Domain Decomposition Algorithms for Mortar discretization

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Outline

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Domain Decomposition
FETI-DP algorithms
BDDC algorithm
Mortar Discretization
BDDC and FETI-DP for Mortar
Connection
Numerical Results
Extensions
Conclusions

Domain Decomposition Algorithms in general

✔ A model elliptic problem

$$\begin{aligned} -\Delta u(x) &= f, \quad x \in \Omega, \\ u(x) &= 0, \quad x \in \partial\Omega. \end{aligned}$$

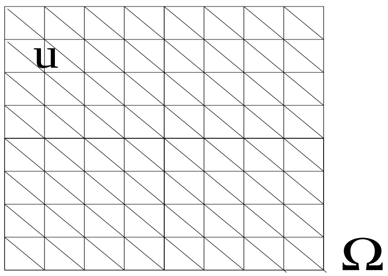


Figure 1: A triangulation on Ω

✔ A discrete problem

$$Au = f.$$

Iterative methods

- ✔ **Cost for the exact solver to $Au = f$**
  - ✘ $O(n^3)$ when $A \in R^{n\times n}$
  - ✘ Inefficient when n gets larger
- ✔ **Iterative methods with preconditioners**
  - ✘ Reduce the cost and memory for sparse large matrix A
  - ✘ Preconditioners to accelerate convergence

Ex. CGM, GMRES

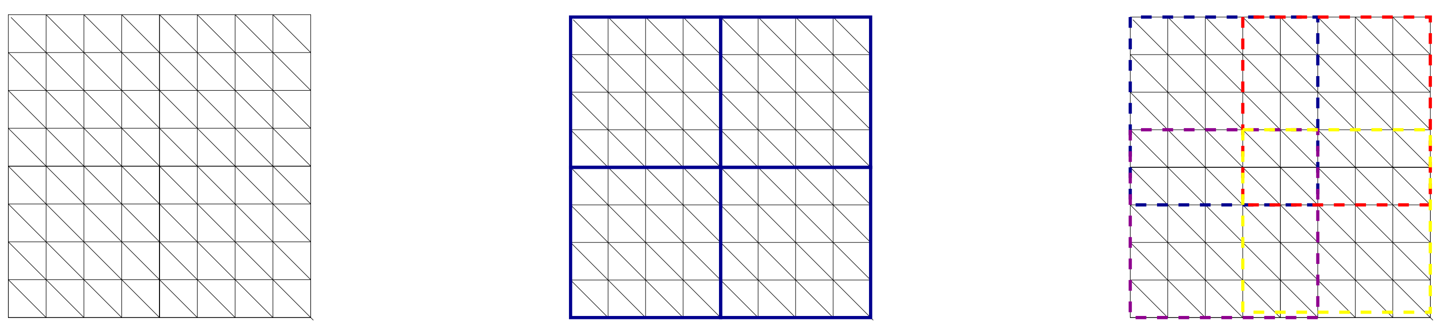
Preconditioners by Domain Decomposition

Domain decomposition methods are intended for construction of scalable preconditioners easily adaptable to parallel processors.

✔ **Decomposition of the domain and local problems**

$$\Omega = \bigcup_{i=1}^{N} \Omega_i, \quad A_i$$

Nonoverlapping (center), Overlapping (right)



Preconditioners by Domain Decomposition

- ✔ **Preconditioner**

$$\widehat{A}^{-1} = \sum_{i=1}^{N} R_i^t D_i A_i^{-1} D_i R_i$$

R_i : restriction, D_i : weights

- ✔ **Convergence of iterative methods** depends on

$$\kappa(\widehat{A}^{-1}A) = \frac{\lambda_{Max}(\widehat{A}^{-1}A)}{\lambda_{Min}(\widehat{A}^{-1}A)}$$

- ✔ **The number of subdomains** increases
  - ✘ The size of A_i gets smaller, i.e., less cost for computing $\widehat{A}^{-1}$
  - ✘ $\kappa(\widehat{A}^{-1}A)$ gets larger, i.e., slow down the convergence

Preconditioners by Domain Decomposition

- ✔ **Coarse finite element space**
  A special finite element space depending on the subdomain partition $\{\Omega_i\}_{i=1}^N$ with the dimension comparable to N.
- ✔ **Preconditioner enriched with a coarse component**

$$\widehat{A}^{-1} = R_0^t A_0^{-1} R_0 + \sum_{i=1}^N R_i^t D_i A_i^{-1} D_i R_i$$

- ✔ **Scalable preconditioner $\kappa(\widehat{A}^{-1}A)$ gets stable as N increases.**

$$\kappa(\widehat{A}^{-1}A) \leq C(1 + log(H/h))^2$$

H/h: local problem size

Domain Decomposition Algorithms

The main goal of Domain Decomposition Algorithms is providing

- ✔ **appropriate weights and local problems**
- ✔ **appropriate coarse problem**

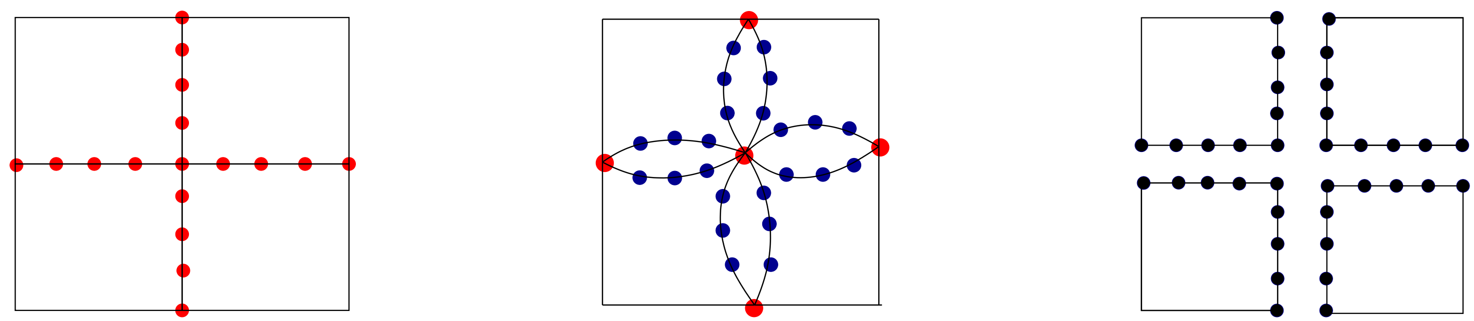
leading to a scalable preconditioner.

