problem:

$$\begin{aligned}
-\nabla \cdot (\alpha(x, y, z) \nabla u) &= f \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial\Omega,
\end{aligned}$$

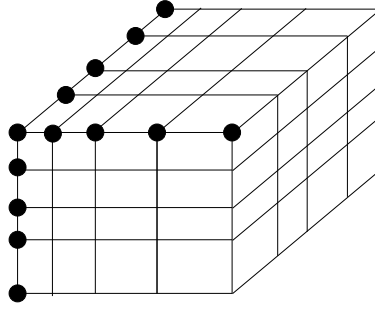


FIG. 2. Hexahedral elements generated by the quasi-uniform nodes (black dots) along each axis when $n = 5$

where $\Omega = (0, 1)^3$ is the unit cube, $u(x, y, z) = \sin(\pi x)y(1-y) \sin(\pi z)$ is the exact solution, and $\alpha(x, y, z) = 1$.

We divide the domain Ω into $N \times N \times N$ cubical subdomains with side length $H = 1/N$. Each subdomain is discretized by conforming trilinear finite elements and these elements are non-matching across subdomain interfaces. To make them non-matching, we generate hexahedral elements in each subdomain as follows. In a subdomain Ω_i , we choose n random quasi-uniform nodes along each axis including its end points. From these nodes, we generate nonuniform structured grids, which consists of hexahedrons with mesh parameter h_i (see Figure 2). Since the finite elements are obtained from the quasi-uniform nodes, the mesh parameter h_i is comparable to $H/(n-1)$. The corresponding Lagrange multiplier is given by the tensor product of two dimensional multipliers considered in [20]. Even though the theory provided in the previous section was developed for tetrahedral finite elements, it extends to the approximation described above without difficulty.

To see the scalability of the preconditioner, we perform two types of experiments. First, we keep the number of subdomains fixed and increase the number of nodes n along each axis. In the second test, we have the number of subdomains increasing with a fixed subdomain problem size. We solve the FETI-DP equation using conjugate gradient method with and without the Neumann-Dirichlet preconditioner \widehat{F}_{DP}^{-1} . The conjugate gradient iteration continues until the relative residual norm is reduced by 10^{-6} .

In Table 1, the number of CG iterations and condition numbers are shown when the number of nodes n increases with the fixed number of subdomains $N^3 = 4^3$. From the result, we observe the \log^2 -growth of the condition number for the proposed preconditioner. It also shows that the preconditioner effectively reduces CG iterations. Table 2 shows the numerical results when we fix $n = 5$ and increase the number of subdomains. As proved in our analysis, the condition number becomes stable as the number of subdomains is increasing.

We have tested the FETI-DP algorithm for the simple elliptic problem. Further computational work should be done for the elliptic problems with discontinuous constant coefficients. It is also important to find the optimal ratio of meshes $h_i/h_j \simeq (\rho_i/\rho_j)^\gamma$ for some $0 \leq \gamma \leq 1$

$n - 1$	Without preconditioner		With preconditioner	
	Iter	Cond	Iter	Cond
4	26	1.75e+1	17	6.90
8	67	2.17e+2	23	1.29e+1
12	75	3.15e+2	25	1.56e+1
16	83	3.89e+2	26	1.74e+1
24	92	5.26e+2	29	2.04e+1
32	99	6.39e+2	30	2.24e+1

TABLE 1

The number of CG iterations (*Iter*) and corresponding condition numbers (*Cond*) for the FETI-DP operator with or without the Neumann-Dirichlet preconditioner when subdomain problem size n^3 increases with the fixed number of subdomains N^3 ($N = 4$)

N^3	Without preconditioner		With preconditioner	
	Iter	Cond	Iter	Cond
2^3	21	1.63e+1	15	5.72
3^3	24	2.14e+1	17	7.13
4^3	26	1.75e+1	17	6.90
6^3	26	2.09e+1	18	7.91
8^3	27	2.22e+1	18	8.41

TABLE 2

The number of CG iterations (*Iter*) and corresponding condition numbers (*Cond*) for the FETI-DP operator with or without the Neumann-Dirichlet preconditioner when number of subdomains N^3 increases with the fixed subdomain problem size n^3 ($n = 5$)

as performed in [21] by applying an appropriate adaptivity strategy.

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