Domain Decomposition Algorithms

- ✔ Overlapping Schwarz algorithms
- ✔ Iterative substructuring
- ✔ Balancing Neumann-Neumann
- ✔ FETI and **FETI-DP algorithms**
- ✔ **BDDC algorithms**

FETI-DP (Dual-Primal Finite Element Tearing and Interconnecting method)

Space $\widehat{X}$ (left), Interconnecting $\widetilde{X}$ (center), Tearing X (right)



Interface unknowns : dual and primal unknowns

$$\widehat{X} \subset \widetilde{X} \subset X$$

Corresponding problem matrix

$$(\widehat{X}, \widehat{K}),\ (\widetilde{X}, \widetilde{K}),\ (X, K), \quad K = diag_i(K_i)$$

Equivalent Problems

✔ **Minimization in $\widehat{X}$**

$$\widehat{K}u = \widehat{f} \Longleftrightarrow \min_{v \in \widehat{X}} \left(\frac{1}{2} v^t \widehat{K} v - v^t \widehat{f} \right)$$

✔ **Constraint minimization in $X = \prod_{i=1}^{N} X_i$**

$$\min_{v \in X} \left(\frac{1}{2} v K v - v^t f \right),$$

with **continuity constraints** $v_i = v_j$ on interfaces.

Constraint minimization after enforcing primal constraints

- ✔ **Space $\widetilde{X}$: continuous at primal unknowns**
  v_Π (primal), v_Δ (dual)
- ✔ **Constraint Minimization problem in $\widetilde{X}$**

$$\min_{v \in \widetilde{X}} \left(\frac{1}{2} v^t \widetilde{K} v + v^t \widetilde{f} \right)$$

  with constraints $B_\Delta v_\Delta = 0$

- ✔ **Mixed Problem**

$$\begin{aligned} K_{\Delta\Delta} u_\Delta + K_{\Delta\Pi} u_\Pi + B_\Delta^t \lambda &= f_\Delta, \\ K_{\Pi\Delta} u_\Delta + K_{\Pi\Pi} u_\Pi &= f_\Pi, \\ B_\Delta u_\Delta &= 0. \end{aligned}$$

FETI-DP algorithm

- ✔ Elimination of unknowns other than λ

$$B\widetilde{K}^{-1}B^t\lambda = d, \tag{1}$$

$$B = \begin{pmatrix} B_\Delta & 0 \end{pmatrix}, \quad \widetilde{K} = \begin{pmatrix} K_{\Delta\Delta} & K_{\Delta\Pi} \\ K_{\Pi\Delta} & K_{\Pi\Pi} \end{pmatrix}$$

- ✔ FETI-DP solves (1) iteratively using

$$\widehat{M}^{-1} = B \begin{pmatrix} D_\Delta & 0 \\ 0 & 0 \end{pmatrix} \widetilde{K} \begin{pmatrix} D_\Delta & 0 \\ 0 & 0 \end{pmatrix} B^t$$

- ✔ Dual form to the standard DD

$$\widetilde{K}^{-1} = \begin{pmatrix} I & 0 \\ -K_{\Pi\Delta}K_{\Delta\Delta}^{-1} & I \end{pmatrix} \begin{pmatrix} K_{\Delta\Delta}^{-1} & 0 \\ 0 & S_{\Pi\Pi}^{-1} \end{pmatrix} \begin{pmatrix} I & -K_{\Delta\Delta}^{-1}K_{\Delta\Pi} \\ 0 & I \end{pmatrix}$$

$$K_{\Delta\Delta}^{-1} = diag_i\left((K_{\Delta\Delta}^{(i)})^{-1}\right) \quad S_{\Pi\Pi} = K_{\Pi\Pi} - K_{\Pi\Delta}K_{\Delta\Delta}^{-1}K_{\Delta\Pi}$$

Balancing Domain Decomposition with Constraints (BDDC)

- ✔ Primal problem assembled by local problems

$$\widehat{K}u = f,\ u \in \widehat{X}$$

By using restriction $R_i : \widehat{X} \to X_i$,

$$\sum_{i=1}^{N} R_i^t K_i R_i u = \sum_{i=1}^{N} R_i^t f_i \tag{2}$$

- ✔ BDDC solves (2) iteratively using

$$\widehat{M}^{-1} = \sum_{i=1}^{N} R_i^t D_i (K_L^{(i)})^{-1} D_i R_i + R_0^t K_0^{-1} R_0$$

$K_L^{(i)}$ local problems, K_0 coarse problem

Coarse finite element space X_0 well connected to FETI-DP

$X_0 \subset \widetilde{X}$: continuous at primal unknowns,
its dimension is the number of primal unknowns

- ✔ Each primal unknown $x_{\Pi,l}$, we consider a basis function $\phi_l \in \widetilde{X}$ such that
  1. $\phi_l(x_{\Pi,k}) = \delta_{lk}$
  2. minimizes the energy-norm (scalability)

$$E(\phi_l) = \phi_l^t \widetilde{K} \phi_l$$

- ✔ Space X_0 spanned by

$$R_0 = \begin{pmatrix} \phi_1 & \cdots & \phi_{N_\Pi} \end{pmatrix}, \; (\, note : \; K_0 = R_0^t \widetilde{K} R_0)$$

Coarse finite element space X_0 well connected to FETI-DP

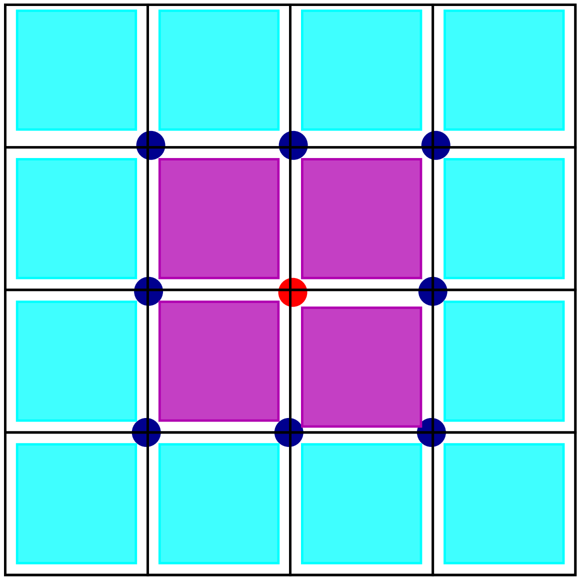


Figure 2: Support (purple subdomains) of the coarse basis associated to the red node

The basis has value 1 at the red node and has value 0 at the all blue nodes.

Local problems

- Local finite element spaces
  $X_\Delta^{(i)} \subset X_i$: zero at the primal unknowns
- Local problems

  Separate unknowns $v_i \in X_i$
  $v_\Pi^{(i)} \in X_\Pi^{(i)}$: primal, $v_\Delta^{(i)} \in X_\Delta^{(i)}$: dual

$$K_i = \begin{pmatrix} K_{\Delta\Delta}^{(i)} & K_{\Delta\Pi}^{(i)} \\ K_{\Pi\Delta}^{(i)} & K_{\Pi\Pi}^{(i)} \end{pmatrix}$$

- Preconditioner

$$\widehat{M}^{-1} = \sum_{i=1}^{N} (R_\Delta^{(i)})^t D_\Delta^{(i)} (K_{\Delta\Delta}^{(i)})^{-1} D_\Delta^{(i)} (R_\Delta^{(i)})^t + R_0^t K_0^{-1} R_0$$

$R_\Delta^{(i)}$ restriction to $X_{\Delta,i}$

The BDDC and FETI-DP with the same primal unknowns

- ✔ (Preconditioner*Problem) forms

$$\begin{aligned} B_{DDC} &= (R_D^t \widetilde{K}^{-1} R_D) R^t \widetilde{K} R \\ F_{DP} &= (B_\Sigma \widetilde{K} B_\Sigma^t) B \widetilde{K}^{-1} B^t, \end{aligned}$$

  $R : \widehat{X} \longrightarrow \widetilde{X}$, D, Σ : weights

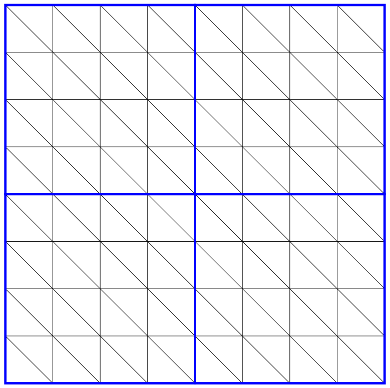
- ✔ Share the same eigenvalues except 1
  when the weights D and Σ satisfy certain relation
- ✔ Scalable
  There exist weights D and Σ such that

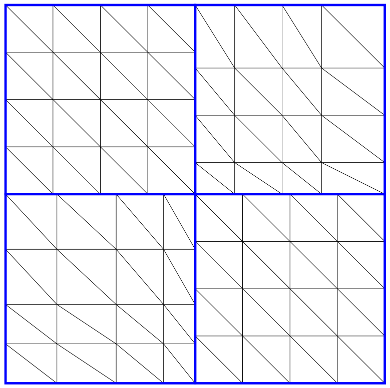
$$\kappa(B_{DDC}), \kappa(F_{DP}) \leq C(1 + log(H/h))^2$$

  (H/h local problem size)

Conforming and nonconforming finite element spaces

Matching (left) and Nonmatching (right)





$H_0^1(\Omega)$ the space where the solution of PDE exists
left one $\subset H_0^1(\Omega)$ (conforming)
right one is not contained in $H_0^1(\Omega)$ (nonconforming)
We need a special tool to approximate the solution in the nonconforming case.
The tool is **mortar discretization**.

Generalization BDDC and FETI-DP to the nonconforming case

- ✔ Form two linear systems from mortar discretization

  1. primal form
  2. dual form

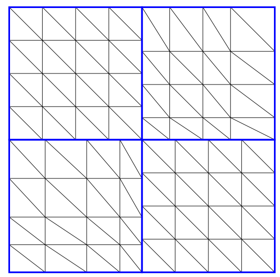
- ✔ Apply BDDC and FETI-DP to the linear systems
  BDDC to the primal form
  FETI-DP to the dual form

- ✔ Providing scalable preconditioners as efficient as the ones in the conforming case

$$\kappa(B_{DDC}), \kappa(F_{DP}) \leq C(1 + log(H/h))^2$$

Mortar Discretization

✔ Mortar matching condition

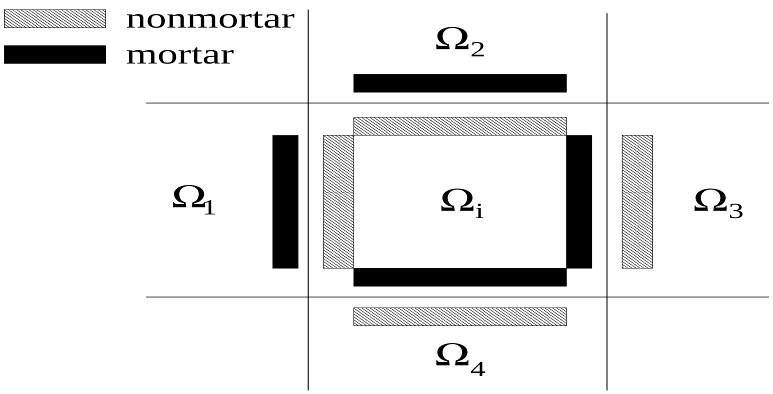


$$\int_{\Gamma_{ij}} (v_i - v_j)\psi \, ds = 0, \; \forall \psi \in M_{ij}$$

Mortar finite element space is a subspace of $\prod_{i=1}^{N} X_i$ that satisfies the mortar matching condition.
Mortar discretization is to approximate the solution in **the mortar finite element space**.

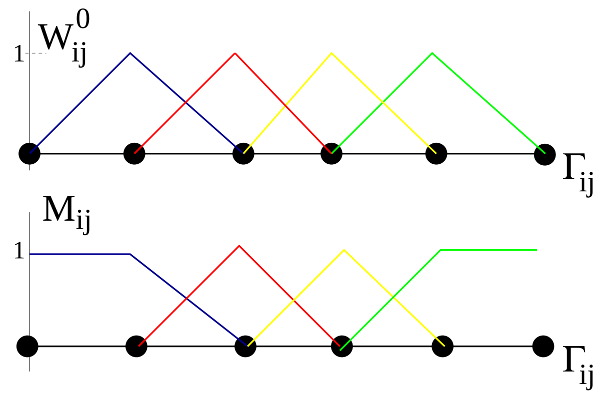
Lagrange multipliers spaces

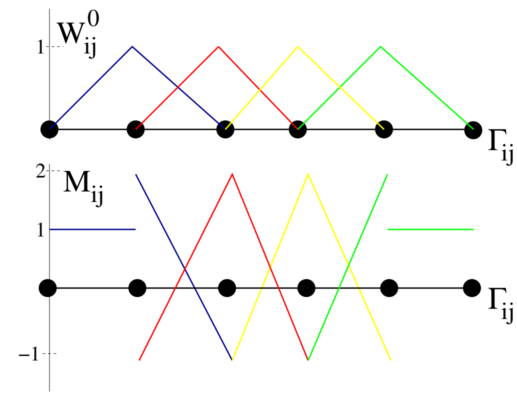
✔ Mortar and nonmortar sides



- ✔ M_{ij} on each nonmortar side Γ_{ij}
  - ✘ the same dimension as that of finite element functions supported in Γ_{ij}
  - ✘ contains constant functions

Examples of M_{ij}





✔ **Finite Element spaces**

X_i : subdomain

W_i : subdomain boundary

$X = \prod_{i=1}^N X_i$, $W = \prod_{i=1}^N W_i$
(can be **discontinuous** across interfaces)

$\widehat{X} \subset X$, $\widehat{W} \subset W$
(**mortar matching condition**)

Primal Constraints

- ✔ **Select some primal constraints from the mortar condition**

$$\int_{\Gamma_{ij}} (v_i - v_j)\psi \, ds = 0 \quad \forall \psi \in M_{ij}$$

Example.
$\psi = 1$, **average matching across** Γ_{ij}

- ✔ **Spaces with primal constraints** $\widetilde{X}$, $\widetilde{W}$

$$\widehat{X} \subset \widetilde{X} \subset X$$

$$\widehat{W} \subset \widetilde{W} \subset W$$

Change of bases (unknowns) associated to the primal constraints

- ✔ **Primal unknowns for $w \in \widetilde{W}$**

$$\widehat{w}_{\Pi} = \int_{\Gamma_{ij}} w \, ds$$

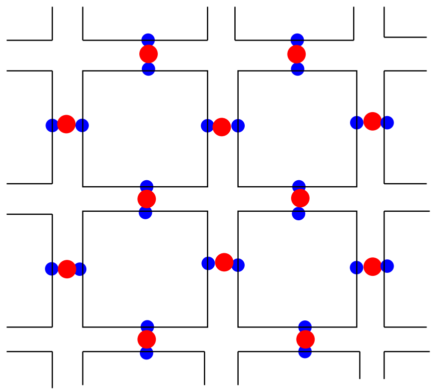
- ✔ **Change of bases (unknowns)**
  To make the primal unknowns explicitly
  To derive a similar operator form to the conforming case

$$w|_{\Gamma_{ij}} = T_{ij} \begin{pmatrix} \widehat{w}_{\Delta} \\ \widehat{w}_{\Pi} \end{pmatrix}.$$

$\widehat{w}_{\Delta}$ (dual), $\widehat{w}_{\Pi}$ (primal)

Change of bases (unknowns) associated to the primal constraints

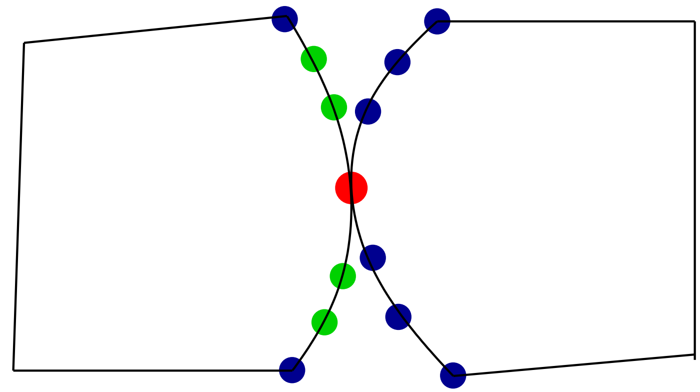
✔ Global and local primal unknowns



w_Π : global primal unknowns (red nodes)
average of functions on the interface

$w_\Pi^{(i)}$: local primal unknowns (blue nodes)
average of functions on each edge

Space Decomposition



✔ **Decomposition of $\widetilde{W}$**

Π (primal), Δ (dual)

$\widetilde{W} = W_\Delta \times W_\Pi = W_{\Delta,n} \times W_{\Delta,m} \times W_\Pi,$

We further decompose the dual part into
n : nonmortar, m : the remaining

Representation of $\widehat{W}$

- ✔ **Mortar matching condition**

  $B_\triangle w_\triangle + B_\Pi w_\Pi = 0$

  $B_{\Delta,n} w_{\Delta,n} + B_{\Delta,m} w_{\Delta,m} + B_\Pi w_\Pi = 0$

  $$w_{\Delta,n} = -B_{\Delta,n}^{-1}(B_{\Delta,m} w_{\Delta,m} + B_\Pi w_\Pi)$$

  W_G : space of unknowns $(w_{\Delta,m},\ w_\Pi)$

- ✔ **Mortar Map** $R_G^t : W_G \rightarrow \widehat{W}$

$$\begin{pmatrix} w_{\Delta,n} \\ w_{\Delta,m} \\ w_{\Delta,\Pi} \end{pmatrix} = \begin{pmatrix} -B_{\Delta,n}^{-1} B_{\Delta,m} & -B_{\Delta,n}^{-1} B_\Pi \\ I & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} w_{\Delta,m} \\ w_\Pi \end{pmatrix}$$

Mortar Discretization

- ✔ K_i : local stiffness matrices
- ✔ S_i : Schur complement (eliminating interior unknowns)
- ✔ Subassembly at primal unknowns

$$S_i = \begin{pmatrix} S^{(i)}_{\Delta\Delta} & S^{(i)}_{\Delta\Pi} \\ S^{(i)}_{\Pi\Delta} & S^{(i)}_{\Pi\Pi} \end{pmatrix} \Longrightarrow \widetilde{S} = \begin{pmatrix} S_{\Delta\Delta} & S_{\Delta\Pi} \\ S_{\Pi\Delta} & S_{\Pi\Pi} \end{pmatrix}$$

- ✔ **Mortar discretization**

Note that $R_G^t : W_G \rightarrow \widehat{W} (\subset \widetilde{W})$

$$R_G \widetilde{S} R_G^t w_G = R_G \widetilde{g}.$$

$R_G^t w_G \in \widehat{W}$ is the desired solution in the mortar discretization.

Equivalent dual problem

- ✔ **Constraint minimization problem**

  Primal problem : $R_G\widetilde{S}R_G^t w_G = R_G\widetilde{g}$

  $\min_{w\in\widetilde{W}} \left\{\frac{1}{2}w^t\widetilde{S}w + w^t\widetilde{g}\right\}$ **with** $Bw = 0$**,**

  $B = \left(\begin{array}{ccc} B_{n,\Delta} & B_{m,\Delta} & B_{\Pi} \end{array} \right).$

- ✔ **Mixed form**

$$\left(\begin{array}{cc} \widetilde{S} & B^t \\ B & 0 \end{array}\right)\left(\begin{array}{c} w \\ \lambda \end{array}\right) = \left(\begin{array}{c} \widetilde{g} \\ 0 \end{array}\right)$$

- ✔ **Dual problem**

$$B\widetilde{S}^{-1}B^t\lambda = B\widetilde{S}^{-1}\widetilde{g}$$

BDDC and FETI-DP for Mortar discretization

- ✔ **BDDC algorithm solves**

$$R_G \widetilde{S} R_G^t w_G = R_G \widetilde{g}$$

with a preconditioner (Coarse + Local problems)

- ✔ **FETI-DP algorithm solves**

$$B^t \widetilde{S}^{-1} B \lambda = B^t \widetilde{S}^{-1} \widetilde{g}$$

with a preconditioner (Local problems)

Coarse and local problems

✔ **Coarse problem**

$W_\Pi \subset \widetilde{W}$: a basis $\phi_k(x) \in \widetilde{W}$ for each primal unknown $x_{\Pi,k}$ on an edge,

1. $\phi_k(x_{\Pi,l}) = \delta_{kl}$
   average one on the edge and zero on the other edges
2. minimizing energy $E(\phi_k) = \phi_k^t \widetilde{S} \phi_k$

$F_{\Pi\Pi}$: coarse problem matrix

$$F_{\Pi\Pi} = S_{\Pi\Pi} - S_{\Pi\Delta} S_{\Delta\Delta}^{-1} S_{\Delta\Pi}$$

Note $\widetilde{S} = \begin{pmatrix} S_{\Delta\Delta} & S_{\Delta\Pi} \\ S_{\Pi\Delta} & S_{\Pi\Pi} \end{pmatrix}$

Coarse and local problems

✔ **Local problems**

$W_{\Delta}^{(i)}$: zero at the primal unknowns (zero averages on edges)

$S_{\Delta\Delta}^{(i)}$: Local problem matrix

$$S_i = \begin{pmatrix} S_{\Delta\Delta}^{(i)} & S_{\Delta\Pi}^{(i)} \\ S_{\Pi\Delta}^{(i)} & S_{\Pi\Pi}^{(i)} \end{pmatrix}$$

BDDC preconditioner

- ✔ BDDC Preconditioner

$$R^t_\Delta D_\Delta S^{-1}_{\Delta\Delta} D_\Delta R_\Delta + R^t_\Pi D_\Pi F^{-1}_{\Pi\Pi} D_\Pi R_\Pi$$

$R_\Delta : W_\Delta \times W_\Pi \rightarrow W_\Delta$
$R_\Pi : W_\Delta \times W_\Pi \rightarrow W_\Pi$
Note : $\widetilde{W} = W_\Delta \times W_\Pi$

- ✔ Weight affects the convergence
  We aim at finding weight leading to

$$\kappa(B_{DDC}) \leq C(1 + log(H/h))^2$$

FETI-DP preconditioner

- ✔ FETI-DP preconditioner

$$B_\Delta \Sigma_\Delta S_{\Delta\Delta} \Sigma_\Delta B_\Delta^t$$

- ✔ How we choose weight Σ_Δ affects the convergence of iterative methods
- ✔ Partition of B_Δ
  n : nonmortar, m : the remaining

$$B_\Delta = \begin{pmatrix} B_{\Delta,n} & B_{\Delta,m} \end{pmatrix}$$

Note: $B_{\Delta,n}$ is invertible

Neumann-Dirichlet Type weights for FETI-DP

- ✔ Neumann-Dirichlet weight

$$\Sigma_\Delta = \begin{pmatrix} \Sigma_{\Delta,n} & 0 \\ 0 & \Sigma_{\Delta,m} \end{pmatrix}$$

$$\Sigma_{\Delta,n} = (B^t_{\Delta,n} B_{\Delta,n})^{-1}, \Sigma_{\Delta,m} = 0.$$

- ✔ Resulting local problems

$$B_\Delta \Sigma_\Delta S_{\Delta\Delta} \Sigma_\Delta B^t_\Delta \lambda, \quad S^{(i)}_{\Delta\Delta} \Sigma^{(i)}_\Delta (B^{(i)}_\Delta)^t \lambda$$

$$S^{(i)}_{\Delta\Delta} \begin{pmatrix} {B^{(i)}_{\Delta,n}}^{-1} \lambda \\ 0 \end{pmatrix},$$

$$S^{(i)}_{\Delta\Delta} = K^{(i)}_{\Delta\Delta} - K^{(i)}_{\Delta I} (K^{(i)}_{II})^{-1} K^{(i)}_{I\Delta}$$

Neumann-Dirichlet weight for FETI-DP

- ✔ Condition number bound

$$\kappa(F_{DP}) \leq C(1 + log(H/h))^2$$

- ✔ The most efficient one for problems with jump coefficients

$$-\nabla \cdot (\rho(x)\nabla u) = f$$

  $\rho(x) = \rho_i(> 0)$ for $x \in \Omega_i$
  The convergence rate is independent of jumps though the preconditioner does not reflect any information of jump.

- ✔ Our goal is to provide the BDDC algorithm with weight D that performs as good as the FETI-DP algorithm.

Block factorization of $\widetilde{S}$

$$\begin{pmatrix} I & 0 \\ S_{\Pi\Delta}S_{\Delta\Delta}^{-1} & I \end{pmatrix}\begin{pmatrix} S_{\Delta\Delta} & 0 \\ 0 & F_{\Pi\Pi} \end{pmatrix}\begin{pmatrix} I & S_{\Delta\Delta}^{-1}S_{\Delta\Pi} \\ 0 & I \end{pmatrix}$$

$$F_{\Pi\Pi} = S_{\Pi\Pi} - S_{\Pi\Delta}S_{\Delta\Delta}^{-1}S_{\Delta\Pi}$$

- ✔ F_{DP} and B_{DDC} operators

$$F_{DP} = (B\Sigma\widetilde{S}\Sigma B^t)B\widetilde{S}^{-1}B^t,$$

$$B_{DDC} = (R_G D\widetilde{S}^{-1}DR_G^t)R_G\widetilde{S}R_G^t.$$

- ✔ Jump and Average operators
  $P_\Sigma = \Sigma B^t B$, $E_D = R_G^t R_G D$

Theorem

If P_Σ and E_D satisfy

$$P_\Sigma + E_D = I$$
$$E_D^2 = E_D, \quad P_\Sigma^2 = P_\Sigma,$$
$$E_D P_\Sigma = P_\Sigma E_D = 0,$$

then the operators B_{DDC} and F_{DP} have the same spectra except the eigenvalue 1.

- ✔ First shown by Li and Widlund for conforming finite discretization
- ✔ We are able to extend the result to mortar discretization

Weight D for the BDDC algorithm

- ✔ The weight D satisfies the properties in Theorem.

$$D = \begin{pmatrix} D_n & 0 & 0 \\ 0 & D_m & 0 \\ 0 & 0 & D_\Pi \end{pmatrix}, \quad \begin{array}{l} D_n = 0 \\ D_m = I \\ D_\Pi = I \end{array}$$

- ✔ The BDDC algorithm with the weight D shares the same spectra with the FETI-DP algorithm.

$$\kappa(B_{DDC}) \leq C(1 + log(H/h))^2$$

Comparison of the BDDC and FETI-DP

- ✔ Model problem

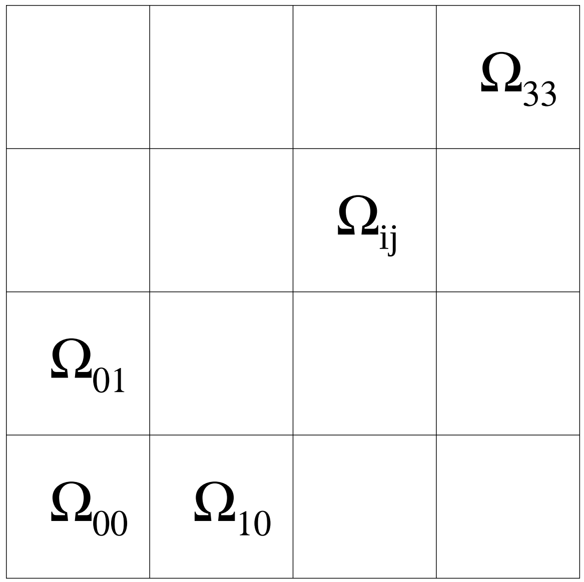
$$-\Delta u(x,y) = f(x,y) \quad (x,y) \in \Omega = [0,1]^2,$$
$$u(x,y) = 0 \quad (x,y) \in \partial\Omega.$$

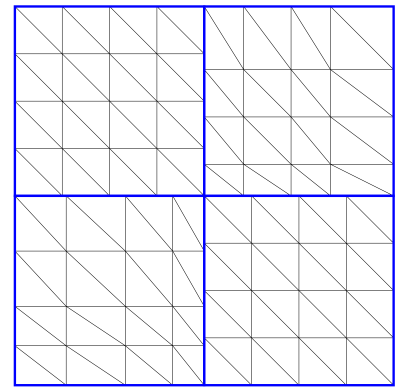
Exact solution: $u(x,y) = y(1-y)\sin \pi x$

- ✔ CGM: relative residual norm $\leq$ 1.0e-6
- ✔ N: the number of subdomains
- ✔ n: the number of nodes on the subdomain edge including end points (H/h)

Comparison of the BDDC and FETI-DP

✔ Subdomain partition and triangulation





Comparison of the BDDC and FETI-DP

- ✔ Local problem size

$N = 4 \times 4$ / $n - 1$	F_{DP} λ_{min}	F_{DP} λ_{max}	B_{DDC} λ_{min}	B_{DDC} λ_{max}
4	1.40	4.09	1.00	4.09
8	1.01	5.72	1.00	5.72
16	1.00	7.72	1.00	7.72
32	1.01	1.00e+1	1.00	1.00e+1
64	1.01	1.28e+1	1.00	1.28e+1

- ✔ The number of subdomains

$n = 5$ / N	F_{DP} λ_{min}	F_{DP} λ_{max}	B_{DDC} λ_{min}	B_{DDC} λ_{max}
4×4	1.40	4.09	1.00	4.09
8×8	1.37	4.41	1.00	4.41
16×16	1.32	4.49	1.00	4.49
32×32	1.30	4.57	1.00	4.62

Performance of the suggested preconditioner

✔ Discontinuous Coefficients

$$-\nabla \cdot (\rho(x) \nabla u(x)) = f(x)$$

where $\rho(x) = \rho_i (> 0)$ for $x \in \Omega_i$.

Preconditioners for F_{DP}

1.Neumann-Dirichlet

$$\widehat{M}_{ND}^{-1} = \begin{pmatrix} {B_{\Delta,n}}^{-1} \\ 0 \end{pmatrix}^t S_{\Delta\Delta} \begin{pmatrix} {B_{\Delta,n}}^{-1} \\ 0 \end{pmatrix}$$

2.Neumann-Neumann

$$\widehat{M}_{NN}^{-1} = (B_\Delta B_\Delta^t)^{-1} B_\Delta S_{\Delta\Delta} B_\Delta^t (B_\Delta B_\Delta^t)^{-1}$$

3.Neumann-Neumann with weight

$$\widehat{M}_{NNW}^{-1} = (B_\Delta D_\Delta^{-1} B_\Delta^t)^{-1} B_\Delta D_\Delta^{-1} S_{\Delta\Delta} D_\Delta^{-1} B_\Delta^t (B_\Delta D_\Delta^{-1} B_\Delta^t)^{-1}$$

Note D_Δ depends on ρ_i

$D_\Delta = \frac{1}{2} I$ when $\rho_i = 1$ for all i.

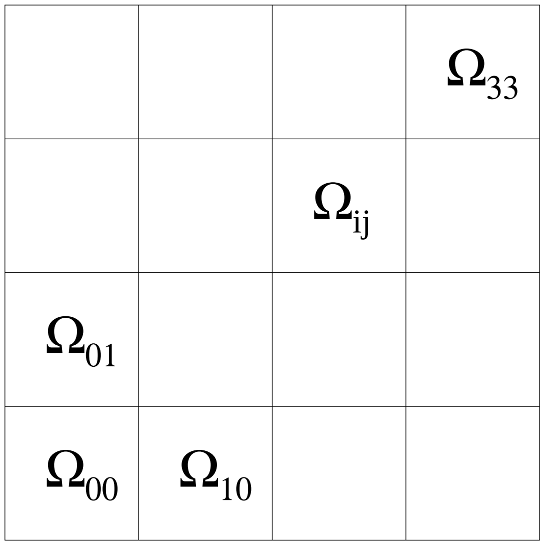
Performance of the suggested preconditioner

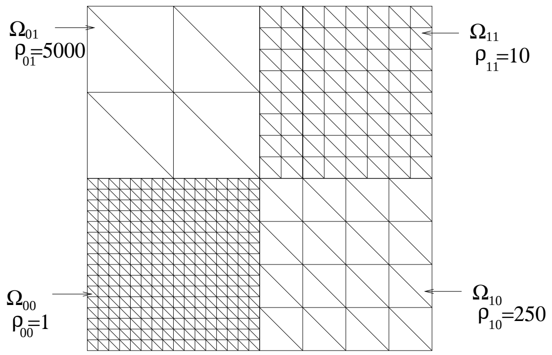
✔ Constant coefficient case $\rho_i = 1$ for all i

$n-1$	$\widehat{M}_{ND}^{-1}$	$\widehat{M}_{NN}^{-1}$	N	$\widehat{M}_{ND}^{-1}$	$\widehat{M}_{NN}^{-1}$
4	10	7	4×4	10	7
8	13	8	8×8	11	8
16	15	10	16×16	12	8
32	16	10	32×32	12	8

The winner is Neumann-Neumann $\widehat{M}_{NN}^{-1}$ for constant coefficient problems.

Performance of the suggested preconditioner





$$\alpha(x,y) = \begin{cases} 1 & (i,j) = (even, even) \\ 250 & (i,j) = (odd, even) \\ 5000 & (i,j) = (even, odd) \\ 10 & (i,j) = (odd, odd) \end{cases}$$

Ratio of meshes: $\frac{h_{ij}}{h_{kl}} \simeq \left(\frac{\rho_{ij}}{\rho_{kl}}\right)^{1/4}$

Performance of the suggested preconditioner

N	$\max(H_{ij}/h_{ij})$	$\widehat{M}_{NN}^{-1}$	$\widehat{M}_{ND}^{-1}$	$\widehat{M}_{NNW}^{-1}$
2×2	16	17	3	3
	32	26	3	3
	64	39	4	3
	128	50	4	4
	256	60	4	4
4×4	16	75	4	3
	32	81	4	4
	64	111	4	4
	128	130	4	4
8×8	16	113	3	3
	32	136	4	4
	64	168	4	4

Extensions to more general PDEs

✔ Choice of primal unknowns is important in scalability

$$\kappa(B_{DDC}), \kappa(F_{DP}) \leq C(1 + log(H/h))^2$$

1. $3D$ elliptic problems, $2D$ Stokes problem
   face (edge) average constraints
   $\int_{F_{ij}} (v_i - v_j)\psi \, ds = 0, \quad \psi = 1$

2. $3D$ elasticity problems
   Six primal constraints on each face F_{ij}
   $\{\mathbf{r}_m\}_{m=1}^{6}$: rigid body motions
   $\int_{F_{ij}} (\mathbf{v}_i - \mathbf{v}_j) \cdot \psi \, ds = 0, \quad \psi = P(\mathbf{r}_m)$
   $P(\mathbf{r}_m)$: L^2-projection onto M_{ij}

Extensions to more general partitions

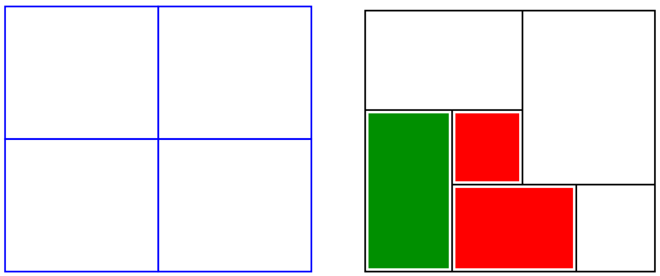


Figure 3: Geometrically conforming (left) and nonconforming (right) partitions

A nonmortar $F \subset \partial\Omega_i$ partitioned by its mortar neighbors Ω_j,

$$F = \cup_j F_{ij}, \quad F_{ij} = \partial\Omega_i \cap \partial\Omega_j.$$

Extensions to more general partitions

Mortar matching condition is

$$\int_F (w_i - \phi)\, \psi \, ds = 0, \quad \forall \psi \in M(F).$$

$M(F)$: Lagrange multiplier space

ϕ is the function from its mortar neighbors

$$\phi = w_j \; on \; F_{ij}.$$

Extensions to more general partitions

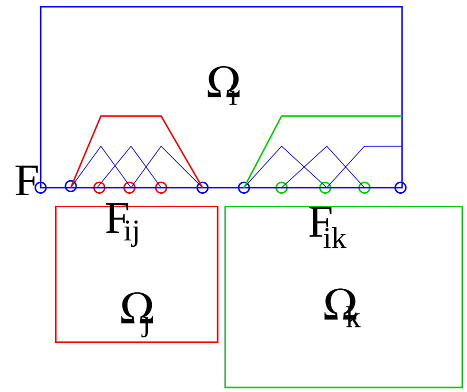


Figure 4: Examples of ψ

✔ Primal constraints on F_{ij}

$$\int_{F_{ij}} (w_i - w_j)\psi \, ds = 0,$$

ψ is **the sum of Lagrange multipliers bases** supported in F_{ij}.

Extensions to more general partitions

- ✔ Primal constraints on F
  We recall the mortar matching condition

$$\int_F (w_i - \phi)\,\psi\,ds = 0, \quad \forall \psi \in M(F).$$

$$\psi = 1, \quad \int_F w_i\,ds = \int_F \phi\,ds$$

$$\int_F w_i\,ds = \sum_j \int_{F_{ij}} w_j\,ds.$$

- ✔ Slightly weaker condition number bound,

$$C(1 + log(H/h))^3$$

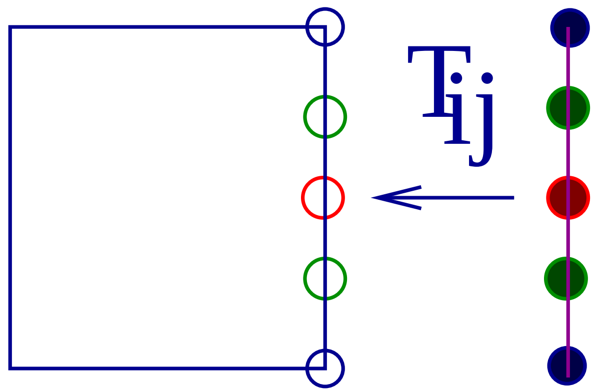
Conclusions

1. FETI-DP with the Neumann-Dirichlet preconditioner
   - ▷ Elliptic problems in $2D$, $3D$
   - ▷ Stokes problem in $2D$
   - ▷ $3D$ compressible elasticity
   - ▷ The most efficient for the problems with coefficient jumps
2. A BDDC algorithm well connected to FETI-DP with ND-preconditioner
3. Geometrically nonconforming subdomain partitions

The end

Thank you!

Change of bases (unknowns) associated to the primal constraints



$$\begin{pmatrix} w_{c,1} \\ w_{\Delta,1} \\ w_{\Pi} \\ w_{\Delta,2} \\ w_{c,2} \end{pmatrix} = a \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} \widehat{w}_{\Pi} + \begin{pmatrix} w_{C,1} \\ \widehat{w}_{\Delta,1} \\ \widehat{z}_{\Pi} \\ \widehat{w}_{\Delta,2} \\ w_{C,2} \end{pmatrix}$$

$$\widehat{z}_{\Pi} = c_1 w_{C,1} + r_1 \widehat{w}_{\Delta,1} + r_1 \widehat{w}_{\Delta,2} + c_2 w_{C,2}$$

$$\begin{pmatrix} w_{C,1} \\ w_{\Delta,1} \\ w_{\Pi} \\ w_{\Delta,2} \\ w_{C,2} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & a & 0 & 0 \\ c_1 & r_1 & a & r_2 & c_2 \\ 0 & 0 & a & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} w_{C,1} \\ \widehat{w}_{\Delta,1} \\ \widehat{w}_{\Pi} \\ \widehat{w}_{\Delta,2} \\ w_{C,2} \end{pmatrix}$$

$\widehat{w}_{\Pi}$: average of w on Γ_{ij}